On irreducibility of modules of Whittaker type

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Basics

Let $(V, Y, 1, \omega)$ be a VOA and g be an automorphism of V with $o(g) = T < \infty$. Then V is a direct sum of the eigenspaces V^j of g,

$$V = \coprod_{j \in \mathbb{Z}/T\mathbb{Z}} V^j,$$

where $V^j = \left\{ v \in V \mid gv = \eta^j v \right\}$ for η a fixed primitive T-th root of unity.

The action of Aut(V) on twisted modules

Aut(V) : the group of automorphisms g of V

$g, h \in \mathrm{Aut}(V)$ with $o(g) < \infty$

(M, Y_M): a weak g-twisted V-module

There is a weak hgh^{-1}-twisted V-module $(h \circ M, Y_{h \circ M})$ where $h \circ M \cong M$ as a vector space and

$$Y_{h \circ M}(v, z) = Y_M\left(h^{-1}v, z\right) \text{ for } v \in V.$$

This defines a left action of Aut(V) on weak twisted V-modules and on the isomorphism classes of weak twisted V-modules. We write

$$h \circ (M, Y_M) = (h \circ M, Y_{h \circ M}) = h \circ M.$$

We say M is ***h-stable*** if M and $h \circ M$ are isomorphic.

Note that $g \circ M$ and M are isomorphic if M is an admissible g-twisted V-module [Dong-Li-Mason; 2000].

Backgroud: Quantum Galois Theory

Theorem. [Dong-Mason; 1997]

Let (M, Y_M) be an irreducible ordinary module for the VOA V. Let g be an automorphism of V of prime order p such that $M \circ g \ncong M$. Then M is an irreducible module for the orbifold subalgebra $V^{\langle g \rangle}$.

Goal:

- Extend this result to modules that are not ordinary
- Applications: **Constuction of irreducibles modules for orbifold vertex (super)algebras**

Previous results on cyclic orbifold

Theorem. [Adamovic-Lam-Pedic-Y.; 2019] Let V be a VOA, $g \in \mathrm{Aut}(V)$ with $o(g) = T < \infty$. Let W be an irreducible weak V-module such that $W, W \circ g, \cdots, W \circ g^{T-1}$ are inequivalent irreducible modules. Then W is an irreducible weak $V^{\langle g \rangle}$-module.

These conditions can be checked for a large class of Whittaker modules for certain infinite dimensional Lie algebras.

Lie algebra associated to the VOA

The Lie algebra $\mathfrak{g}(V)$ associated to the VOA V is realized on the vector space [Borcherds; 1986][Dong-Li-Mason; 1998]

$$\mathfrak{g}(V) = \frac{V \otimes \mathbb{C}[t, t^{-1}]}{(L(-1) \otimes 1 + 1 \otimes \frac{d}{dt}).V \otimes \mathbb{C}[t, t^{-1}]},$$

where the commutator is given by

$$[a \otimes t^n, b \otimes t^m] = \sum_{i=0}^{\infty} \binom{n}{i} (a_i b) \otimes t^{n+m-i}.$$

Let V be a vertex operator algebra. Then [Dong-Li-Mason; 1998]

- Every weak V-module W is a $\mathfrak{g}(V)$-module with the action $v \otimes t^n \mapsto v_n \quad (v \in V,\ n \in \mathbb{Z})$.
- If W is an irreducible weak V-module, then W is also an irreducible $\mathfrak{g}(V)$-module.

Whittaker modules

Let V be a vertex algebra. Assume that the Lie algebra $\mathfrak{L}$ is one of the following:

(i) $\mathfrak{L} = g(V)$, or

(ii) $\mathfrak{L}$ is the Lie algebra of modes of generating fields of the vertex algebra V .

Then every weak V-module is a module for the Lie algebra $\mathfrak{L}$. We also assume the following:

- Let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{L}$.
- Let Wh$(\mathfrak{L}, \mathfrak{n})$ denotes the full category of $\mathfrak{L}$-modules M such that $\mathfrak{n}$ acts locally finitely on M (that is, $U(\mathfrak{n})\, v$ is finite dimensional for all $v \in M$ [Batra-Mazorchuk; 2011]).

Whittaker modules

Whittaker modules Lie algebras: Virasoro algebra [Ondrus-Wiesner; 2009, 2013][Guo-Lu-Zhao; 2011], Heisenberg-Virasoso[Lu-Zhao; 2013], Affine Lie algebra [Adamovic-Lu-Zhao; 2016]...

Heisenberg VOAs: [Tanabe; 2017][Hartwig-Y.; 2019]...

Definition. Let $\lambda : \mathfrak{n} \to \mathbb{C}$ be a Lie algebra homomorphism which will be called a ***Whittaker function***. Let $U_\lambda = \mathbb{C}u_\lambda$ be the 1-dimensional $\mathfrak{n}$-module such that

$$xu_\lambda = \lambda(x)u_\lambda \quad (x \in \mathfrak{n}).$$

Consider the standard (universal) Whittaker $\mathfrak{L}$-module

$$M_\lambda = U(\mathfrak{L}) \otimes_{U(\mathfrak{n})} U_\lambda.$$

Definition. We say that an irreducible V-module W is of **Whittaker type** λ if W is an irreducible quotient of the standard Whittaker module M_λ.

Previous applications to Whittaker modules

[Adamovic-Lam-Pedic-Y.; 2019]

Theorem. Let W be an irreducible weak V-module such that all $W_i = g^i \circ W$ are Whittaker modules whose Whittaker functions $\lambda^{(i)} : \mathfrak{n} \to \mathbb{C}$ are mutually distinct. Then W is an irreducible weak $V^{\langle g \rangle}$-module.

Applications to Whittaker moduls of Heisenberg and Weyl vertex algebras.

Questions

Whether Dong-Mason theorem is true for

(1) weak twisted modules of a vertex operator superalgebra;

(2) for non-abelian orbifolds V^G, when G is a non-abelian finite group;

(3) for orbifolds V^G, when G is an infinite-group.

We give a positive answer to generalizations of types (1) and (2).

Remark. [Adamovic-Tomic, 2022] proved that Whittaker modules $M_1(\lambda, \mu)$ of Weyl vertex algebra is reducible for an infinite orbifolds of M. So it seems that there are no natural generalizations to infinite groups.

Main Results

Theorem. [Adamovic-Lam-Tomic-Y.; 2022] Let V be vertex superalgebra with a countable dimension, $G \leq \mathrm{Aut}(V)$ a finite group (not necessarily abelian) and $h \in Z(G)$, the center of the group G. Let M be an irreducible weak h-twisted module of V such that $M \ncong g \circ M$ for all $g \in G$. Then M is also irreducible as a V^G-module.

In the case $g = h$ and $G = \langle g \rangle$, this theorem gives a twisted version of Theorem in [Adamovic-Lam-Pedic-Y.; 2019].

Proof of Theorem

Let $G < \mathrm{Aut}(V)$ be a finite subgroup (not necessarily abelian). Consider the h-twisted V-module

$$\mathcal{M} = \oplus_{g \in G} g \circ M.$$

Definition. For each $\chi \in \mathrm{Irr}(G)$, define $P_\chi \in \mathbb{C}[G]$ by

$$P_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g.$$

Denote P_χ^V and $P_\chi^{\mathcal{M}}$ to be the corresponding actions of P_χ on V and $\mathcal{M}$, respectively.

Then

$$V = \oplus_{\chi \in \mathrm{Irr}(G)} V_\chi = \oplus_{\chi \in \mathrm{Irr}(G)} P_\chi^V(V),$$

and

$$\mathcal{M} = \oplus_{\chi \in \mathrm{Irr}(G)} P_\chi^{\mathcal{M}}(\mathcal{M}) = \oplus_{\chi \in \mathrm{Irr}(G)} \mathcal{M}_\chi,$$

where $\mathcal{M}_\chi = P_\chi^{\mathcal{M}}(\mathcal{M})$.

- **Lemma.** For each $\chi \in \mathrm{Irr}(G)$, we have $V_\chi \cdot \mathcal{M}_1 \subset \mathcal{M}_\chi$.

Main Results

Let V be a vertex superalgebra with a countable dimension.

Theorem. Assume $g \in \mathrm{Aut}(V)$ and $o(g) = T < \infty$. Assume that W is an irreducible weak g-twisted V-module such that $g \circ W \cong W$. Then W is a completely reducible weak V^0-module such that

(1) $W = \oplus_{i=0}^{T} W^i$, $V^i.W^j \subset W^{i+j \bmod T}$, where W^j, $j = 1, \cdots, T$, are eigenspaces of certain linear isomorphism $\phi(g) : W \to W$.

(2) Each W^i is an irreducible weak V^0-module.

(3) The modules W^i, $i = 0, \cdots, T-1$, are non-isomorphic as weak V^0-modules.

Neveu-Schwarz vertex superalgebras

Neveu-Schwarz algebra is the Lie superalgebra

$$\mathfrak{ns} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}L(i) \oplus \bigoplus_{r \in \frac{1}{2}+\mathbb{Z}} \mathbb{C}G(r) \oplus \mathbb{C}C$$

which satisfies:

$$\begin{aligned}
[L(m), L(n)] &= (m-n)L(m+n) + \tfrac{1}{12}(m^3 - m)\delta_{n+m,0}C \\
[L(m), G(r)] &= (\frac{m}{2} - r)G(m+r) \\
[G(r), G(s)] &= 2L(r+s) + (r^2 - \tfrac{1}{4})\delta_{r+s,0}C \\
[\mathfrak{ns}, C] &= 0
\end{aligned}$$

for all $m, n \in \mathbb{Z}$, $r, s \in \frac{1}{2} + \mathbb{Z}$.

Let $V_c(\mathfrak{ns})$ be the universal Neveu-Schwarz vertex superalgebra of central charge $c \in \mathbb{C}$. The category of weak $V_c(\mathfrak{ns})$-modules coincides with the category of restricted $\mathfrak{ns}$-module.

Whittaker modules of Neveu-Schwarz vertex superalgebras

Let

$$\mathfrak{p} = \bigoplus_{i>0} L(i) \oplus \bigoplus_{i>0} \mathbb{C}G(i+\tfrac{1}{2})$$

and let $\Psi : \mathfrak{p} \to \mathbb{C}$ be a Lie superalgebra homomorphism. Assume that Ψ is non-zero. Then Ψ is uniquely determined by $(a, b) \in \mathbb{C}^2$, $a \cdot b \neq 0$ such that

$$\Psi(G(i+\tfrac{1}{2})) = \Psi(L(i+2)) = 0 \text{ for each } i \in \mathbb{Z}_{>0}$$
$$\Psi(L(1)) = a \text{ and } \Psi(L(2)) = b.$$

Let $\mathbb{C}w$ be the 1-dimensional $(\mathfrak{p} + \mathbb{C}C)$-module such that

$$xw = \Psi(x)w \quad (x \in \mathfrak{p}), \quad Cw = cw.$$

The universal Whittaker module associated to Whittaker function Ψ is defined as

$$\mathrm{Wh}(\Psi, c) = U(\mathfrak{ns}) \otimes_{U(\mathfrak{p}+\mathbb{C}C)} \mathbb{C}w.$$

It was proved in [Liu-Pei-Xia, 2020] that $\mathrm{Wh}(\Psi, c)$ is an irreducible, restricted $\mathfrak{ns}$-modules if $\Psi \neq 0$.

Orbifold of Neveu-Schwarz vertex superalgebras

Note that $V_c(\mathfrak{ns})$ has the canonical automorphism σ of order two which is lifted from the automorphism of Lie superalgebra $\mathfrak{ns}$ such that

$$L(n) \mapsto L(n), G(n + \tfrac{1}{2}) \mapsto -G(n + \tfrac{1}{2}), C \mapsto C \quad (n \in \mathbb{Z}).$$

Let $V_c^+(\mathfrak{ns})$ be the fixed point vertex subalgebra.

- $V_c^+(\mathfrak{ns})$ is a W-algebra of type $W(2, 4, 6)$ [Li-Milas-Wauchope; 2021][Milas-Penn; 2020].

Note thtat $\sigma \circ \mathrm{Wh}(\Psi, c) \cong \mathrm{Wh}(\Psi, c)$.

Proposition. [Adamovic-Lam-Tomic-Y.; 2022] $\mathrm{Wh}(\Psi, c)$ is an irreducible $V_c(\mathfrak{ns})$-module for each nonzero Whittaker function Ψ. Moreover, $\mathrm{Wh}(\Psi, c)$ is a direct sum of two non-isomorphic irreducible modules for $V_c^+(\mathfrak{ns})$.

The $N = 1$ Ramond algebra

The $N = 1$ Ramond algebra is the Lie superalgebra

$$\mathfrak{R} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}L(i) \oplus \bigoplus_{r \in \mathbb{Z}} \mathbb{C}G(r) \oplus \mathbb{C}C$$

which satisfies:

$$\begin{aligned}
[L(m), L(n)] &= (m-n)L(m+n) + \tfrac{1}{12}(m^3 - m)\delta_{n+m,0}C \\
[L(m), G(r)] &= (\frac{m}{2} - r)G(m+r) \\
[G(r), G(s)] &= 2L(r+s) + (r^2 - \tfrac{1}{4})\delta_{r+s,0}C \\
[\mathfrak{ns}, C] &= 0
\end{aligned}$$

for all $m, n \in \mathbb{Z}$, $r, s \in \mathbb{Z}$.

- The category of weak σ-twisted $V_c(\mathfrak{ns})$-modules coincides with the category of restricted $\mathfrak{R}$-module.

Whittaker modules of $N = 1$ Ramond algebra

Let

$$\mathfrak{p}^{tw} = \bigoplus_{i>0} L(i) \oplus \bigoplus_{i>0} \mathbb{C}G(i).$$

Let $\Psi : \mathfrak{p}^{tw} \to \mathbb{C}$ be the Lie superalgebra homomorphism. Assume that Ψ is non-zero. Then Ψ is uniquely determined by $(a, b) \in \mathbb{C}^2$, $a \cdot b \neq 0$ such that $\Psi(G(i+1)) = \Psi(L(i+2)) = 0$ for each $i \in \mathbb{Z}_{>0}$ and

$$\Psi(G(1)) = a, \ \Psi(L(1)) = b, \ \Psi(L(2)) = a^2.$$

Let $\mathbb{C}w$ be the 1-dimensional $\left(\mathfrak{p}^{tw} + \mathbb{C}C\right)$-module such that

$$xw = \Psi(x) \ (x \in \mathfrak{p}^{tw}), \ Cw = cw.$$

The universal Whittaker module associated to Whittaker function Ψ is defined as

$$\mathrm{Wh}^{tw}(\Psi, c) = U(\mathfrak{ns}) \otimes_{U(\mathfrak{p}^{tw}+\mathbb{C}C)} \mathbb{C}w.$$

Let $L^{tw}(\Psi, c)$ be its simple quotient.

Remark. One can show that $\mathrm{Wh}^{tw}(\Psi, c)$ is an irreducible, restricted $\mathfrak{R}$-modules if $\Psi \neq 0$, and therefore $L^{tw}(\Psi, c) = \mathrm{Wh}^{tw}(\Psi, c)$.

Theorem. [Adamovic-Lam-Tomic-Y.; 2022] Assume that Ψ is an non-zero Whittaker function such that $\Psi(G(1)) = c \neq 0$. Then the irreducible Whittaker module $L^{tw}(\Psi, c)$ is an irreducible module for the orbifold vertex algebra $V_c^+(\mathfrak{ns})$.

Permutation orbifold of Heisenberg VAs

$\mathfrak{h}$: a complex ℓ-dimensional vector space with a non-degenerate form $(\cdot,\cdot)$ equipped with the structure of a commutative Lie algebra.

$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K$: Heisenberg algebra

$M(\ell)$: the Heisenberg VOA of rank ℓ associated to the $\hat{\mathfrak{h}}$.

Then the automorphism group of $M(1)$ is isomorphic to the orthogonal group $O(\ell)$.

The symmetric group S_ℓ is a subgroup of $O(\ell)$ and acts on $M(1)$ as permutations on an orthonormal basis of $\mathfrak{h}$.

Let $\mathfrak{n} = \mathfrak{h} \otimes \mathbb{C}[t]$ be its nilpotent Lie subalgebra. Define the Whittaker function $\boldsymbol{\lambda} \in \mathfrak{n}^*$ such that for each $h \in \mathfrak{h}$

$$\boldsymbol{\lambda}(h(n)) = 0 \quad \text{for } n >> 0.$$

Let $M(1, \boldsymbol{\lambda})$ be the standard Whittaker modules for $\hat{\mathfrak{h}}$ of level 1 associated to the Whittaker function $\boldsymbol{\lambda}$.

For $g \in O(\ell)$ we define:

$$g \circ M(1, \boldsymbol{\lambda}) = M(1, g \circ \boldsymbol{\lambda}).$$

Theorem. [Adamovic-Lam-Tomic-Y.; 2022] Let G be any finite subgroup of $O(\ell)$ such that

$$g \circ \boldsymbol{\lambda} \neq \boldsymbol{\lambda}, \quad \forall g \in G.$$

Then $M(1, \boldsymbol{\lambda})$ is irreducible $M(1)^G$-module.

Corollary. Assume that $\sigma \circ \boldsymbol{\lambda} \neq \boldsymbol{\lambda}$ for any 2-cycle $\sigma \in S_\ell$. Then $M(1, \boldsymbol{\lambda})$ is an irreducible $M(1)^{S_\ell}$-module.

Remark. We believe that $M(1, \boldsymbol{\lambda})$ is also irreducible for affine W-algebras realized as subalgebras of $M(1)^{S_\ell}$. But our current techniques are not sufficient to prove this claim.

Permutation orbifold of Virasoro VOA

Vir : the Virasoro Lie algebra with generators $L(n)$, $n \in \mathbb{Z}$, and the central element C

Vir^c : the universal Virasoro VOA of central charge c

Vir^+ : the Lie subalgebra of Vir generated by $L(n)$, $n \in \mathbb{Z}_{>0}$.

The classical *Whittaker module* for the Virasoro algebra of central charge c is generated by the Whittaker function $\Psi : \mathrm{Vir}^+ \to \mathbb{C}$ such that

$$\Psi(L(1)) = a, \Psi(L(2)) = b, \Psi(L(n)) = 0 \quad (n > 3).$$

Let $\mathrm{Wh}_c(\Psi)$ be the universal Whittaker Virasoro module of central charge c.

- $\mathrm{Wh}_c(\Psi)$ is irreducible [Lu-Guo-Zhao; 2011].
- Since $\mathrm{Wh}_c(\Psi)$ is a restricted Vir-module, it is an irreducible module for the Vir^c.

Let $U = \mathrm{Vir}^c$, and consider $V = U^{\otimes \ell}$ for $\ell \in \mathbb{Z}_{>0}$. Then the symmetric group S_ℓ acts naturally on V as a group of automorphisms.

Let $G \leq S_\ell = \mathrm{Aut}(V)$. Then

Theorem. [Adamovic-Lam-Tomic-Y.; 2022] Assume that $\Psi_i : \mathrm{Vir}^+ \to \mathbb{C}$, $i = 1, \ldots, \ell$ are Whittaker functions such that

$$\Psi_i(L(1)) = a_i, \Psi_i(L(2)) = b_i, \Psi_i(L(n)) = 0 \quad (n > 3).$$

Assume that for each $g \in G$:

$$(a_{g(1)}, \cdots, a_{g(\ell)}) \neq (a_1, \ldots, a_\ell) \quad \text{or} \quad (b_{g(1)}, \cdots, b_{g(\ell)}) \neq (b_1, \ldots, b_\ell).$$

Then

$$\mathrm{Wh}(\Psi_1, \ldots, \Psi_\ell) := \mathrm{Wh}_c(\Psi_1) \otimes \cdots \otimes \mathrm{Wh}_c(\Psi_\ell)$$

is an irreducible V^G-module.

Recent related results

[Dong-Ren-Yang; 2023] Every irreducible σ-twisted V-module is a direct sum of finitely many irreducible V^G-modules and irreducible V^G-modules appearing in different G-orbits are inequivalent.

[Tanabe; 2023] Generalize the Schur-Weyl type duality to twisted weak modules for a vertex algebra.

Thanks !