# On the operad structure of moduli spaces and the consistency of non-chiral conformal field theory

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Riken

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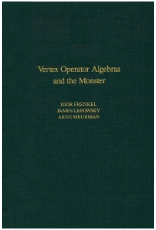
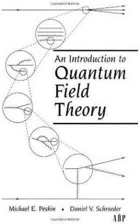
# Motivation

- 2019: I was asked "What is the relation between vertex operator algebra and quantum field theory?"



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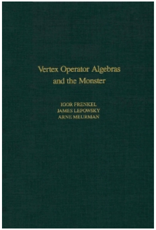


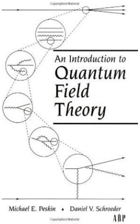


$$\text{(vertex algebra)} \qquad Y(a,z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1},$$

$$\text{(full field algebra)} \qquad \mathbb{Y}(a,z) = \sum_{r,s\in\mathbb{R}} a(r,s) z^{-r-1}\bar{z}^{-s-1},$$

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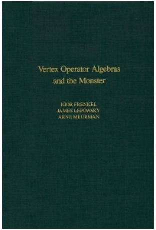


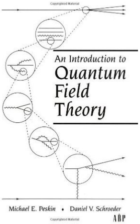


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- (Huang-Kong 2005) is very important, yet only a few papers have been written on full field algebra even after almost 20 years...

# Tensor category structure revisited

- In order to understand non-chiral conformal field theories, it is essential to understand intertwining operators in depth. (tensor category structure)

**Theorem 0.1 (Huang-Lepowsky, Huang-Lepowsky-Zhang)**

*If $V$ is a regular ($C_2$-cofinite) vertex operator algebra, the category of $V$-modules has a structure of a braided tensor category.*

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- We give new proof for braided tensor category structure.
  - shorter (117 pages) and geometric.

# Plan of this talk

1. Explain our proof of braided tensor category structure on $V$-module
2. Review of non-chiral conformal field theory (full vertex algebra)
3. state the main result and proof
   - Vertex operator algebra and colored parenthesized braid operad, arXiv:2209.10443.
   - Two-dimensional conformal field theory, full vertex algebra and current-current deformation, Adv. Math, 427, (2023)
   - to appear

## Section 1: Braided tensor category structure

# Intuitive understanding of BTC structure – higher non-commutativity

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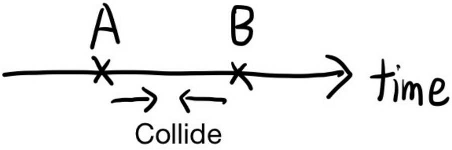
- Non-commutativity in 1d QFT (aka QM), $[A, B] \neq 0$.



# Intuitive understanding of BTC structure – higher non-commutativity

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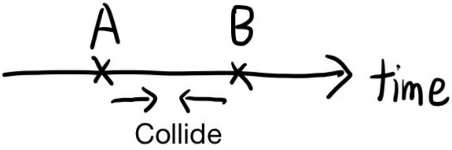
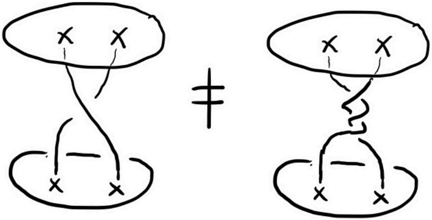



- Higher non-commutativity in 2d QFT:



- Braided tensor category comes from the property of two-dimensional space!

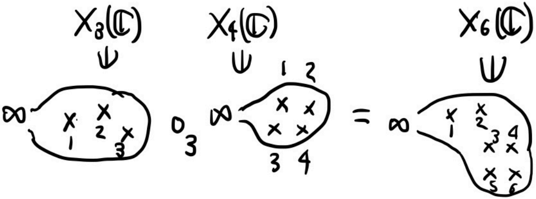
# Property of two-dimensional space?

- $$X_r(\mathbb{C}) = \{(z_1, \dots, z_r) \in \mathbb{C}^r \mid z_i \neq z_j\}$$

    - For a vertex algebra $V$, $a_1, \dots, a_r \in V$ and $u \in V^\vee$,

    $$\langle u, Y(a_1, z_1) Y(a_2, z_2) \dots Y(a_r, z_r) 1 \rangle$$

    is a holomorphic function on $X_r(\mathbb{C})$.



# Many operads!

- There are many ways to define an operad structure on $X_r(\mathbb{C})$.
- Fulton-MacPherson operad [Getzler-Jones, Kontsevich], Deligne-Mumford-Knudsen operad
- or formal tubed Riemman sphere [Huang], which is used to define the vertex tensor category.

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**Question**

Which operad is the best to show the braided tensor category structure?

# Which operad should we choose?

- What we want to consider is a line/up to homotopy of the configuration space.

$$\Pi_1(X_r(\mathbb{C})) \quad = \quad \text{the fundamental groupoid}$$

  - $\Pi_1$ is symmetric monoidal functor:

$$\Pi_1 : \underline{\text{category of topological spaces}} \to \underline{\text{category of groupoid}}$$

- $\mathrm{CPaB} \subset \{\Pi_1(X_r)\}_r$ is an operad, called a colored parenthesized braid operad [Tamarkin].

# Colored parenthesized braid operad?

**Theorem 1.1 (B.Fresse)**

*Let $C$ be a $\mathbb{C}$-linear category $C$. Then, there is a bijection:*

$$\{\textit{action of } \mathrm{CPaB} \textit{ on } C\} \leftrightarrow \{\textit{Braided tensor category structure on } C \quad \textit{such that} \quad M \boxtimes I = M = I \boxtimes M\}$$

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- (A) It is a property of space-time (operad structure of configuration space).

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- (Q) Why does $\underline{V\text{-mod}_f}$ have BTC structure?
- (A) It is a property of space-time (operad structure of configuration space).
- Mathematically, it is enough to show that the monodromy of conformal block determines the action of $\mathrm{CPaB}$.

# weak 2-category, 2-action and 2d-CFT

**Proposition 1.1 (M)**

*Let $C$ be a $\mathbb{C}$-linear category $C$. Then, there is a bijection:*

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- 2-action is a feature of 2d CFT.
- By the Cobordism hypothesis by Jacob Lurie, we can expect the appearance of D-action and $(\infty, D)$-category for D-dimensional CFT for any $D \geq 2$.

# Conformal block I

- $V = \bigoplus_{n \geq 0} V_n$: VOA
  - we do not assume $V$ is $C_2$-cofinite nor rational.
- $\underline{V\text{-mod}_f}$: category of $V$-module $M$ such that
  1. $M$ is $C_1$-cofinite;
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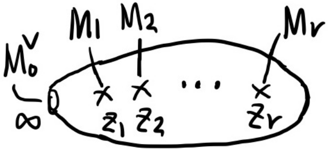
  1. $M$ is $C_1$-cofinite;
  2. The action of $L(0)$ on $M$ is locally finite and generalized eigenvalues are bounded below.
- For any modules $\vec{M} = (M_0, M_1, \dots, M_r) \in \underline{V\text{-mod}_f}$, we can associate a conformal block.




which is a multivalued holomorphic function on $X_r(\mathbb{C})$.

# Conformal block II

- $D_{X_r}$-module [Tsuchiya-Kanie,Tsuchiya-Nagatomo]

$$D_{\vec{M}} = M_0^\vee \otimes M_1 \otimes \cdots \otimes M_r \otimes O_{X_r}^{\mathrm{hol}}/\text{relation}$$

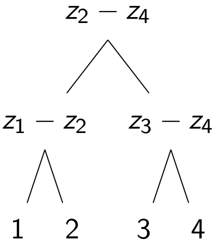
  - This relation is the same with Hao Li's talk.
- the conformal block is the holomorphic solution sheaf on $X_r$:

$$CB_{\vec{M}} = \mathrm{Hom}(D_{\vec{M}}, O_{X_r}^{\mathrm{hol}})$$

- Huang does not introduce D-module, however, he essentially prove that $D_{\vec{M}}$ is holonomic under $C_1$-cofinite [Huang].
  - Hao Li shows holonomic under the quasi-lisse condition!
- Thus, $CB_{\vec{M}}$ is locally free sheaf on $X_r$, i.e., any solution has the analytic continuation.

# Combinatorics of singularities

- $D_{\vec{M}}$ has singularities along the diagonals $\{z_i = z_j\}$.
- We need to see the behavior of D-module along $\{z_i = z_j\}$.
  - Huang's paper regularity is proved only for $r = 3$, which is one-variable case, $X_4(\mathbb{C})/\mathrm{PSL}_2\mathbb{C} \cong \mathbb{C}P^1 \setminus \{0, 1, \infty\}$.
- In general, $z_1 \to z_2$ and $z_3 \to z_4$ simultaneously... or even more complicated way.




Each tree corresponds to iterated vertex operators.

$$Y(Y(a_1, z_{12})a_2, z_{24})Y(a_3, z_{34})a_4$$

for tree (12)(34).

# Convergence region

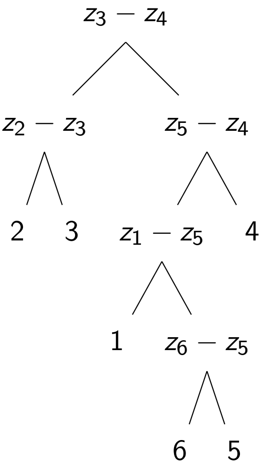
- For each tree, we can associate the region:

$$U_{(12)(34)} = \{|z_1 - z_2| < |z_2 - z_4|, \\ |z_3 - z_4| < |z_2 - z_4|\},$$

which is the convergent region of

$$Y(Y(a_1, z_{12})a_2, z_{24})Y(a_3, z_{34})a_4.$$

# Convergent region




$$U_{(23)((1(65))4)} = \{|z_{23}| < |z_{34}|, |z_{54}| < |z_{34}|, |z_{65}| < |z_{15}| < |z_{54}| < |z_{34}|\}.$$

# Formal solution

**Theorem 1.2 (M)**

*For $M_0, \dots, M_r \in \underline{V\text{-mod}_f}$ and any tree $A$ with $r$ leaves,*

$$\mathrm{Hom}(D_{\vec{M}}, O_{X_r}(U_A)) \cong \mathrm{Hom}(D_{\vec{M}}, \mathbb{C}[[\vec{\zeta}_A]][\vec{\zeta}_A^{\mathbb{C}}, \log \vec{\zeta}_A])$$

- That is any logarithmic formal solution will converge and gives an analytic solution.
- $\zeta_A$ is new formal variable.

$$\vec{\zeta}_{(12)(34)} = (\frac{z_{12}}{z_{24}}, \frac{z_{34}}{z_{24}}),$$

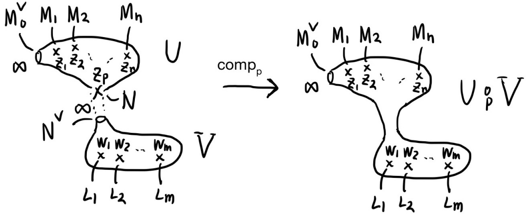
then

$$U_{(12)(34)} = \{|\frac{z_{12}}{z_{24}}| < 1, |\frac{z_{34}}{z_{24}}| < 1\}.$$

# Gluing of conformal block

- For $A \in \mathrm{Tr}_{[n]}$ and $B \in \mathrm{Tr}_{[m]}$, we construct a natural transformation:

$$\mathrm{CB}(U_A) \otimes \mathrm{CB}(U_B) \xrightarrow{\mathrm{comp}_p} \mathrm{CB}(U_A \circ_p U_B).$$



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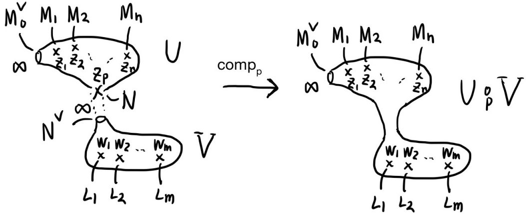


  - In the rational $C_2$-cofinite case, this morphism was constructed by Tsuchiya-Nagatomo. Our construction is an operad theoretical generalization of [TN].

- What is $U_A \circ_p U_B \subset X_{n+m-1}$?

# Gluing of open subset

- $U_{3((12)4)} \circ_2 U_{2(13)} = U_{5(1(3(24))6)}$

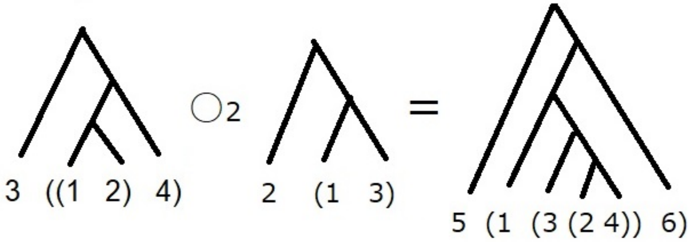


Figure: $3((12)4) \circ_2 2(13)$

# Main theorem I

- For any path $\gamma : U_A \to U_{A'}$, we can define the analytic continuation along the path $\gamma$, $A(\gamma) : \mathrm{CB}(U_A) \to \mathrm{CB}(U_{A'})$.

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## Theorem 1.3 (M)

*For any path $\gamma : U_A \to U_{A'}$ and $\mu : U_B \to U_{B'}$, the following diagram commute:*

$$\begin{array}{ccc} \mathrm{CB}(U_A) \otimes \mathrm{CB}(U_B) & \overset{\mathrm{comp}_p}{\to} & \mathrm{CB}(U_{A \circ_p B}) \\ {\scriptstyle A(\gamma)\otimes A(\mu)} \downarrow & & \downarrow {\scriptstyle A(\gamma \circ_p \mu)} \\ \mathrm{CB}(U_{A'}) \otimes \mathrm{CB}(U_{B'}) & \overset{\mathrm{comp}_p}{\to} & \mathrm{CB}(U_{A' \circ_p B'}), \end{array}$$

# Main Theorem II

- The above theorem can be rephrased as follows:

**Theorem 1.4 (M)**

*The monodromy of conformal blocks gives an operad lax 2-morphism* $\rho : \mathrm{CPaB} \to \mathrm{PEnd}_{\underline{V\text{-}mod_f}}$.

- group $G$ acts on a vector space $V$ iff monoid homomorphism $G \to \mathrm{End}(V)$.
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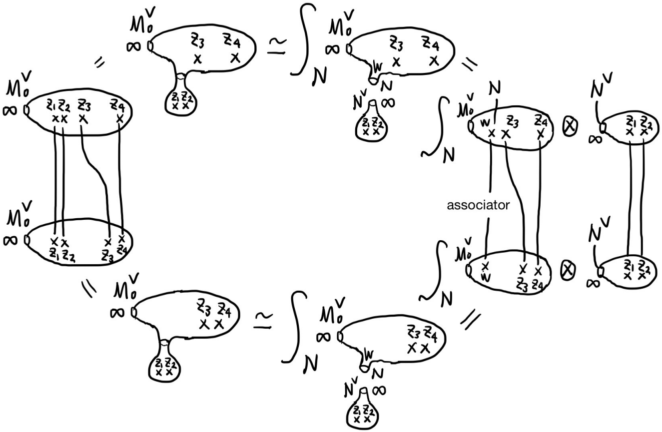
**Theorem 1.5 (M)**

*If $V$ is rational $C_2$-cofinite, then the above map lifts to 2-morphism* $\rho : \mathrm{CPaB} \to \mathrm{End}_{\underline{V\text{-}mod_f}}$. *Thus,* $\underline{V\text{-}mod_f}$ *is a braided tensor category.*

# CPaB action implies BTC: Sketch of proof

- This diagram calculate the associator $((12)3)4 \to (12)(34)$, which implies the pentagon identity.




- For more detail, see Introduction of our paper.

# Section 2: Non-chiral conformal field theory

# Why consider non-chiral CFT?

- chiral CFT could not deform, but non-chiral CFT can deform!
  - VOA is $\mathbb{Z}$-graded $V = \bigoplus_{n\in\mathbb{Z}} V_n$, while a full VOA is $\mathbb{R}^2$-graded $F = \bigoplus_{h,\bar{h}} F_{h,\bar{h}}$.
  - Physically, $h + \bar{h}$ is an energy of state, must be continuously varied.
- lattice full vertex algebra [M]:

$$\mathbb{Y}(e_\alpha, z, \bar{z}) = \exp\Big(\sum_{n\geq 1} \alpha(-n)\frac{z^n}{n} + \bar{\alpha}(-n)\frac{\bar{z}^n}{n}\Big)$$

$$\exp\Big(\sum_{n\geq 1} \alpha(n)\frac{z^{-n}}{-n} + \bar{\alpha}(n)\frac{\bar{z}^{-n}}{-n}\Big) e_\alpha z^{\alpha(0)} \bar{z}^{\alpha(0)}$$

  - $(z-w)^{(\alpha,\beta)}(\bar{z}-\bar{w})^{(\bar{\alpha},\bar{\beta})}$.

# Recall the definition of $\mathbb{Z}$-graded vertex algebra

- Consider $Y(-,z) : V \to \mathrm{End}\, V[[z^{\pm}]]$ and set

$$a \cdot_z b = Y(a,z)b$$

- $(V, Y, 1)$ is a vertex algebra if the following formal power series convergent to the same holomorphic function:

$$\begin{aligned} a_1 \cdot_{z_1} (a_2 \cdot_{z_2} a_3) &= Y(a_1, z_1) Y(a_2, z_2) a_3, \\ a_2 \cdot_{z_2} (a_1 \cdot_{z_1} a_3) &= Y(a_2, z_2) Y(a_1, z_1) a_3, \\ (a_1 \cdot_{z_{12}} a_2) \cdot_{z_2} a_3 &= Y(Y(a_1, z_{12}) a_2, z_2) a_3, \end{aligned}$$

  and ...
  - a vertex algebra is an associative commutative algebra up to the analytic continuation [Lepowsky-Li].

# Definition of full vertex algebra [M]

- $a \cdot_z b = \mathbb{Y}(a,z)b = \sum_{r,s\in\mathbb{R}} a(r,s)bz^{-r-1}\bar{z}^{-s-1}$
- $(V, Y, 1)$ is a vertex algebra if the following formal power series convergent to the same holomorphic function:

$$a_1 \cdot_{z_1} (a_2 \cdot_{z_2} a_3) = \mathbb{Y}(a_1,z_1)\mathbb{Y}(a_2,z_2)a_3,$$
$$a_2 \cdot_{z_2} (a_1 \cdot_{z_1} a_3) = \mathbb{Y}(a_2,z_2)\mathbb{Y}(a_1,z_1)a_3,$$
$$(a_1 \cdot_{z_{12}} a_2) \cdot_{z_2} a_3 = \mathbb{Y}(\mathbb{Y}(a_1,z_{12})a_2,z_2)a_3,$$

and ...

- The axioms of commutative associative algebra

$$a(bc) = (ab)c \quad \text{and} \quad ab = ba$$

is nice. Because, it implies that any iterated product is the same, e.g., $(b(dc))a = a(b(cd))$.

# Consistency of full vertex algebra

- For any tree $A$, we can consider the iterated vertex operators $\mathbb{Y}_A$:

**Theorem 2.1 (M, to appear)**

*Let $F$ be a full vertex algebra and assume that there is a filtration $F = \cup_n F^n$ such that $F^n$ is $C_1$-cofinite. Then, for any $a_1, \dots, a_r \in F$ and trees $A, B \in \mathrm{Tr}_{[r]}$,*

$$\mathbb{Y}_A(a_1, \dots, a_r) \underset{a.c.}{=} \mathbb{Y}_B(a_1, \dots, a_r).$$

*Moreover, there is a unique $S : F^\vee \otimes F^{\otimes r} \to C^{real\ analytic}(X_r)$ such that:*

$$S(a_0^*, a_1, \dots, a_r)|_{U_A} = \langle a_0^*, \mathbb{Y}_A(a_1, \dots, a_r)\rangle.$$

- This is a proof of the bootstrap hypothesis of 2d CFT in physics.

# Examples

- full framed VOA

$$L(\frac{1}{2},0)^{\otimes l} \otimes \overline{L(\frac{1}{2},0)}^{\otimes r} \subset F.$$

- twisted WZW model

$$F = \bigoplus_{\lambda \in P^+} L_{\mathfrak{g},k}(\lambda) \otimes \overline{L_{\mathfrak{g},k}(\lambda^*)}$$

  - used to prove Creutzig-Gaiotto conjectures.

- Deformation: If $U(1)^n \subset F_{1,0}$ and $U(1)^m \subset F_{0,1}$, then we construct family of full VOAs parametrized by

$$D_F \backslash \mathrm{O}(n,m;\mathbb{R})/\mathrm{O}(n;\mathbb{R}) \times \mathrm{O}(m;\mathbb{R}).$$

  - Please visit my arXiv!

# Section 3: Quantum field theory v.s. full vertex algebra

# System of correlators

- For nice full VOA, we construct a system of correlation functions:

$$S_r : F^\vee \otimes F^{\otimes r} \to C(X_r(\mathbb{C}))^{\text{real analytic}}.$$

- We give a characterization of this system of correlators $\{S_r\}_{r=0,1,2,\dots}$, namely, from $\{S_r\}_{r=0,1,2,\dots}$ we can recover a full vertex algebra $(F, Y, 1)$.
- What is the axioms of a system of correlators?

# Axioms of quantum field theory

1. Wightman axioms (Minkowski): operator-valued distribution
2. Osterwalder-Schrader axioms (Euclidian): system of correlators
3. Haag-Kastler axioms (conformal net): von Neumann algebra of fields
4. factorization algebra: cosheaf of fields
5. full vertex operator algebra: algebra without completion!

# axioms of correlators

$$\{S_r : F^\vee \otimes F^{\otimes r} \to C(X_r(\mathbb{C}))^{\text{real analytic}}\}_{r=0,1,\ldots}$$

permutation) For any $\sigma \in S_r$ and $a_1, \ldots, a_r \in F$, $u \in F^\vee$,

$$S_r(u, (a_1, z_1) \ldots, (a_r, z_r)) = S_r(u, (a_{\sigma 1}, z_{\sigma 1}), \ldots, (a_{\sigma r}, z_{\sigma r}))$$

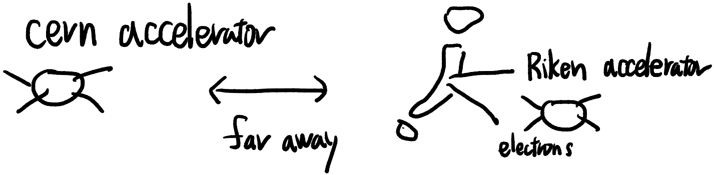
covariance) $S_r$ is covariant under the action of the left-right Virasoro actions $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$.

vacuum)

$$S_{r+1}(u, (a_1, z_1) \ldots, (a_r, z_r), (1, z_{r+1})) = S_r(u, (a_1, z_1, \ldots, (a_r, z_r$$

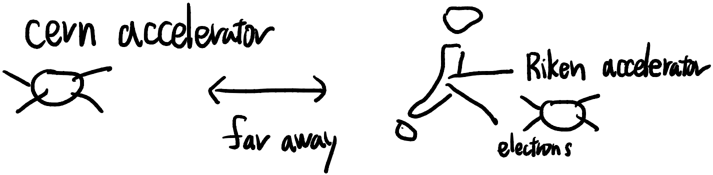
- This condition does not appear in OS axioms, but is replaced by the cluster decomposition property.

# Cluster decomposition in physics




- Two experiments should be independent, i.e. there is no correlation.

# Cluster decomposition in physics




- Two experiments should be independent, i.e. there is no correlation.
- This can be expressed as follows:

$$\lim_{t\to\infty} S_{n+m}(1,(a_1,z_1),\dots,(a_n,z_n),(a_{n+1},z_{n+1}+t),\dots,(a_{n+m},z_{n+m}+t)$$
$$= \underbrace{S_n(1,(a_1,z_1),\dots,(a_n,z_n))}\,\underbrace{S_m(1,a_{n+1},z_{n+1}),\dots,(a_{n+m},z_{n+m})).}$$

  - which is the part of Osterwalder-Schrader axioms.

# Strong cluster decomposition

Let $\{e_\alpha \in F\}_{\alpha \in I}$ be a basis of $F$ and $\{e^\alpha \in F^\vee\}_{\alpha \in I}$ the dual basis. We assume that

$$S_{n+m}(u,(a_1,z_1),\dots,(a_n,z_n),(a_{n+1},z_{n+1}),\dots,(a_{n+m},z_{n+m}))$$
$$= \sum_{i \in I} S_n(u,(a_1,z_1),\dots,(e_i,z_n))S_m(e^i,a_{n+1},z_{n+1}),\dots,(a_{n+m},z_{n+m})).$$

Here, the right-hand-side is absolutely convergent in following region (roughly),

$$\{(z_1,\dots,z_{n+m}) \in X_{n+m}(\mathbb{C}) \mid \min_{i \in \{1,\dots,n\}}\{|z_i - z_{n+m}|\} > \max_{k,l \in \{n+1,\dots,n+m\}}\{|z_k - z_l|\}\}.$$

# Result

**Theorem 3.1 (M, to appear)**

*Let $F$ be a nice $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$-module. If $\{S_r : F^\vee \otimes F^{\otimes r} \to C(X_r)^{\text{real analytic}}\}_{r \geq 0}$ satisfy (permutation), (covariance), (vacuum) and (strong cluster decomposition) properties. Then, one can define a vertex operator*

$$Y_F : F \to \mathrm{End} F[[z^{\pm}]][(z\bar{z})^{\mathbb{C}}]$$

*such that $(F, Y, 1)$ is a full vertex algebra. Conversely, if $(F, Y, 1)$ is a full vertex operator algebra with asymptotically $C_1$-cofinite filtration, then we can construct the system of correlators which satisfies the axioms.*

- {system of correlators} $\leftrightarrow$ {full vertex operator algebra}.
- A similar result can be found in [Huang-Kong] with different ways.