Borcherds' Lie algebra and C_2-cofiniteness of moonshine VOAs

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Introduction

Classifi. of hol. VOA of $c = 24$ was accomplished except for the moonshine type. The remaining is to prove the uniqueness of moonshine type, i.e.,

[Uniqueness conj [FLM]] If V is VOA, with non-sing inv $\langle , \rangle$, $c = 24$, $\sum \dim V_n q^{n-1} = j(\tau) - 744 = q^{-1} + 196884q + \ldots$, then $V \cong V^\natural$?

With additional assumps, there are serveral results. e.g., if we have an iso. $V/C_2(V) \cong V^\natural/C_2(V^\natural)$ of Poisson algebras keeping grades and inner products, then we have $V \cong V^\natural$ by a Griess' result and easy calculation.

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Conversely, what we can get from the assump of uniqueness conj? From now on, V is a VOA satisfying these conditions.
Given data are few. Under this assumption, the known result we can use is

iso: $B(V) \cong B(V^\natural)$ of Borchers' Lie algebras. (explain later)

$B(V^\natural)$ is called Monster Lie algebra. However, we can not use the Monster actions and so we just call it Borcherds' Lie algebra.

Borchers Lie algebra

Borcherds has also shown: $B(V)$ is GKM algebra with simple roots

$\{(1,-1),(1,1),(1,2),(1,3),...\} \subseteq \mathrm{II}_{1,1}$, Lorentzian lattice with Gram matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

with multiplicities $\dim V_{n+1}$ for $(1,n)$. In particular,

$B(V)$ is generated by root spaces $B(V)^{(1,m)} \cong (V_{m+1}, \langle , \rangle)$ freely.

e.g. $[B(V)^{(1,1)}, B(V)^{(1,1)}] \cong \mathrm{Skew}_{\dim B(V)^{(1,1)}}$

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Remark 1

Borcherds has shown: $B(V^\natural)$ is GKM alg. with the above simple roots. The critical point of the proof is that he used only nonsingular inv. $\langle , \rangle$, $\mathrm{ch}_{V^\natural}(\tau) = J(\tau)$, and $c = 24$. Hence as they mentioned in [B86] and [J10], $B(V)$ has the same simple roots and $\cong B(V^\natural)$.

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Because of that, my first impression was "we could not get back useful inf. of V from $B(V)$.

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Today, I will show you one of the results from this fact. Namely, we can prove the following theorem.

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More precisely,

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The proof consists of three steps: (1) $V_5 \subseteq C_2(V)$, (2) $V_6 \subseteq C_2(V)$, and (3) $V_n \subseteq C_2(V)$ for $n \geq 7$ by induction.
Today I will show only the first step (1) because the others are similar. If I have time, then (3).

Physical states and Borchers Lie algebra

Set $\tilde{V} = V \otimes V_{\mathrm{II}_{1,1}}$, which has $\mathrm{II}_{1,1}$-graded structure.
$L(n)$, $\tilde{L}(n)$ denotes Virasoro ops of V and $\tilde{V}$, respectively. Set

$$P^1 = \{u \in \tilde{V}_1 \mid \tilde{L}(n)u = 0 \ \forall n > 0\}$$

the space of physical states. Then we define

a Borcherds Lie algebra $B(V) = P^1/\mathrm{Rad}(P^1)$,

where $\mathrm{Rad}(P^1)$ is the null space of $\langle \cdot, \cdot \rangle$.

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where $\mathrm{Rad}(P^1)$ is the null space of $\langle \cdot, \cdot \rangle$.
Let $P^1_{(m,n)} \subseteq P^1$ of degree $(m,n) \in \mathrm{II}_{1,1}$ and
$B(V)^{(m,n)} = P^1_{(m,n)}/\mathrm{Rad}(P^1_{(m,n)})$.

No ghost theorem

We will recall the following results from [B86].

Theorem 2 (The no-ghost theorem)

Let $V = \oplus_{n=0}^{\infty} V_n$ *be a VOA of* $c = 24$ *with non-sing bi. form* $\langle , \rangle$ *and* $G \leq \mathrm{Aut}(V)$ *(we view* $G = G \times 1$ *on* $V \otimes V_{\mathrm{II}_{1,1}}$*). Then* $B(V)^{(m,n)} \cong V_{1+mn}$ *as* G*-mods with an inv bi form if* $mn \neq 0$ *and* $B(V)^{(0,0)} \cong V_1 \oplus \mathbb{C}^2$ *and* $B(V)^{(m,n)} = 0$ *for else.*

For $V^\natural$, $B(V^\natural)^{(m,n)} \cong V^\natural_{mn+1}$ as the monster simple group modules for $mn \neq 0$.

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What a waste of good theorem!

A large group and its homogeneous spaces

P^1, $B(V)$ are defined by Virasoro ops $\{L(n)+L'(n) \mid n \in \mathbb{Z}\}$,
We will introduce a group

Definition 1

$$G = \{g = \prod_m g_m \in \prod_m O(V_m, \langle , \rangle) \mid g_m L(n) = L(n) g_{m-n} \ {}^\forall n, m \in \mathbb{Z}\}$$

(Note: G is not auto gr. of $B(V)$, just set of orth transf comm with $\mathrm{Vir.}$)
and extend it to an auto. $G \otimes 1_{V_{\mathrm{II}_{1,1}}}$ of $\tilde{V}$.

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$P^1_{(m,n)}$ and $B(V)^{(m,n)}$ are G-modules.

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Definition 2 (homogeneous G-submodules)

Set $V_k^p = \{v \in V_k \mid L(n)v = 0 \ {}^\forall n \geq 1\}$ $\quad$ Vir primary states
$\Rightarrow V = \oplus_{k \in \mathbb{N}} U(\mathrm{Vir}) V_k^p$, $\quad$ V_k^p is simple G-mod. and $G = \prod_{m=0}^{\infty} O(V_m^p)$.
Define **projection:** $\pi_k : \tilde{V} \to (U(\mathrm{Vir}) \cdot V_k^p) \otimes V_{\mathrm{II}_{1,1}}$
$P^1_{(m,n),k} := \pi_k(P^1_{(m,n)})$, $\quad$ $B(V)^{(m,n),k} := \pi_k(B(V)^{(m,n)})$

Root space of Borchers Lie algebra as G-modules

Lemma 3

For $(m,n) = (1,-1),(1,1),(1,3),(2,2)$, $V^{(m,n)} \cong V_{mn+1}$ as G-modules.

Remark 2

In Borcherds' proof of the no-ghost theorem, his ops were given by elts in $V_{\mathrm{II}_{1,1}}$, and Vir, which implies $B(V)^{(m,n)} \cong V_{mn+1}$ as G-mods for $mn \neq 0$.

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For $v \in V^p_{mn+1}$, we have $v \otimes e^{(m,n)} \in P^1$. It is easy to see $V^p_{mn+1} \otimes e^{(m,n)} = P^1_{(m,n),mn+1}$ and $\mathrm{Rad}(P^1_{(m,n),mn+1}) = 0$. Similarly, $V^p_{mn} \otimes \delta^{(m,-n)}_{(-1)} e^{(m,n)} \subseteq P^1_{(m,n)}$ and $\mathrm{Rad}(P^1_{(m,n),mn}) = \tilde{L}(-1)(V^p_{mn} \otimes e^{(m,n)})$, where $\delta^{(m,n)}_{(-1)} \mathbf{1}' = (m,n)(-1)\mathbf{1} \in V_{\mathrm{II}_{1,1}}$ for $(m,n) \in \mathrm{II}_{1,1}$. i.e.,

The both of mult. of V^p_{mn+1} and V^p_{mn} in $B(V)^{(m,n)}$ are one.

[Proof] By direct calculation, we have

$$P^1_{(1,-1)} = \mathbb{C}\mathbf{1}\otimes e^{(1,-1)} \cong V_0 \quad \text{and} \quad \mathrm{Rad}(P^1_{(1,-1)}) = 0$$

$$P^1_{(1,1)} = \underset{\substack{\wr\Vert \\ V_2^p}}{\mathbb{C}(V_2^p\otimes e^{(1,1)})} \oplus \underset{\substack{\wr\Vert \\ V_0}}{\mathbb{C}[L(-2)\mathbf{1}\otimes e^{(1,1)} - 6\mathbf{1}\otimes\delta^{(1,-1)}_{(-1)}e^{(1,1)}]} \oplus \mathrm{Rad} \cong V_2.$$

From $P^1_{(1,3),4} = V_4^p$, $P^1_{(1,3),3} = V_3^p + \mathrm{Rad}.$ $\mathrm{mult}(P^1_{(1,3),0}) < \dim V_2^p$, $\dim B(V)^{(1,3)} = \dim V_4$, we have $B^{(1,3)}(V) \cong V_4$.

We will give a proof later for the case $(m,n) = (2,2)$. □

The useful fact and products by $P^1_{(1,-1)}, 0)$ and $\pi_0(P^1)$

$$\{B(V)^{(1,n)} : n = -1, 0, ...\} \text{ generates } \oplus_{m=1}^{\infty} \oplus_{n\in\mathbb{Z}} B(V)^{(m,n)}.$$

e.g. we will use

$$B(V)^{(2,2)} = [B(V)^{(1,-1)}, B(V)^{(1,3)}] + [B(V)^{(1,1)}, B(V)^{(1,1)}]$$

Coming back to physical states, we have

$$P^1_{(2,2)} \subseteq (P^1_{(1,-1)})_0(P^1_{(1,3)}) + (P^1_{(1,1)})_0(P^1_{(1,1)}) + \mathrm{Rad}(P^1_{(2,2)})$$

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Using $P^1_{(1,1)} = P^1_{(1,1),0} \oplus P^1_{(1,1),2}$ $\quad (B(V)^{(1,1)} = B(V)^{(1,1),0} \oplus B(V)^{(1,1),2})$,

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Proposition 4

$(\mathbf{1} \otimes e^{(1,-1)})_0 = 1 \otimes (e^{(1,-1)})_0$ *is G-isomorphism.*
For $\alpha \in \pi_0(P^1)$, 0 th product α_0 is G-homo.

Hence $[B(V)^{(1,-1)}, B(V)^{(1,k)}]$ and $[B(V)^{(1,1),0}, B(V)^{(1,k)}]$ are holo. images of $B(V)^{(1,k)}$ as G-modules.

Product and projection to $U(\mathrm{Vir})V_n^p$

For $v, u \in V_2^p$, we set

$$\Phi^1(v,u) = (v \otimes e^{(1,1)})_0(u \otimes e^{(1,1)}) \quad \in P^1_{(2,2)}$$

Then we have expressions:

$$\Phi^1(v,u) = v_{-2}u \otimes o_0(e^{(1,1)})e^{(2,2)} + v_{-1}u \otimes o_1(e^{(1,1)})e^{(2,2)}$$
$$+ v_0u \otimes o_2(e^{(1,1)})e^{(2,2)} + v_1u \otimes o_3(e^{(1,1)})e^{(2,2)} + v_3u \otimes o_5(e^{(1,1)})e^{(2,2)}.$$

We note $\quad Y(e^\gamma, z) = E^-(-\gamma, z)E^+(-\gamma, z)e^\gamma z^{\mathrm{wt}(e^\gamma)+\gamma}$

$$= \exp(\textstyle\sum_{n=1}^\infty \gamma(-n)/nz^n) \exp(\sum_{n=1}^\infty -\gamma(n)/nz^{-n})e^\gamma z^{\mathrm{wt}(e^\gamma)+\gamma}.$$

In particular, $\{o_k(e^{(1,1)})e^{(2,2)} \mid k\}$ do not depend on the choices of v, u.

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In particular, $\{o_k(e^{(1,1)})e^{(2,2)} \mid k\}$ do not depend on the choices of v, u.
Clearly, $v_{3-k}u \in V_k$. Since V_k^p is the only primary states in $\pi_k(V)$,

If $\pi_k(v_{3-k}u) = 0$, then $\pi_k(v_{3-k-j}u) = 0$ $\forall j \in \mathbb{N}$.

Since $c = 24$, $U(\mathrm{Vir})V_k^p$ is a Verma module for $k \neq 0$. Expressing $L(-n_1)\cdots L(-n_k)z$ with $n_1 \geq \ldots \geq n_k$, we have

Explicit expression by Virasoro operators

Lemma 5

For $(k,m) \in \mathbb{N} \times \mathbb{N}$, *operators* ${}^{\exists}Q^{(2,2),k}(-m) \in U(\mathrm{Vir})$ *of degree* $-m$ *s.t.* $\pi_k(v_{3-k-m}u) = Q^{(2,2),k}(-m)\pi_k(v_{3-k}u)$

We will show the exact formula for a few $Q^{a,b}(-m)$.

Lemma 6

For $v, u \in V_2^p$ *and* $w \in V_3^p$, *we have:*

$$\pi_0(v_{-1}u) = \frac{1}{71}\{3L(-4)\pi_0(v_3u) + \frac{11}{12}L(-2)^2\pi_0(v_3u),$$
$$\pi_0(v_{-3}u) = \frac{1}{196883}\{3492L(-6)\pi_0(v_3u) + \frac{15623}{12}L(-4)L(-2)\pi_0(v_3u)\}$$
$$+ \frac{1}{196883}\{\frac{1271}{2}L(-3)^2\pi_0(v_3u) + 124L(-2)^3\pi_0(v_3u)\},$$
$$\pi_2(v_0w) = \frac{1}{41}\{6L(-2)\pi_2(v_1u) + \frac{1}{4}L(-1)^2\pi_2(v_1u)\},$$
$$\pi_3(v_{-2}u) = \frac{1}{47}\{\frac{17}{3}L(-2)\pi_3(v_2w) + \frac{11}{2}L(-1)^2\pi_3(v_2w)\}, \quad \text{and}$$
$$\pi_4(v_{-2}w) = \frac{1}{8}L(-2)\pi_4(v_0w) + \frac{1}{16}L(-1)^2\pi_4(v_0w).$$

In particular, $L(-2)Z \in \pi_3(C_2(V))$, $L(-2)^3\mathbf{1} \in \pi_0(C_2(V))$, *and* $L(-2)Y \in \pi_2((V_2)_0(V_3) + L(-1)V_3)$ *for* $Z \in V_4^p$ *and* $Y \in V_2^p$.

Physical state defined by V_k^p

Conversely, we can define physical states as follows:

Lemma 7

For $Z \in V_k^p$, *define*

$$\Phi_k^1(Z) = \quad \sum_{t=0}^{5-t} Q^{(2,2),k}(-t)Z \otimes o_t(e^{(1,1)})e^{(2,2)} \quad \in P^1_{(2,2),k}$$

Then $\Phi^1(v,u) = \sum_{k=0}^{\infty} \Phi_k^1(\pi_k(v_{3-k}u))$.

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Namely, $\Phi^1(v,u)$ is uniquely determined by elements

$$\begin{array}{ccccc} (\pi_0(v_3 u), & \pi_2(v_1 u), & \pi_3(v_0 u), & \pi_4(v_{-1} u), & \pi_5(v_{-2} u)) \\ \Cap & \Cap & \Cap & \Cap & \Cap \\ V_0 & \oplus V_2^p & \oplus V_3^p & \oplus V_4^p & \oplus V_5^p. \end{array}$$

i.e. The structure of V on $B(V)$ works only on $\oplus V_k^p$.

Lemma 8

In $\{\pi_k(\Phi^1(v,u)) \mid v,u \in V_2^p\}$, physical states over V_k^p appears at most onece for each k modulo $\mathrm{Rad}(P^1)$.

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Then $\Phi^1(v,u) = \sum_{k=0}^{\infty} \Phi_k^1(\pi_k(v_{3-k}u))$.

Namely, $\Phi^1(v,u)$ is uniquely determined by elements

$$\begin{array}{ccccc} (\pi_0(v_3u), & \pi_2(v_1u), & \pi_3(v_0u), & \pi_4(v_{-1}u), & \pi_5(v_{-2}u)) \\ \cap & \cap & \cap & \cap & \cap \\ V_0 & \oplus V_2^p & \oplus V_3^p & \oplus V_4^p & \oplus V_5^p. \end{array}$$

i.e. The structure of V on $B(V)$ works only on $\oplus V_k^p$.

Lemma 8

In $\{\pi_k(\Phi^1(v,u)) \mid v,u \in V_2^p\}$, physical states over V_k^p appears at most onece for each k modulo $\mathrm{Rad}(P^1)$.

By this lemma, we have $B(V)^{(2,2)} \cong V_5$ as G-modules.

Definition 3

$X := [B(V)^{(1,1),2}, B(V)^{(1,1),2}]$ may not be a G-module (just a subspace). Since
$Z := [B(V)^{(1,-1)}, B(V)^{(1,3)}] + [B(V)^{(2,2),0}, B(V)^{(2,2)}]$ is a G-submod of G-mod. $B(V)^{(2,2)} = Z + X$, X contains a complement of Z in $B(V)^{(2,2)}$. Expressing $B(V)^{(2,2)}/Z$ as a direct sum of V_k^p, we say

"X covers a G-mod. $B(V)^{(2,2)}/Z$."

Since $\mathrm{Rad}()$ is also G-mod, we also say

$$(P^1_{(1,1),2})_0(P^1_{(1,1),2}) \text{ covers a } B(V)^{(2,2)}/Z.$$

Since $L(-m)V \subseteq C_2(V)$ for $2 \neq m \geq 1$, $V_5 = V_5^p + L(-2)V_3^p + C_2(V)_5$.
So we forcus our study to V_3^p and V_5^p

Proposition 9

$[B(V)^{(1,1)}, B(V)^{(1,1)}]$ covers $V_5/V_4 \cong V_5^p + V_3^p + V_2^p$ at least and $(V_2^p \otimes e^{(1,1)})_0(V_2^p \otimes e^{(1,1)})$ covers $V_5^p + V_3^p$ at least .

[Proof] $B(V)^{(2,2)} \cong V_5$, $[B(V)^{(1,-1)}, B(V)^{(1,3)}] \cong B(V)^{(1,3)} \cong V_4$, $\pi_3([B(V)^{(1,1),0}, B(V)^{(1,1)}]) = \pi_5([B(V)^{(1,1),0}, B(V)^{(1,1)}]) = 0$. □

Covering

Therefore, we have

Lemma 10

Assume that $\{\Phi^1(v,u) : v,u \in V_2^p\}$ *covers* V_k^p *in* $B(V)^{(2,2)}/J$ *for some* G*-mod* J*. If* $\pi_k(\Phi^1(v,u)) \in J$ *for some* $v,u \in V_2^p$*, then* $\pi_k(\Phi^1(v,u)) = 0$*, that is,* $v_{3-k}u = 0$*.*

Then we can translate Proposition 9 into the following:

Lemma 11

$\mathrm{Span}_{\mathbb{C}}\{(\pi_3(v_0u), \pi_5(v_{-2}u)) \in V_3^p \oplus V_5^p \mid v,u \in V_2^p\}$ *covers* $V_5^p + V_3^p$ *at least.*

Weight five

As a corollary of Lemma 6 and 11, we have:

Lemma 12

$V_5 \subseteq C_2(V)$.

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[Proof] By Lemma 11, for $z \in V_5^p$, ${}^\exists v^i, u^i \in V_2^p$ s.t. $z = \pi_5(\sum v^i_{-2}u^i)$ and $\pi_3(\Phi^1(\sum v^i_0 u^i)) \in J$. By Lemma 10, $\pi_3(\sum v^i_0 u^i) = 0$. Hence $\pi_3(\sum v^i_{-2}u^i) = 0$ and $\sum v^i_{-2}u^i - z \in \pi_0(V_5) + \pi_2(V_5) + \pi_4(V_5)$. Since $\pi_0(V_5) + \pi_2(V_5) + \pi_4(V_5) \subseteq C_2(V)$, we have $z \in C_2(V)$.

Weight five

As a corollary of Lemma 6 and 11, we have:

Lemma 12

$V_5 \subseteq C_2(V)$.

[Proof] By Lemma 11, for $z \in V_5^p$, $^\exists v^i, u^i \in V_2^p$ s.t. $z = \pi_5(\sum v^i_{-2}u^i)$ and $\pi_3(\Phi^1(\sum v^i_0 u^i)) \in J$. By Lemma 10, $\pi_3(\sum v^i_0 u^i) = 0$. Hence $\pi_3(\sum v^i_{-2}u^i) = 0$ and $\sum v^i_{-2}u^i - z \in \pi_0(V_5) + \pi_2(V_5) + \pi_4(V_5)$. Since $\pi_0(V_5) + \pi_2(V_5) + \pi_4(V_5) \subseteq C_2(V)$, we have $z \in C_2(V)$.

Similarly, for $x \in V_3^p$, $^\exists v^i, u^i \in V_2^p$ s.t. $0 = \pi_5(\sum v^i_{-2}u^i)$ and $\pi_3(\sum v^i_0 u^i) = x$. Then since $\pi_3(\sum v^i_{-2}u^i) = \frac{17}{141}L(-2)x + \frac{11}{94}L(-1)^2 x$ by Lemma 6,
$\sum v^i_{-2}u^i - (\frac{17}{141}L(-2)x + \frac{11}{94}L(-1)^2 x) \in \pi_0(V_5) + \pi_2(V_5) + \pi_4(V_5) \subseteq C_2(V)$.
Therefore we have $L(-2)x \in C_2(V)$. □

This completes the proof of $V_5 \subseteq C_2(V)$ and my talk.

Thank you !!

Proposition 13

For $n \geq 7$, we have $V_n \subseteq C_2(V)$.

[Proof]
[Case $n = 2m+1$] We note $V^p_{2m+1} \otimes e^{(2,m)} \subseteq P^1_{(2,m),2m+1}$ and $\mathrm{Rad}(P^1_{(2,m),2m+1}) = 0$. As we explained, $\sum \pi_{2m+1}(P^1_{(1,k)})_0(P^1_{(1,m-k)})$ covers V^p_{2m+1}. Since $P^1_{(1,k)} \subseteq (\oplus_{j=0}^{k+1} V_j) \otimes V_{\mathrm{II}_{1,1}}$, to get elet $v_j w \in V_{2m+1}$ by product of $v \in \oplus_{j=0}^{k+1} V_j$ and $w \in \oplus_{j=0}^{m-k+1} V_j$, we have $j \leq -2$.
Therefore, $V^p_{2m+1} \subseteq \pi_{2m+1}(C_2(V))$ and
$V^p_{2m+1} \subseteq C_2(V) + \oplus_{j=0}^{2m} \mathrm{Vir} V_j) = C_2(V) + \oplus_{k=1}^{m} L(-2)^k V_{2m+1-k}$. Since $L(-2)V_{2m-1} \subseteq C_2(V)$ by induction, we have $V_{2m+1} \subseteq C_2(V)$.

[Case $n = 2m$] $V^p_{2m} \otimes \delta^{(2,-m)}_{(-1)} e^{(2,m)} \subseteq P^1_{(2,m),2m}$ and $\mathrm{Rad}(P^1_{(2,m),2m}) = (L(-1) \otimes 1 + 1 \otimes \delta^{(2,m)}_{(-1)}) V^p_{2m} \otimes e^{(2,m)}$. By the same argument as above, $\pi_{2m}(C_2(V))$ covers V^p_{2m}. Since $L(-2)V_{2m-2} \subseteq C_2(V)$ by induction, we have $V_{2m} \subseteq C_2(V)$. □