Collapsing chains

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Let $\mathfrak{g}$ be a simple Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is reductive and $\mathfrak{g}$ admits a nondegenerate even supersymmetric invariant form $(\cdot|\cdot)$.

- Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\bar{0}}$ and a positive set of roots for Δ^+ in the set Δ of roots for $(\mathfrak{g}, \mathfrak{h})$ such that the highest root θ is even.
- Choose f to be a root vector $e_{-\theta}$.
- The invariant bilinear form $(.\,|\,.)$ is normalized so that $(\theta|\theta) = 2$.
- Choose the root vector $e = e_\theta \in \mathfrak{g}_\theta$ in such a way that $(e_\theta|e_{-\theta}) = \frac{1}{2}$. Set $x = [e_\theta, e_{-\theta}]$.

The minimal gradation

The gradation induced by $\mathrm{ad}x$ is

$$\mathfrak{g} = \mathbb{C}f \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathbb{C}e$$

Moreover

$$\mathfrak{g}_0 = \mathbb{C}x \oplus \mathfrak{g}^\natural$$

with $\mathfrak{g}^\natural = \mathfrak{g}_0^f$. Moreover

$$\mathfrak{g}^\natural = \oplus_i \mathfrak{g}_i^\natural$$

with $\mathfrak{g}_i^\natural$ simple or abelian.

The minimal W-algebra $W^k(\mathfrak{g},\theta)$ is the VOA freely generated by fields

$$J^{\{a^i\}}\ (\{a^i\} \text{ a basis of } \mathfrak{g}^\natural), G^{\{u^i\}}\ (\{u^i\} \text{ a basis of } \mathfrak{g}_{-1/2}),\ L$$

λ-bracket

- L is a conformal vector with central charge

$$\frac{k\,\dim\mathfrak{g}}{k+h^\vee}-6k+h^\vee-4$$

- $J^{\{a\}}$, $a\in\mathfrak{g}^\natural$ are primary of conformal weight $\Delta=1$ and generate an affine vertex algebra $V^{\beta_k}(\mathfrak{g}^\natural)$, where

$$\beta_k(a,b)=\delta_{ij}(k+\frac{h^\vee-h_i^\vee}{2})(a|b),\ a\in\mathfrak{g}_i^\natural,\ b\in\mathfrak{g}_j^\natural.$$

  $h_i^\vee$ is half the eigenvalue of the Casimir of $\mathfrak{g}^\natural$ acting on $\mathfrak{g}_i^\natural$ relative to $(\cdot|\cdot)_{|\mathfrak{g}^\natural\times\mathfrak{g}^\natural}$

- $[J^{\{a\}}{}_\lambda G^{\{v\}}]=G^{\{[a,v]\}}$ for $v\in\mathfrak{g}_{-1/2}$, $a\in\mathfrak{g}^\natural$;

λ-bracket

- $G^{\{v\}}$, $v \in \mathfrak{g}_{-1/2}$ are primary of conformal weight $\Delta = \frac{3}{2}$;
- Set $\langle u, v\rangle = (e|[u,v])$ for $u, v \in \mathfrak{g}_{-1/2}$. In all cases $\mathfrak{g}^\natural = \oplus_i \mathfrak{g}_i^\natural$ with $\mathfrak{g}_i^\natural$ simple or abelian ideal. Let $a^\natural$ (resp. $a_i^\natural$) denote the projection of $a \in \mathfrak{g}_0$ on $\mathfrak{g}^\natural$ (resp. $\mathfrak{g}_i^\natural$) w.r.t. $\mathfrak{g}_0 = \oplus \mathfrak{g}_i^\natural \oplus \mathbb{C}x$. Then

$$[G^{\{u\}}{}_\lambda G^{\{v\}}] = -2(k+h^\vee)\langle u,v\rangle L + \langle u,v\rangle \sum_{\alpha=1}^{\dim \mathfrak{g}^\natural} :J^{\{u^\alpha\}} J^{\{u_\alpha\}}:$$

$$+2\sum_{\alpha,\beta} \langle [u_\alpha, u], [v, u^\beta]\rangle :J^{\{u^\alpha\}} J^{\{u_\beta\}}: +2(k+1)(\partial + 2\lambda) J^{\{[[e,u],v]^\natural\}}$$

$$+2\lambda \sum_{\alpha,\beta} \langle [u_\alpha, u], [v, u^\beta]\rangle J^{\{[u^\alpha, u_\beta]\}} + 2\lambda^2 \langle u, v\rangle p(k)|0\rangle$$

Table for $p(k)$ in the classical cases

$\mathfrak{g}$	$p(k)$
$sl(m\|n)$, $n \neq m$	$(k+1)(k+(m-n)/2)$
$psl(m\|m)$	$k(k+1)$
$osp(m\|n)$	$(k+2)(k+(m-n-4)/2)$
$spo(n\|m)$	$(k+1/2)(k+(n-m+4)/4)$

Table for $p(k)$: exceptional cases

$\mathfrak{g}$	$p(k)$
E_6	$(k+3)(k+4)$
E_7	$(k+4)(k+6)$
E_8	$(k+6)(k+10)$
F_4	$(k+5/2)(k+3)$
G_2	$(k+4/3)(k+5/3)$
$D(2,1;a)$	$(k-a)(k+1+a)$
$F(4)$, $\mathfrak{g}^\natural = so(7)$	$(k+2/3)(k-2/3)$
$F(4)$, $\mathfrak{g}^\natural = D(2,1;2)$	$(k+3/2)(k+1)$
$G(3)$, $\mathfrak{g}^\natural = G_2$	$(k-1/2)(k+3/4)$
$G(3)$, $\mathfrak{g}^\natural = osp(3\|2)$	$(k+2/3)(k+4/3)$

The minimal simple W-algebra $W_k(\mathfrak{g}, \theta)$

If $k \neq -h^\vee$ then $W^k(\mathfrak{g}, \theta)$ admits a unique simple quotient denoted by $W_k(\mathfrak{g}, \theta)$.

Quantum reduction functor

The quantum reduction functor is a functor from categories of $V^k(\mathfrak{g})$-modules to the category of $W^k(\mathfrak{g},\theta)$-modules such that

$$\mathcal{H}(V^k(\mathfrak{g})) = W^k(\mathfrak{g},\theta)$$

and, if $k \notin \mathbb{Z}_+$,

$$\mathcal{H}(V_k(\mathfrak{g})) = W_k(\mathfrak{g},\theta).$$

In particular $\mathcal{H}$ maps $V_k(\mathfrak{g})$-modules to $W_k(\mathfrak{g},\theta)$-modules.

Let $\mathfrak{h}^\natural$ be the Cartan subalgebra of $\mathfrak{g}^\natural$, and let

$$\mathfrak{g}^\natural = \mathfrak{n}^\natural_- \oplus \mathfrak{h}^\natural \oplus \mathfrak{n}^\natural_+$$

be the triangular decomposition corresponding to a choice of the set of positive roots. For $\nu \in (\mathfrak{h}^\natural)^*$ and $\ell_0 \in \mathbb{C}$, a highest weight module for $W^k(\mathfrak{g}, \theta)$ of highest weight (ν, ℓ_0) is a module generated by a vector v_{ν,ℓ_0} (the highest vector) such that

1. $J_0^{\{h\}} v_{\nu,\ell_0} = \nu(h) v_{\nu,\ell_0}$ for $h \in \mathfrak{h}^\natural$
2. $L_0 v_{\nu,\ell_0} = \ell_0 v_{\nu,\ell_0}$
3. $J_n^{\{u\}} v_{\nu,\ell_0} = G_n^{\{v\}} v_{\nu,\ell_0} = L_n v_{\nu,\ell_0} = 0$ for $n > 0$, $u \in \mathfrak{g}^\natural$, $v \in \mathfrak{g}_{-1/2}$
4. $J_0^{\{u\}} v_{\nu,\ell_0} = 0$ for $u \in \mathfrak{n}^\natural_+$.

Fix a basis $\{v_i \mid i \in I\}$ of $\mathfrak{g}_{-1/2}$ and a basis $\{u_i \mid i \in J\}$ of $\mathfrak{n}_-^\natural$. Set $A^{\{i\}} = J^{\{u_i\}}$ if $i \in J$, $A^{\{i\}} = G^{\{v_i\}}$ if $i \in I$, and $A^{\{0\}} = L$.

A highest weight module M of highest weight (ν, ℓ_0) is called a Verma module if

$$\mathcal{B} = \left\{ \left(A^{\{1\}}_{-m_1}\right)^{b_1} \cdots \left(A^{\{s\}}_{-m_s}\right)^{b_s} v_{\nu,\ell_0} \right\}$$

where $b_i \in \mathbb{Z}_+$, $b_i \leqslant 1$ if $i \in I$, $m_i \geqslant 0$ is a basis of M.

The Verma module of highest weight (ν, ℓ_0) exists, is unique up to isomorphism, and is denoted by $M^W(\nu, \ell_0)$.

If M is a highest weight module of highest weight (ν, ℓ_0) then M is a quotient of $M^W(\nu, \ell_0)$. The unique irreducible quotient is denoted by $L^W(\nu, \ell_0)$.

Reduction of highest weight modules

- If M is a highest weight module for $V^k(\mathfrak{g})$ of highest weight Λ, then $\mathcal{H}(M)$ is either zero or a highest weight module over $W^k(\mathfrak{g},\theta)$ of highest weight (ν,ℓ) with

$$\nu = \Lambda_{|\mathfrak{h}^\natural}, \quad \ell = \frac{(\Lambda|\Lambda+2\rho)}{2(k+h^\vee)} - \Lambda(x).$$

- The functor $\mathcal{H}$ maps Verma modules to Verma modules.
- Let $\lambda \in \widehat{\mathfrak{h}}_k^*$. If $(\lambda|\delta-\theta) \in \{0,1,2,\dots\}$, then $\mathcal{H}(L(\lambda)) = \{0\}$. Otherwise, $\mathcal{H}(L(\lambda))$ is an irreducible highest weight $W^k(\mathfrak{g},\theta)$-module.

Collapsing levels

The roots of $p(k)$ are called collapsing levels.

If k is a collapsing level then $W_k(\mathfrak{g},\theta)$ "collapses" to the image of $V^{\beta_k}(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g},\theta)$:

$$W_k(\mathfrak{g},\theta) = V_{\beta_k}(\mathfrak{g}^\natural)$$

Remark

In this case the reduction functor provides a functor from $V_k(\mathfrak{g})$-modules to $V_{\beta_k}(\mathfrak{g}^\natural)$-modules.

Example

$$\mathfrak{g} = D(2,1;n),\ \ n \in \mathbb{N},\ \ k = -\frac{n}{n+1}, W_k(\mathfrak{g},\theta) = V_{n-1}(sl(2)).$$

A highest weight module $L(\Lambda)$ can be a $V_k(\mathfrak{g})$-module only if

$$\Lambda = \ell\theta + \frac{r}{2}\alpha$$

where α is the positive root of $sl(2)$, $r = 0, 1, 2, \dots, n-1$ and

$$\ell = -\frac{r}{2(n+1)},\ \ \ell = \frac{r+2}{2(n+1)},\ \ \ell = -\frac{n}{2(n+1)} - \frac{m}{2}, m \in \mathbb{Z}_+$$

Fun fact

For many collapsing levels the collapsed algebra

$$W_k(\mathfrak{g},\theta) = V_{k'}(\mathfrak{g}_i^\natural)$$

with k' collapsing again.

Take $\mathfrak{g} = so(m)$ and $k = 2 - m/2$. Then

$$W_k(\mathfrak{g}, \theta) = V_{4-m/2}(so(m-4)).$$

In such a case one has a collapsing chain:

$$V_k(\mathfrak{g}), \mathcal{H}(V_k(\mathfrak{g})), \mathcal{H}^2(V_k(\mathfrak{g})), \dots,$$

Collapsing chains

Definition

We say that a sequence $(\mathfrak{g}_0, k_0), (\mathfrak{g}_1, k_1), \ldots, (\mathfrak{g}_n, k_n)$ is a collapsing chain and write $(\mathfrak{g}_0, k_0) \triangleright (\mathfrak{g}_1, k_1) \triangleright \ldots \triangleright (\mathfrak{g}_n, k_n)$ if

$$\mathcal{H}(V_{k_i}(\mathfrak{g}_i)) = V_{k_{i+1}}(\mathfrak{g}_{i+1}), \;\; i = 0, \ldots, n-1$$

Example

Let $(\mathfrak{g}, k) = (so(m), 2 - m/2)$. There is a collapsing chain $(\mathfrak{g}, k) = (\mathfrak{g}_0, k_0) \triangleright \ldots \triangleright (\mathfrak{g}', k')$ with

$$(\mathfrak{g}', k') = \begin{cases} \mathbb{C} & \text{if } m \equiv 0, 1 \mod 4, \\ M(1) & \text{if } m \equiv 2 \mod 4, \\ (sl(2), 1) & \text{if } m \equiv 3 \mod 4. \end{cases}$$

Definition

We denote by $KL^k(\mathfrak{g})$ the category of weak modules for $V^k(\mathfrak{g})$, which

(1) are locally finite as $\mathfrak{g}$-modules;

(2) admit a decomposition into generalized eigenspaces for $L_{\mathfrak{g}}(0)$ whose eigenvalues are bounded below.

We also denote by $KL_k(\mathfrak{g})$ the full subcategory of $KL^k(\mathfrak{g})$ consisting of the $V_k(\mathfrak{g})$-modules.

Definition

We denote by $KL^k_{weight}(\mathfrak{g})$ (resp. KL_k^{weight}) the full subcategory of modules in $KL^k(\mathfrak{g})$ (resp. $KL_k(\mathfrak{g})$) on which $L_{\mathfrak{g}}(0)$ and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ act semisimply.

Semisimplicity of KL_k

Main goal

Use the collapsing chains to prove that KL_k is semisimple.

Main strategy

First prove that KL_k^{weight} is semisimple then prove that $KL_k = KL_k^{weight}$.

How to prove that KL_k^{weight} is semisimple

Theorem

Suppose that we have a collapsing chain

$$(\mathfrak{g}_0, k_0) \triangleright (\mathfrak{g}_1, k_1) \triangleright \cdots \triangleright (\mathfrak{g}_n, k_n)$$

If $KL_{k_n}^{weight}(\mathfrak{g}_n)$ is semisimple then $KL_{k_i}^{weight}(\mathfrak{g}_i)$ is semisimple for all i.

One proves by induction that $KL_{k_i}^{weight}(\mathfrak{g}_i)$ is semisimple for all i. The proof of the inductive step is a collection of various facts:

- If $M \in KL_{k_i}^{weight}(\mathfrak{g}_i)$ is highest weight then $\mathcal{H}(M)$ is highest weight and nonzero. Since $KL_{k_{i+1}}^{weight}(\mathfrak{g}_{i+1})$ is semisimple $\mathcal{H}(M)$ is irreducible.
- $\mathcal{H}$ is exact on category $\mathcal{O}$. This implies that M is irreducible, so every highest weight in $KL_{k_i}^{weight}(\mathfrak{g}_i)$ is irreducible.
- If M is finitely generated then it is in category $\mathcal{O}$. In particular it admits a finite filtration $\{0\} = M_0 \subset M_1 \subset \cdots$ with highest weight subquotients.
- By the first part of the proof M_i/M_{i-1} is irreducible.

Proof continued

At this point one needs

Lemma

If M, N are highest weight modules in $KL_{k_i}^{weight}(\mathfrak{g}_i)$ then $Ext^1(M, N) = 0$.

Using the lemma one proves that every finitely generated module in $KL_{k_i}^{weight}(\mathfrak{g}_i)$ are semisimple using induction on the length of the filtration $\{0\} = M_0 \subset M_1 \subset \cdots$.

Using abstract nonsense one extends semisimplicity to all $KL_{k_i}^{weight}(\mathfrak{g}_i)$.

- There is a contravariant exact functor $M \mapsto M^\sigma$ of finite order on $KL_{k_i}^{weight}(\mathfrak{g}_i)$ that maps $L(\lambda)$ to $L(\lambda)$.
- If $0 \to L(\lambda) \to M \to L(\mu) \to 0$ is exact and does not split, then, since every highest weight module is irreducible, $\lambda \geqslant \mu$.
- If $0 \to L(\mu) \to M^\sigma \to L(\lambda) \to 0$ splits then we are done.
- If $0 \to L(\mu) \to M^\sigma \to L(\lambda) \to 0$ does not split then $\lambda = \mu$.
- Since $KL_{k_i}^{weight}(\mathfrak{g}_i)$ is a category of weight modules. $Ext(L(\lambda), L(\lambda)) = 0$.

Example

- Let $(\mathfrak{g}, k) = (so(m), 2 - m/2)$. There is a collapsing chain that terminates with

$$\begin{cases} \mathbb{C} & \text{if } m \equiv 0, 1 \mod 4, \\ M(1) & \text{if } m \equiv 2 \mod 4, \\ (sl(2), 1) & \text{if } m \equiv 3 \mod 4. \end{cases}$$

  so $KL^{weight}_{2-m/2}(so(m))$ is semisimple.
- Let $(\mathfrak{g}, k) = (so(n|1), -1)$, $n \geqslant 4$. $\mathfrak{g}$ collapses to $M(1)$ so $KL^{weight}_{-1}(sl(n|1))$ is semisimple.

When is KL_k semisimple?

It is clear that $KL_k(\mathfrak{g})$ is semisimple if and only if $KL_k^{weight}(\mathfrak{g})$ is semisimple and that $KL_k(\mathfrak{g}) = KL_k^{weight}(\mathfrak{g})$.

A neat trick

Assume that we have a non-split extension

$$0 \to M \to M^{ext} \to N \to 0$$

with M and N simple. Then $N \simeq M$.

Proof.

In KL_k $L(0) = L(0)_{ss} + L(0)_{nil}$ with $L(0)_{ss}$ semisimple and $L(0)_{nil}$ locally nilpotent. If $L(0)_{nil} \neq 0$, then $L(0)_{nil}$ provides the isomorphism between M and N. If $L(0)_{nil} = 0$ then there is h in the Cartan subalgebra such that $h(0) = h(0)_{ss} + h(0)_{nil}$ with $h(0)_{nil} \neq 0$. Repeat the argument with $h(0)_{nil}$.

□

Application

Theorem

Assume that the category $KL_k^{weight}(\mathfrak{g})$ *is semisimple and that for any irreducible* $V_k(\mathfrak{g})$*–module* M *in* $KL_k(\mathfrak{g})$ *we have* $Ext^1(M_{top}, M_{top}) = \{0\}$ *in the category of finite-dimensional* $\mathfrak{g}$*–modules. Then* $KL_k(\mathfrak{g})$ *is semisimple.*

Proof.

Assume that we have a non-split extension $0 \to M \to M^{ext} \to N \to 0$. Then $M = N$. It follows that $0 \to M_{top} \to M^{ext}_{top} \to M_{top} \to 0$ is nonsplit.

□

Example

If $\mathfrak{g}$ is even and $KL_k^{weight}(\mathfrak{g})$ is semisimple then $KL_k(\mathfrak{g})$ is semisimple.

Another example

If $\mathfrak{g} = sl(m|1)$ then $KL_{-1}(\mathfrak{g})$ is semisimple.

Proof.

Using a free field realization of $V_{-1}(sl(m|1))$ and fusion rules one can show that an irreducible $M \in KL_{-1}(sl(m|1))$ must have an atypical top. Then one uses a result of Germoni that shows that atypical modules do not have self extensions. □

A realization of $V_1(sl(m|1))$

The vertex algebra $V_1(sl(m|1))$ is realized as a subalgebra of $W_1 \otimes F_m$, where W_1 is the Weyl vertex algebra generated by $a^{\pm} = a_1^{\pm}$, and F_m the Clifford vertex algebra generated by $\Psi_i^{\pm}$, $i = 1, \dots, m$.
Let $L = \mathbb{Z}c + \mathbb{Z}d$ be the rank two lattice such that

$$\langle c, d \rangle = 2, \ \langle c, c \rangle = \langle d, d \rangle = 0.$$

Let $V_L = M(1) \otimes \mathbb{C}[L]$ be the associated lattice vertex algebra and set

$$\Pi(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}c].$$

There is an embedding of W_1 into $\Pi(0)$ so $V_1(sl(m|1))$ embeds in

$$\Pi(0) \otimes F_m.$$

Theorem

Assume $\mathfrak{g} = sl(m|1)$. *Define*

$$\widetilde{w} := e^{-mc} \otimes : \Psi_1^+ \cdots \Psi_m^+ : \in \Pi(0) \otimes F_m.$$

Then we have:

- $\widetilde{W} = V_1(\mathfrak{g})\widetilde{w}$ *is a highest weight* $V_1(\mathfrak{g})$*–module in the category* $KL_1^{weight}(\mathfrak{g})$.
- $\widetilde{W}$ *is reducible and it contains a proper submodule isomorphic to* $V_1(\mathfrak{g})$.

Extension to $KL_k^{weight}(sl(m|1))$, $k \in \mathbb{N}$, $k > 0$

Set $\mathfrak{g} = sl(m|1)$. It is clear that there is a diagonal action of $V^k(\mathfrak{g})$ on $V_1(\mathfrak{g})^{\otimes k}$. It is known that

$$V_k(\mathfrak{g}) \cong V^k(\mathfrak{g}).(\underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{k \text{ times}}) \subset V_1(\mathfrak{g})^{\otimes k}.$$

As a consequence, we have that $\widetilde{W} \otimes V_1(\mathfrak{g})^{\otimes(k-1)}$ is a $V_k(\mathfrak{g})$–module. Define

$$\widetilde{w}^{(k)} = \widetilde{w} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(k-1) \text{ times}}$$

$$\widetilde{W}^{(k)} = V_k(\mathfrak{g}).\widetilde{w}^{(k)} \subset \widetilde{W} \otimes V_1(\mathfrak{g})^{\otimes(k-1)}.$$

$\widetilde{W}^{(k)}$ is indecomposable non irreducible.