Functorial constructions of double Poisson vertex algebras

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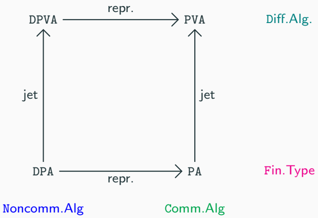
A **double Poisson vertex algebra** is a noncommutative differential algebra, with a *double λ-bracket* $\{\!\{-_\lambda-\}\!\}$ compatible with the product and the differential.

Aims:

- Produce examples of double Poisson vertex algebras.
- Compare different categories:



- Consider related constructions: *Poisson and Hamiltonian reductions*.
- Discuss connections with vertex algebras.

Poisson algebras

Recall that a **Poisson algebra** is a commutative associative algebra equipped with a Lie bracket compatible with the multiplication (Leibniz rule).

Naive observation: only few examples of noncommutative algebras equipped with a "Poisson bracket", except

$$\{a, b\} = ab - ba.$$

- *How to define a more interesting noncommutative version?*

Idea: use the representation functor [KONTSEVICH-ROSENBERG]:

$$\texttt{Assoc.Alg.} \longrightarrow \texttt{Comm.Alg.}, \quad \mathbb{A} \longmapsto \mathbb{A}_N,$$

where $\mathbb{A}_N$ is the **N-th representation algebra**:

$$\mathbb{A}_N := \mathbb{C}[\mathrm{Rep}(\mathbb{A}, N)].$$

Definition (Van den Bergh, 2008)

A **double Poisson algebra** is an associative algebra $\mathbb{A}$ with a linear map

$$\{\!\{-,-\}\!\} : \mathbb{A} \otimes \mathbb{A} \longrightarrow \mathbb{A} \otimes \mathbb{A}, \quad a \otimes b \longmapsto \{\!\{a,b\}\!\}, \quad \text{s.t.}$$

$$\begin{aligned}
&\text{(skewsymmetry)}: && \{\!\{a,b\}\!\} = -\{\!\{b,a\}\!\}^{\sigma}, \qquad (a\otimes b)^{\sigma} := b \otimes a,\\
&\text{(left Leibniz rule)}: && \{\!\{a,bc\}\!\} = \{\!\{a,b\}\!\}\, c + b\,\{\!\{a,c\}\!\},\\
&\text{(Jacobi identity)}: && \{\!\{a,\{\!\{b,c\}\!\}\}\!\} - \{\!\{b,\{\!\{a,c\}\!\}\}\!\} - \{\!\{\{\!\{a,b\}\!\},c\}\!\} = 0 \,/
\end{aligned}$$

Theorem (Van den Bergh)

A double Poisson algebra structure on $\mathbb{A}$ induces a Poisson algebra structure on $\mathbb{A}_N$.

Example 1

$\mathbb{A} = \mathbb{C}\langle a \rangle$ is a double Poisson algebra structure with

$$\{\!\{a, a\}\!\} = a \otimes 1 - 1 \otimes a.$$

It induces the standard Poisson bracket on $\mathbb{A}_N = \mathbb{C}[\mathrm{Mat}_N] \cong \mathbb{C}[(\mathrm{Mat}_N)^*]$.

$\mathbb{A} = \mathbb{C}\langle a \rangle/(a^N) \rightsquigarrow$ standard Poisson bracket on $\mathbb{C}[\{x \in \mathrm{Mat}_N \colon x^N = 0\}]$.

Example 2

$\mathbb{A} := \mathbb{C}\langle a, b \rangle$, with double bracket: $\{\!\{a, a\}\!\} = 0 = \{\!\{b, b\}\!\}$, $\{\!\{b, a\}\!\} = 1 \otimes 1$.

Then:

$$\mathbb{A}_N \simeq \mathbb{C}[\mathrm{Mat}_N] \otimes \mathbb{C}[\mathrm{Mat}_N] \simeq \mathbb{C}[T^*\mathrm{Mat}_N].$$

Example 3

$\mathbb{A} = \mathbb{C}\langle a, b^{\pm} \rangle$ with: $\{\!\{b, b\}\!\} = 0$, $\{\!\{a, b\}\!\} = b \otimes 1$, $\{\!\{a, a\}\!\} = a \otimes 1 - 1 \otimes a$.

Then:

$$\mathbb{A}_N \simeq \mathbb{C}[\mathrm{Mat}_N^*] \otimes \mathbb{C}[\mathrm{GL}_N] \simeq \mathbb{C}[T^*\,\mathrm{GL}_N].$$

Double quivers

Let Q be a quiver with vertex set S. The double quiver $\overline{Q}$ is obtained by adjoining a new arrow $a^*: s \to r$ for each $r \xrightarrow{a} s$.

Fact (Van den Bergh)

The path algebra $\mathbb{C}\overline{Q}$ is a double Poisson algebra by:

$$\{\!\{a, a^*\}\!\} = \epsilon(a)\, e_{h(a)} \otimes e_{t(a)}.$$

where $\epsilon(a) = +1$ if $a \in Q$ and $\epsilon(a) = -1$ otherwise. Moreover,

$$(\mathbb{C}\overline{Q})_{\underline{n}} \simeq \mathbb{C}[T^* \operatorname{Rep}(\mathbb{C}Q, \underline{n})], \quad \text{with} \quad \underline{n} = (n_s) \in \mathbb{N}^S.$$

Remark (Crawley-Boevey–Etingof–Ginzburg, 2007)

This is a special case of **noncommutative cotangent algebras**:

$$\mathbb{C}\overline{Q} \simeq T^*\mathbb{C}Q := T(\operatorname{Der} \mathbb{C}Q).$$

Poisson reduction and Hamiltonian reduction

- GL_N acts on $\mathrm{Rep}(\mathbb{A}, N)$ by conjugation of matrices by Poisson automorphisms.

One can consider the **Poisson reduction** $\mathbb{A}_N^{\mathrm{GL}_N}$ (or $\mathbb{A}_{\underline{n}}^{\mathrm{GL}_{\underline{n}}}$ if $\mathrm{GL}_{\underline{n}} := \prod_s \mathrm{GL}_{n_s}$.)

- If the action is Hamiltonian with moment map μ, one can also consider the **Hamiltonian reduction**

$$\mathbb{C}[\mu^{-1}(\xi)/\,\mathrm{GL}_N] = (\mathbb{A}_N/\langle \mu^*(x) - \xi(x) \colon x \in \mathfrak{gl}_N \rangle)^{\mathrm{GL}_N},$$

for ξ is a single GL_N-orbit.

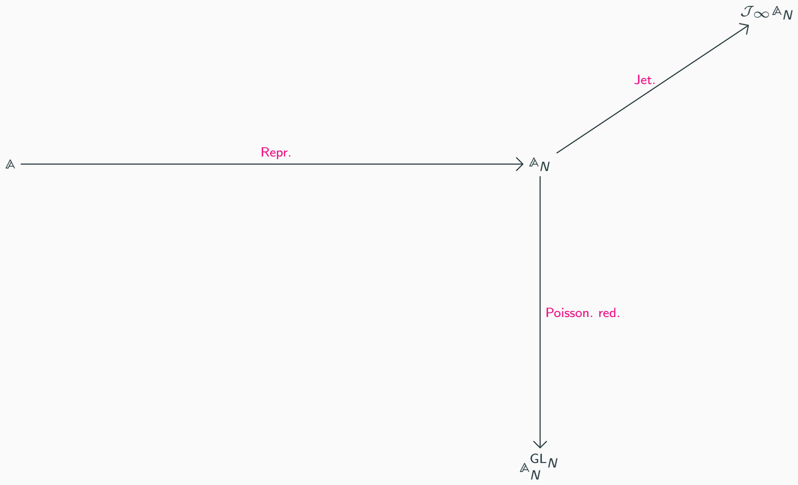
Fact (Van den Bergh)

There is a notion of *noncommutative moment map* μ on a double Poisson algebra $\mathbb{A}$ which induces a moment map $X(\mu)\colon \mathrm{Rep}(\mathbb{A}, N) \to (\mathfrak{gl}_N)^*$.

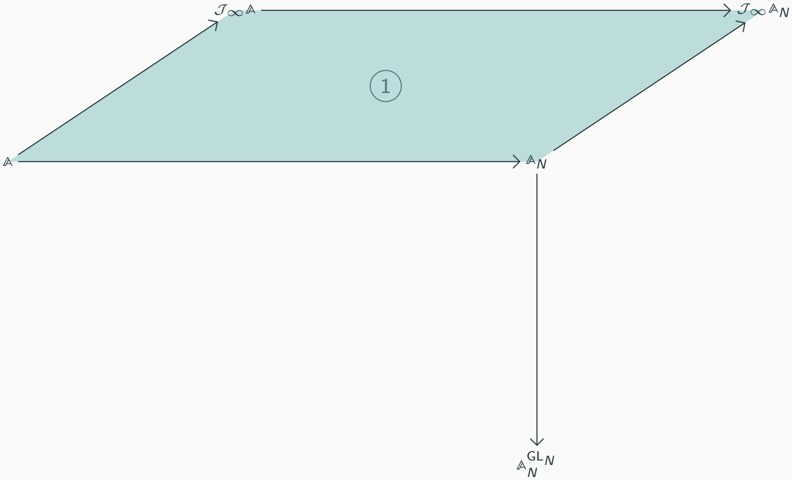
Example

Let $\mathbb{A} := \mathbb{C}\overline{Q}$. Then $\mu = \sum_{a \in \overline{Q}} \epsilon(a) a a^* = \sum_{a \in Q} [a, a^*] \in \mathbb{A}$ is a noncommutative moment map.


$\mathbb{A}$
Repr.
$\mathbb{A}_N$
Poisson. red.
$\mathbb{A}_N^{\mathrm{GL}_N}$



$\mathbb{A}$
Repr.
$\mathbb{A}_N$
Jet.
$\mathcal{J}_\infty \mathbb{A}_N$
Poisson. red.
$\mathbb{A}_N^{\mathrm{GL}_N}$


$\mathcal{J}_\infty \mathbb{A}$
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1
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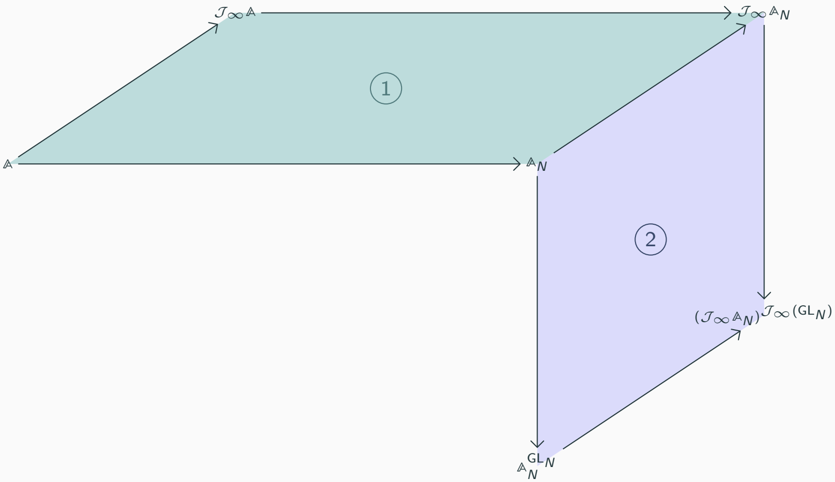
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$\mathbb{A}$
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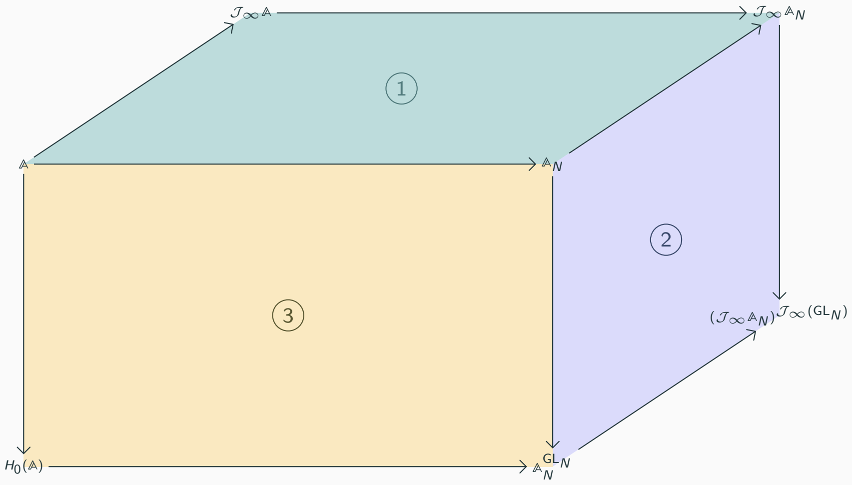
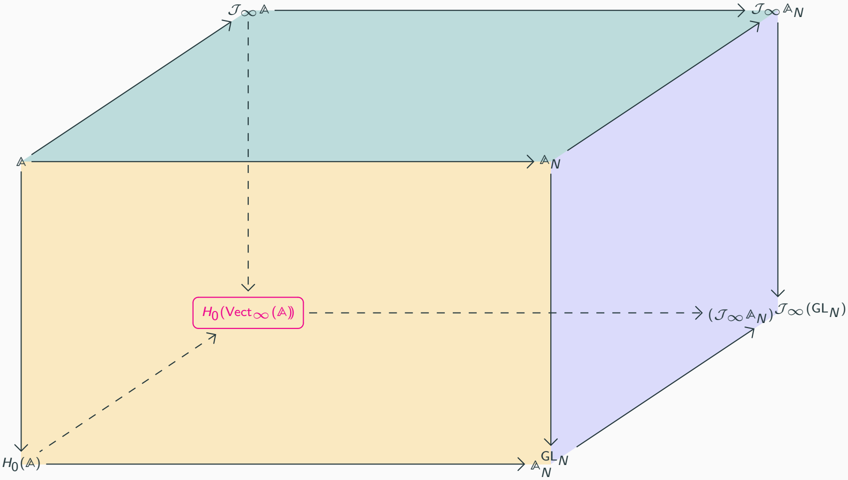
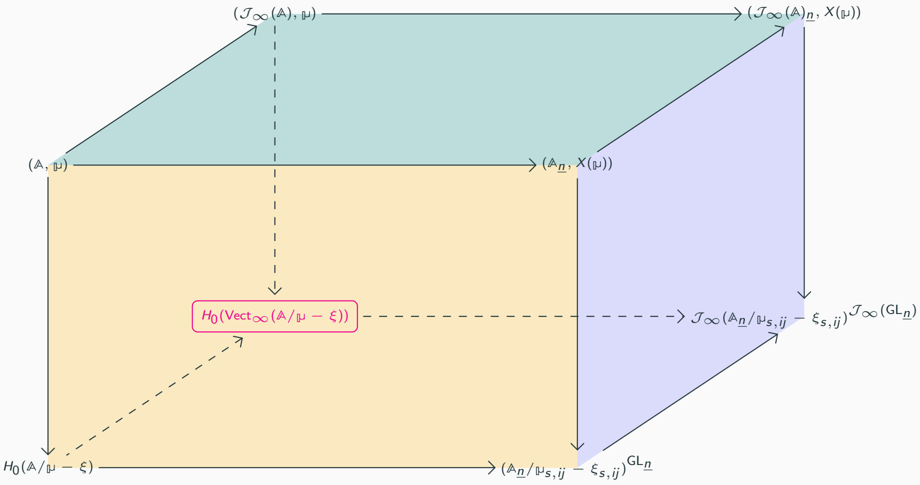
2
$(\mathcal{J}_\infty \mathbb{A}_N)^{\mathcal{J}_\infty(\mathrm{GL}_N)}$
$\mathbb{A}_N^{\mathrm{GL}_N}$



$\mathcal{J}_\infty \mathbb{A}$
$\mathcal{J}_\infty \mathbb{A}_N$
1
$\mathbb{A}$
$\mathbb{A}_N$
2
3
$(\mathcal{J}_\infty \mathbb{A}_N)^{\mathcal{J}_\infty(\mathrm{GL}_N)}$
$H_0(\mathbb{A})$
$\mathbb{A}_N^{\mathrm{GL}_N}$


$\mathcal{J}_\infty \mathbb{A}$
$\mathcal{J}_\infty \mathbb{A}_N$
$\mathbb{A}$
$\mathbb{A}_N$
$H_0(\mathrm{Vect}_\infty(\mathbb{A}))$
$(\mathcal{J}_\infty \mathbb{A}_N)^{\mathcal{J}_\infty(\mathrm{GL}_N)}$
$H_0(\mathbb{A})$
$\mathbb{A}_N^{\mathrm{GL}_N}$



$(\mathcal{J}_\infty(\mathbb{A}), \mathbb{\mu})$
$(\mathcal{J}_\infty(\mathbb{A})_{\underline{n}}, X(\mathbb{\mu}))$
$(\mathbb{A}, \mathbb{\mu})$
$(\mathbb{A}_{\underline{n}}, X(\mathbb{\mu}))$
$H_0(\mathrm{Vect}_\infty(\mathbb{A}/\mathbb{\mu} - \xi))$
$\mathcal{J}_\infty(\mathbb{A}_{\underline{n}}/\mathbb{\mu}_{s,ij} - \xi_{s,ij})^{\mathcal{J}_\infty(\mathrm{GL}_{\underline{n}})}$
$H_0(\mathbb{A}/\mathbb{\mu} - \xi)$
$(\mathbb{A}_{\underline{n}}/\mathbb{\mu}_{s,ij} - \xi_{s,ij})^{\mathrm{GL}_{\underline{n}}}$

Definition (Frenkel–Ben-Zvi, Barakat–De Sole–Kac)

A **Poisson vertex algebra** is a commutative differential algebra (V, ∂) equipped with a linear map

$$\{-_\lambda-\}\colon V \otimes V \to V[\lambda]\,, \quad a \otimes b \mapsto \{a_\lambda b\}, \quad \text{s.t. :}$$

(sesquilinearity): $\{\partial(a)_\lambda b\} = -\lambda\{a_\lambda b\}\,, \quad \{a_\lambda \partial(b)\} = (\lambda + \partial)\{a_\lambda b\}\,,$

(skewsymmetry): $\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\}\,,$

(left Leibniz rule): $\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\}\,,$

(Jacobi identity): $\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} - \{\{a_\lambda b\}_{\lambda+\mu} c\} = 0\,.$

- Integrability of Hamiltonian partial differential equations.
- Study of arbitrary vertex algebras.

The **jet algebra** of a commutative A is the unique (up to isomorphism) differential algebra $\mathcal{J}_\infty A$ such that

$$\mathrm{Hom}_{\mathtt{Diff.Alg}}(\mathcal{J}_\infty A, B) \simeq \mathrm{Hom}_{\mathtt{Alg}}(A, B), \quad \forall\,(B, \partial).$$

Example

- If $A = \mathbb{C}[a]$, then $\mathcal{J}_\infty A = \mathbb{C}[a^{(j)} : j \geqslant 0]$ with $\partial \colon a_\ell^{(i)} \to a_\ell^{(i+1)}$.
- If $A = \mathbb{C}[a_1, \ldots, a_n]/I$, then

$$\mathcal{J}_\infty A = \mathbb{C}[a_\ell^{(j)} : \ell = 1, \ldots, n,\, j \geqslant 0]/(\partial f,\, f \in I).$$

Observation (Arakawa)

If A) is a Poisson algebra, then there exists a unique Poisson vertex algebra structure on $\mathcal{J}_\infty A$ such that

$$\{a_\lambda b\} := \{a, b\}\,, \quad a, b \in A\,.$$

$\rightsquigarrow$ *The geometry of* $\mathrm{Spec}\,\mathcal{J}_\infty A$ *gives information on the singularities of* $\mathrm{Spec}\,A$.

For an arbitrary vertex algebra $\mathcal{V}$, the graded algebra $\operatorname{gr}\mathcal{V}$ with respect to the **Li filtration** is a Poisson vertex algebra.

Observation (Li, Arakawa)

There is a surjective Poisson vertex algebra morphism:

$$\mathcal{J}_\infty R_{\mathcal{V}} \twoheadrightarrow \operatorname{gr}\mathcal{V},$$

where $R_{\mathcal{V}} := \mathcal{V}/C_2(\mathcal{V})$ is the **Zhu C_2-algebra**.

Example: the universal affine vertex algebra associated with $\mathfrak{g}$ and κ

Let $V^\kappa(\mathfrak{g}) := U(\mathfrak{g}) \otimes_{\mathfrak{g}[t]\oplus\mathbb{C}1} \mathbb{C}$. Then:

$$\operatorname{gr} V^\kappa(\mathfrak{g}) \simeq \mathcal{J}_\infty \mathbb{C}[\mathfrak{g}^*] = \mathcal{J}_\infty R_{V^\kappa(\mathfrak{g})}.$$

Example: the vertex algebra of chiral differential operators on $\mathbb{C}^n$

Let $\mathcal{D}^{ch}_{\mathbb{C}^n}$ be the $\beta\gamma$-system generated by fields $\beta_i(z), \gamma_i(z)$ with OPE's: $\beta_i(z)\gamma_j(z) \sim \dfrac{\delta_{i,j}}{z-w}$. Then:

$$\operatorname{gr}\mathcal{D}^{ch}_{\mathbb{C}^n} \simeq \mathcal{J}_\infty \mathbb{C}[T^*\mathbb{C}^n] = \mathcal{J}_\infty R_{\mathcal{D}^{ch}_{\mathbb{C}^n}}.$$

cdo on smooth variety: [Malikov–Schechtman–Vaintrob], [Beilinson–Drinfeld].

Definition (De Sole–Kac–Valeri, 2014)

A **double Poisson vertex algebra** is a differential algebra $\mathbb{V}$,

$$\partial(ab) = a \cdot \partial(b) + \partial(a) \cdot b,$$

equipped with is a linear map

$$\{\!\{-_\lambda-\}\!\} : \mathbb{V} \otimes \mathbb{V} \longrightarrow (\mathbb{V} \otimes \mathbb{V})[\lambda], \quad a \otimes b \longmapsto \{\!\{a_\lambda b\}\!\}, \quad \text{s.t.}$$

(sesquilinearity): $\{\!\{\partial(a)_\lambda b\}\!\} = -\lambda \{\!\{a_\lambda b\}\!\}, \quad \{\!\{a_\lambda \partial(b)\}\!\} = (\partial + \lambda) \{\!\{a_\lambda b\}\!\},$

(left Leibniz rule): $\{\!\{a_\lambda bc\}\!\} = \{\!\{a_\lambda b\}\!\} c + b \{\!\{a_\lambda c\}\!\},$

(skewsymmetry): $\{\!\{a_\lambda b\}\!\} = -\{\!\{b_{-\lambda-\partial} a\}\!\}^\sigma,$

(Jacobi identity): $\{\!\{a_\lambda \{\!\{b_\mu c\}\!\}\}\!\} - \{\!\{b_\mu \{\!\{a_\lambda c\}\!\}\}\!\} - \left\{\!\!\left\{ \{\!\{a_\lambda b\}\!\}_{\lambda+\mu} c \right\}\!\!\right\} = 0.$

- This theory allows to study non-abelian integrable PDEs efficiently.
- [Álvarez-Cónsul–Fernández–Heluani, 2021] This was also used to upgrade the one-to-one correspondence between some (graded) Poisson vertex algebras and Courant-Dorfman algebras to the associative setting.

Theorem (De Sole–Kac–Valeri)

Any double Poisson vertex algebra $\mathbb{V}$ induces a Poisson vertex algebra structure on $\mathbb{V}_N$, and GL_N acts on $\mathbb{V}_N$ by Poisson vertex automorphisms.

Example

$\mathbb{V} = \mathbb{C}\langle a^{(0)}, a^{(1)}, a^{(2)}, \dots \rangle$ is a double Poisson vertex algebra with

$$\{\!\{ a_\lambda a \}\!\} = 1 \otimes a - a \otimes 1 + (1 \otimes 1)\lambda, \quad a := a^{(0)}.$$

Jet algebras

The **jet algebra** $\mathcal{J}_\infty\mathbb{A}$ of an associative algebra $\mathbb{A}$ is the algebra generated by symbols $\partial^k(a)$, $a \in \mathbb{A}$, $k \in \mathbb{N}$, subject to:

$$\partial^k(\alpha a + \beta b) = \alpha\partial^k(a) + \beta\partial^k(b)\,, \quad \partial^k(ab) = \sum_{j=0}^{k} \binom{k}{j} \partial^j(a)\,\partial^{k-j}(b)\,.$$

There is $\partial \in \mathrm{Der}(\mathcal{J}_\infty\mathbb{A})$ uniquely determined by: $\partial(\partial^k(a)) := \partial^{k+1}(a)$.

Example 1

Consider the free algebra $\mathbb{A} = \mathbb{C}\langle x_1, \dots, x_\ell\rangle$. Then,

$$\mathcal{J}_\infty\mathbb{A} = \langle \partial^k(x_1), \dots, \partial^k(x_\ell) \mid k \in \mathbb{N}\rangle,$$

equipped with the derivation ∂: $\partial^k(x_i) \mapsto \partial^{k+1}(x_i)$ for any $1 \leqslant i \leqslant \ell$.

Example 2: jets of path algebras

For Q a quiver, Q_∞ is the quiver with the same vertex set as Q, and arrow set:

$$Q_\infty = \{a^{(\ell)} \mid a \in Q,\, \ell \in \mathbb{N}\} \quad \text{with} \quad t(a^{(\ell)}) = t(a), \quad h(a^{(\ell)}) = h(a).$$

Then $\mathcal{J}_\infty(\mathbb{C}Q) \simeq \mathbb{C}Q_\infty$ as a differential algebra.

Theorem (Bozec–Fairon–M., 2023)

- $\mathcal{J}_\infty \mathbb{A}$ is the unique differential algebra such that

$$\mathrm{Hom}_{\mathtt{Diff.Alg}}(\mathcal{J}_\infty \mathbb{A}, \mathbb{B}) = \mathrm{Hom}_{\mathtt{Alg}}(\mathbb{A}, \mathbb{B})\,, \quad (\mathbb{B}, \partial).$$

- If $\mathbb{A}$ is a double Poisson algebra, then $\mathcal{J}_\infty \mathbb{A}$ has a unique double Poisson structure such that:

$$\{\!\{a_\lambda b\}\!\} = \{\!\{a, b\}\!\}\,, \quad a, b \in \mathbb{A}.$$

- The jet functor is compatible with the representation functor:

$$\begin{array}{ccc} \mathtt{DPVA} & \xrightarrow{(-)_N} & \mathtt{PVA} \\ \uparrow{\scriptstyle \mathtt{J}} & & \uparrow{\scriptstyle \mathtt{J}} \\ \mathtt{DPA} & \xrightarrow{(-)_N} & \mathtt{PA} \end{array} \qquad \begin{array}{ccc} \mathcal{J}_\infty \mathbb{A} & \longrightarrow & \mathcal{J}_\infty \mathbb{A}_N \\ \uparrow & & \uparrow \\ \mathbb{A} & \longrightarrow & \mathbb{A}_N \end{array}$$

$\rightsquigarrow$ *Commutativity of the top face.*

Reduction in the Poisson vertex setting

$(\mathcal{J}_\infty \mathbb{A}_N)^{\mathrm{GL}_N}$ is too big compared with $\mathcal{J}_\infty(\mathbb{A}_N^{\mathrm{GL}_N})$ in most situations!
$\rightsquigarrow$ we consider the $\mathcal{J}_\infty(\mathrm{GL}_N)$-action...

Let G be an affine algebraic group acting on $Y := \mathrm{Spec}(A)$. Then $\mathcal{J}_\infty(\mathrm{G})$ acts on $\mathcal{J}_\infty(Y)$, and so on $\mathcal{J}_\infty(A)$.

In general, the morphism $j_A \colon \mathcal{J}_\infty(A^{\mathrm{G}}) \to (\mathcal{J}_\infty A)^{\mathcal{J}_\infty(\mathrm{G})}$ is neither injective, nor surjective.

Example (Linshaw–Schwarz–Song)

If $\mathrm{G} = \mathrm{GL}_N$ and $Y = (\mathbb{C}^N)^{\oplus p} \oplus ((\mathbb{C}^N)^*)^{\oplus q}$, then

$$\mathcal{J}_\infty(Y /\!\!/ \mathrm{G}) \simeq (\mathcal{J}_\infty Y)^{\mathcal{J}_\infty \mathrm{G}}.$$

Example (Raïs–Tauvel, Belinson–Drinfeld)

If G is a reductive group with Lie algebra $\mathfrak{g}$, then

$$\mathcal{J}_\infty(\mathfrak{g}^* /\!\!/ \mathrm{G}) \simeq (\mathcal{J}_\infty \mathfrak{g}^*)^{\mathcal{J}_\infty \mathrm{G}}.$$

$\mathcal{J}_\infty(\mathrm{G})$ does not act by Poisson vertex algebra automorphisms, even if G acts by Poisson algebra automorphisms on A.

Theorem (Bozec–Fairon–M.)

Assume that G acts by Poisson algebra automorphisms on A.

Then $(\mathcal{J}_\infty A)^{\mathcal{J}_\infty \mathrm{G}}$ is a Poisson vertex subalgebra of $\mathcal{J}_\infty A$.

Idea of the proof

- If G connected, then $(\mathcal{J}_\infty A)^{\mathcal{J}_\infty \mathrm{G}} = (\mathcal{J}_\infty A)^{\mathfrak{g}[\![t]\!]}$, with $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$.

The action of $\mathfrak{g}[\![t]\!]$ on $\mathcal{J}_\infty A$ is entirely determined by the action of $\mathfrak{g}$ on A.

For $a, b \in \mathcal{J}_\infty A$, $x \in \mathfrak{g}$ and $k, n \in \mathbb{Z}_{\geqslant 0}$:

$$(xt^k).\partial a = \partial(xt^k).a + k(xt^{k-1}).a$$

$$(xt^k).(a_{(n)}b) = a_{(n)}((xt^k).b) + \sum_{\ell=0}^{k} \binom{k}{\ell} ((xt^{k-\ell}).a)_{(n+\ell)}b.$$

- If G is finite, then $\mathcal{J}_\infty \mathrm{G} \simeq \mathrm{G}$ acts by Poisson vertex algebra automorphisms. □

$\rightsquigarrow$ *Commutativity of the right face.*

Van den Bergh observed that the Poisson bracket on $\mathbb{A}_N^{\mathrm{GL}_N}$ is completely determined by a Lie bracket on the vector space $H_0(\mathbb{A}) := \mathbb{A}/[\mathbb{A}, \mathbb{A}]$.

Theorem (Crawley-Boevey, Van Den Bergh, Fairon):
The multiplication yields a linear map $\mathrm{m} \circ \{\!\{-,-\}\!\} : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ which descends to a Lie bracket on $H_0(\mathbb{A})$:

$$[a_\sharp, b_\sharp] := (\mathrm{m} \circ \{\!\{a, b\}\!\})_\sharp \,.$$

There is a unique Poisson bracket on $\mathbb{A}_N^{\mathrm{GL}_N}$ such that for any $a, b \in \mathbb{A}$,

$$\{\mathrm{tr}_N(a), \mathrm{tr}_N(b)\} = \mathrm{tr}_N([a_\sharp, b_\sharp]) \,,$$

where $\mathrm{tr}_N(a) := \sum_{j=1}^N a_{jj} \in \mathbb{A}_N$.

$\rightsquigarrow$ *Commutativity of the front face.*

Need a suitable "vertex" analogue of Crawley-Boevey–Van Den Bergh's result.

Idea: replace algebras with differential algebras and Lie brackets with Lie vertex brackets.

Theorem (De Sole–Kac–Valeri)

If $\mathbb{V}$ is a double Poisson vertex algebra, the multiplication on $\mathbb{V}$ induces a linear map $\mathrm{m} \circ \{\!\{-_\lambda-\}\!\} : \mathbb{V} \otimes \mathbb{V} \to \mathbb{V}[\lambda]$ which descends to a Lie vertex bracket on $H_0(\mathbb{V})$:

$$[a_{\sharp\,\lambda} b_\sharp] := (\mathrm{m} \circ \{\!\{a_\lambda b\}\!\})_\sharp\,.$$

- *To complete the cube, we shall consider a variation of the above theorem relative to a subspace of $H_0(\mathbb{V})$.*

Theorem (Bozec–Fairon–M., 2023)

Let $\mathbb{A}$ be endowed with a H_0-Poisson structure $[-,-]$.

$$\mathrm{Vect}_\infty(\mathbb{A}) := \mathrm{span}_{\mathbb{C}}\{\partial^r(a) \colon a \in \mathbb{A},\ r \geqslant 0\} \subset \mathcal{J}_\infty(\mathbb{A}).$$

- There exists a unique H_0-Poisson vertex structure on $\mathrm{Vect}_\infty(\mathbb{A})$ such that:

$$[a_{\sharp\,\lambda} b_\sharp] := [a_\sharp, b_\sharp]\lambda^0\,,$$

with $a := \partial^0(a), b := \partial^0(b) \in \mathrm{Vect}_\infty(\mathbb{A})$.

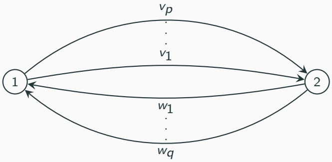
- The following diagram is commutative, for $\mathbb{A}$ a double Poisson algebra:

$$\begin{array}{ccc} \mathbb{A} & \longrightarrow & \mathcal{J}_\infty \mathbb{A} \\ \downarrow & & \downarrow \\ H_0(\mathbb{A}) & \longrightarrow & H_0(\mathrm{Vect}_\infty(\mathbb{A})) \end{array}$$

- The cube commutes! The same goes in the Hamiltonian setting.

Illustrating example

$Q_{p,q}$:



$p, q \geqslant c$, for some $c \geqslant 1$

- $\mathbb{C}Q_{p,q}$ is a double Poisson algebra with: $\{\!\{v_i, w_j\}\!\} = \delta_{i,j}\delta_{(i\leqslant c)}\, e_2 \otimes e_1$.

▶ *The case $p = q = c$ is a double Poisson algebra obtained from a double quiver.*

- $\mathbb{C}(Q_{p,q})_\infty$ is a double Poisson vertex algebra with:

$$\left\{\!\!\left\{ v_i^{(\ell)}{}_\lambda\, w_j^{(m)} \right\}\!\!\right\} = (-1)^\ell\, \lambda^{\ell+m}\, \delta_{i,j}\delta_{(i\leqslant c)}\, e_2 \otimes e_1 .$$

- The Lie bracket on $H_0(\mathbb{C}Q_{p,q})$ is computed as for the *necklace Lie bracket*.

Then we deduce the H_0-Poisson vertex structure on $\mathrm{Vect}_\infty(\mathbb{C}Q_{p,q})$.

▶ Assume $\underline{n} = (n, 1)$. Then $(\mathbb{C}Q_{p,q})_{\underline{n}} \simeq \mathbb{C}[Y]$, where $Y = (\mathbb{C}^n)^{\oplus p} \oplus ((\mathbb{C}^n)^*)^{\oplus q}$.

- The λ-Poisson vertex bracket on $\mathcal{J}_\infty(\mathbb{C}[Y]^{\mathrm{GL}_n}) \simeq \mathcal{J}_\infty(\mathbb{C}[Y])^{\mathcal{J}_\infty(\mathrm{GL}_n)}$ is computed from the Poisson bracket on $\mathbb{C}[Y]^{\mathrm{GL}_n}$.

Problem 1

Let $\mathcal{V}$ be a vertex algebra such that the Poisson structure on $R_{\mathcal{V}}$ comes from a double Poisson algebra, that is,

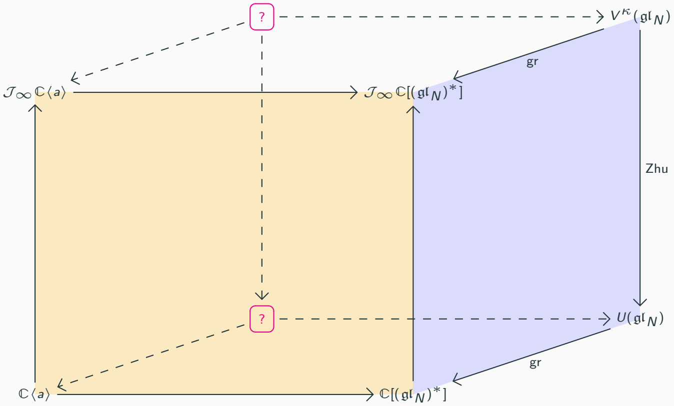
$$R_{\mathcal{V}} \cong \mathbb{A}_N$$

for some double Poisson algebra $\mathbb{A}$ and $N \geqslant 0$, and such that

$$\operatorname{gr} \mathcal{V} \simeq \mathcal{J}_\infty R_{\mathcal{V}}.$$

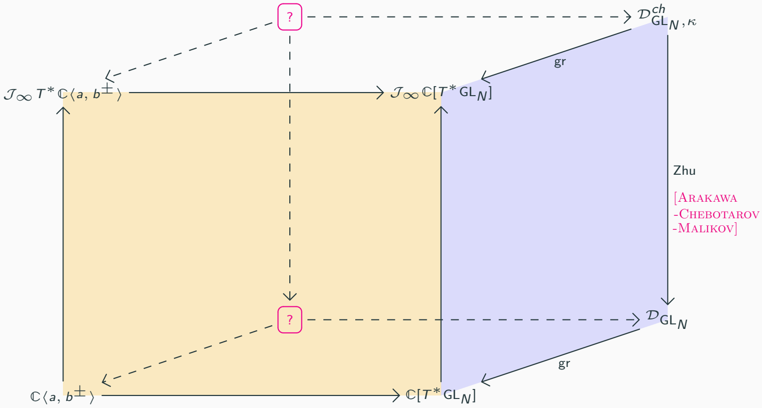
$\rightsquigarrow$ *The Poisson vertex structure on* $\operatorname{gr} \mathcal{V}$ *comes from a double Poisson vertex algebra.*

Is there a double vertex analogue for such a vertex algebra?



How to construct *double* versions of $U(\mathfrak{gl}_N)$ and $V^{\kappa}(\mathfrak{gl}_N)$ corresponding to $\mathbb{C}\langle a\rangle$ and $\mathcal{J}_\infty\mathbb{C}\langle a\rangle$, respectively?

Chiral differential operators



How to construct *double* versions of $\mathcal{D}_{\mathrm{GL}_N}$ and $\mathcal{D}^{ch}_{\mathrm{GL}_N,k}$ corresponding to $\mathbb{C}\langle a, b^\pm\rangle$ and $\mathcal{J}_\infty\mathbb{C}\langle a, b^\pm\rangle$, respectively?

Hamiltonian reduction

A noncommutative moment map $\mathbb{\mu}$ is so that the matrix-valued function

$$X(\mathbb{\mu})\colon \operatorname{Rep}(\mathbb{A}, \underline{n}) \to (\mathfrak{gl}_{\underline{n}})^*$$

is a moment map relative the $\mathrm{GL}_{\underline{n}}$-action by conjugation.

Problem 2

Are there analogues of H_0-structures and noncommutative moment maps that lead to more general Hamiltionian actions on the representation space ?

Hamiltonian actions

There are two commuting Hamiltonian GL_N-actions on $T^*(\mathrm{GL}_N)$:

$$g_L(h,x) = (hg^{-1}, gxg^{-1}), \quad g_R(h,x) = (gh, x), \quad g,h \in \mathrm{GL}_N,\ x \in \mathfrak{gl}_N.$$

with moment maps:

$$\mu_L\colon (h,x) \longmapsto x, \quad \mu_R\colon (h,x) \longmapsto -hxh^{-1}.$$

- *These moment maps admits chiral quantized versions.*

Set $\kappa^* := -\kappa - \kappa_{\mathfrak{gl}_N}$. There are vertex algebra embeddings:

$$\pi_L\colon \mathsf{V}^{\kappa^*}(\mathfrak{gl}_N) \longrightarrow \mathcal{D}^{ch}_{\mathrm{GL}_N,\kappa}, \qquad \pi_R\colon \mathsf{V}^{\kappa}(\mathfrak{gl}_N) \longrightarrow \mathcal{D}^{ch}_{\mathrm{GL}_N,\kappa},$$

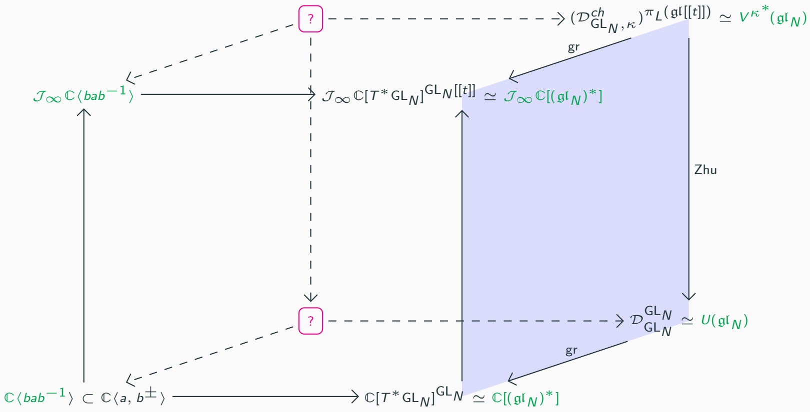
such that [Malikov–Schechtman–Vaintrob]:

$$(\mathcal{D}^{ch}_{\mathrm{GL}_N,\kappa})^{\pi_L(\mathfrak{gl}_N[[t]])} \cong \mathsf{V}^{\kappa^*}(\mathfrak{gl}_N), \qquad (\mathcal{D}^{ch}_{\mathrm{GL}_N,\kappa})^{\pi_R(\mathfrak{gl}_N[[t]])} \cong \mathsf{V}^{\kappa}(\mathfrak{gl}_N).$$

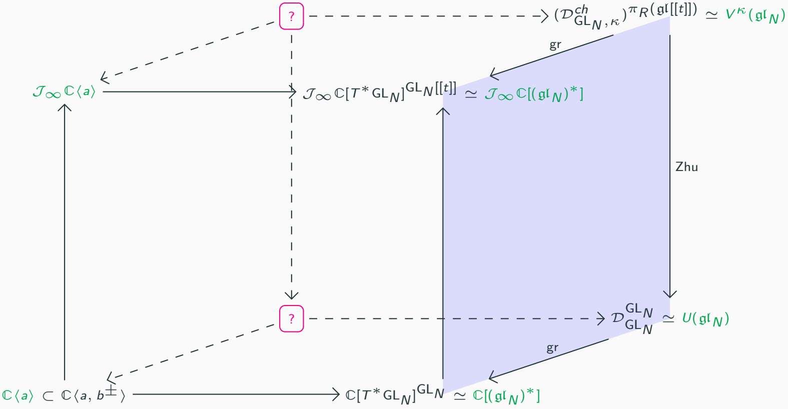
The element

$$\mathbb{\mu} = a - bab^{-1} \in \mathbb{C}\langle a, b^{\pm}\rangle$$

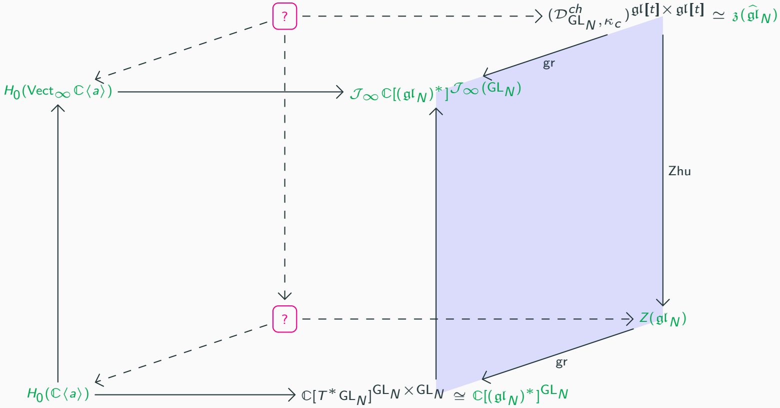
is a noncommutative moment map corresponding to $\mu := \mu_L + \mu_R$.



?
$(\mathcal{D}^{ch}_{\mathrm{GL}_N,\kappa})^{\pi_L(\mathfrak{gl}[[t]])} \simeq V^{\kappa^*}(\mathfrak{gl}_N)$
gr
$\mathcal{J}_\infty \mathbb{C}\langle bab^{-1}\rangle$
$\mathcal{J}_\infty \mathbb{C}[T^*\mathrm{GL}_N]^{\mathrm{GL}_N[[t]]} \simeq \mathcal{J}_\infty \mathbb{C}[(\mathfrak{gl}_N)^*]$
Zhu
?
$\mathcal{D}^{\mathrm{GL}_N}_{\mathrm{GL}_N} \simeq U(\mathfrak{gl}_N)$
gr
$\mathbb{C}\langle bab^{-1}\rangle \subset \mathbb{C}\langle a, b^{\pm}\rangle$
$\mathbb{C}[T^*\mathrm{GL}_N]^{\mathrm{GL}_N} \simeq \mathbb{C}[(\mathfrak{gl}_N)^*]$



?
$(\mathcal{D}^{ch}_{GL_N,\kappa})^{\pi R(\mathfrak{gl}[[t]])} \simeq V^{\kappa}(\mathfrak{gl}_N)$
gr
$\mathcal{J}_\infty \mathbb{C}\langle a \rangle$
$\mathcal{J}_\infty \mathbb{C}[T^* GL_N]^{GL_N[[t]]} \simeq \mathcal{J}_\infty \mathbb{C}[(\mathfrak{gl}_N)^*]$
Zhu
?
$\mathcal{D}^{GL_N}_{GL_N} \simeq U(\mathfrak{gl}_N)$
gr
$\mathbb{C}\langle a \rangle \subset \mathbb{C}\langle a, b^{\pm} \rangle$
$\mathbb{C}[T^* GL_N]^{GL_N} \simeq \mathbb{C}[(\mathfrak{gl}_N)^*]$


?
$(\mathcal{D}^{ch}_{\mathrm{GL}_N,\kappa_c})^{\mathfrak{gl}[t]\times\mathfrak{gl}[t]} \simeq \mathfrak{z}(\widehat{\mathfrak{gl}}_N)$
$H_0(\mathrm{Vect}_\infty \mathbb{C}\langle a\rangle)$
$\mathcal{J}_\infty\mathbb{C}[(\mathfrak{gl}_N)^*]^{\mathcal{J}_\infty(\mathrm{GL}_N)}$
gr
Zhu
?
$Z(\mathfrak{gl}_N)$
gr
$H_0(\mathbb{C}\langle a\rangle)$
$\mathbb{C}[T^*\mathrm{GL}_N]^{\mathrm{GL}_N\times\mathrm{GL}_N} \cong \mathbb{C}[(\mathfrak{gl}_N)^*]^{\mathrm{GL}_N}$

Slodowy slices

• The Poisson structure on **Slodowy slices** $\mathscr{S}_f$ is obtained by Hamiltionian reduction from that of $\mathfrak{g} \simeq \mathfrak{g}^*$.

• For $\mathfrak{g} = \mathfrak{gl}_N$, it was proved by MAFFEI that Slodowy slices can be described in term of quiver varieties.

Problem 3

Can we describe the Poisson structure on Slodowy slices in $\mathfrak{gl}_N$ from double Poisson algebras?

▶ Slodowy slices admit natural quantizations: the **finite W-algebras** $U(\mathfrak{g}, f)$.

▶ Slodowy slices admit natural chiralizations: the **universal affine W-algebras**

$$\mathcal{W}^\kappa(\mathfrak{g}, f) = H^0(\mathcal{V}^\kappa(\mathfrak{g})).$$

We have [DE SOLE–KAC]: $\operatorname{gr} \mathcal{W}^\kappa(\mathfrak{g}, f) \cong \mathcal{J}_\infty \mathbb{C}[\mathscr{S}_f] = \mathcal{J}_\infty R_{\mathcal{W}^\kappa(\mathfrak{g},f)}$.

THE END