Universal vertex algebras beyond the $\mathcal{W}_\infty$-algebras

Andrew Linshaw

University of Denver

Joint with T. Creutzig (Edmonton) and V. Kovalchuk (Denver)

1. Notation

$\mathfrak{g}$ a simple, finite-dimensional Lie (super)algebra.

$\mathcal{W}^k(\mathfrak{g}, f)$ universal $\mathcal{W}$-algebra associated to $\mathfrak{g}$ and an even nilpotent $f \in \mathfrak{g}$.

Simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$.

For this talk: We will replace k with the **shifted level** $\psi = k + h^\vee$.

$\mathcal{W}^\psi(\mathfrak{g}, f)$ will always denote $\mathcal{W}^k(\mathfrak{g}, f)$ with $k = \psi - h^\vee$.

$\mathcal{W}_\psi(\mathfrak{g}, f)$ the simple quotient of $\mathcal{W}_\psi(\mathfrak{g}, f)$.

If $f = f_{\text{prin}}$ is a principal nilpotent, write $\mathcal{W}^\psi(\mathfrak{g}, f) = \mathcal{W}^\psi(\mathfrak{g})$.

1. Notation

$\mathfrak{g}$ a simple, finite-dimensional Lie (super)algebra.

$\mathcal{W}^k(\mathfrak{g}, f)$ universal $\mathcal{W}$-algebra associated to $\mathfrak{g}$ and an even nilpotent $f \in \mathfrak{g}$.

Simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$.

For this talk: We will replace k with the **shifted level** $\psi = k + h^\vee$.

$\mathcal{W}^\psi(\mathfrak{g}, f)$ will always denote $\mathcal{W}^k(\mathfrak{g}, f)$ with $k = \psi - h^\vee$.

$\mathcal{W}_\psi(\mathfrak{g}, f)$ the simple quotient of $\mathcal{W}_\psi(\mathfrak{g}, f)$.

If $f = f_{\text{prin}}$ is a principal nilpotent, write $\mathcal{W}^\psi(\mathfrak{g}, f) = \mathcal{W}^\psi(\mathfrak{g})$.

1. Notation

$\mathfrak{g}$ a simple, finite-dimensional Lie (super)algebra.

$\mathcal{W}^k(\mathfrak{g}, f)$ universal $\mathcal{W}$-algebra associated to $\mathfrak{g}$ and an even nilpotent $f \in \mathfrak{g}$.

Simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$.

For this talk: We will replace k with the **shifted level** $\psi = k + h^\vee$.

$\mathcal{W}^\psi(\mathfrak{g}, f)$ will always denote $\mathcal{W}^k(\mathfrak{g}, f)$ with $k = \psi - h^\vee$.

$\mathcal{W}_\psi(\mathfrak{g}, f)$ the simple quotient of $\mathcal{W}_\psi(\mathfrak{g}, f)$.

If $f = f_{\text{prin}}$ is a principal nilpotent, write $\mathcal{W}^\psi(\mathfrak{g}, f) = \mathcal{W}^\psi(\mathfrak{g})$.

1. Notation

$\mathfrak{g}$ a simple, finite-dimensional Lie (super)algebra.

$\mathcal{W}^k(\mathfrak{g}, f)$ universal $\mathcal{W}$-algebra associated to $\mathfrak{g}$ and an even nilpotent $f \in \mathfrak{g}$.

Simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$.

For this talk: We will replace k with the **shifted level** $\psi = k + h^\vee$.

$\mathcal{W}^\psi(\mathfrak{g}, f)$ will always denote $\mathcal{W}^k(\mathfrak{g}, f)$ with $k = \psi - h^\vee$.

$\mathcal{W}_\psi(\mathfrak{g}, f)$ the simple quotient of $\mathcal{W}_\psi(\mathfrak{g}, f)$.

If $f = f_{\text{prin}}$ is a principal nilpotent, write $\mathcal{W}^\psi(\mathfrak{g}, f) = \mathcal{W}^\psi(\mathfrak{g})$.

1. Notation

$\mathfrak{g}$ a simple, finite-dimensional Lie (super)algebra.

$\mathcal{W}^k(\mathfrak{g}, f)$ universal $\mathcal{W}$-algebra associated to $\mathfrak{g}$ and an even nilpotent $f \in \mathfrak{g}$.

Simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$.

For this talk: We will replace k with the **shifted level** $\psi = k + h^\vee$.

$\mathcal{W}^\psi(\mathfrak{g}, f)$ will always denote $\mathcal{W}^k(\mathfrak{g}, f)$ with $k = \psi - h^\vee$.

$\mathcal{W}_\psi(\mathfrak{g}, f)$ the simple quotient of $\mathcal{W}_\psi(\mathfrak{g}, f)$.

If $f = f_{\text{prin}}$ is a principal nilpotent, write $\mathcal{W}^\psi(\mathfrak{g}, f) = \mathcal{W}^\psi(\mathfrak{g})$.

1. Notation

$\mathfrak{g}$ a simple, finite-dimensional Lie (super)algebra.

$\mathcal{W}^k(\mathfrak{g}, f)$ universal $\mathcal{W}$-algebra associated to $\mathfrak{g}$ and an even nilpotent $f \in \mathfrak{g}$.

Simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$.

For this talk: We will replace k with the **shifted level** $\psi = k + h^\vee$.

$\mathcal{W}^\psi(\mathfrak{g}, f)$ will always denote $\mathcal{W}^k(\mathfrak{g}, f)$ with $k = \psi - h^\vee$.

$\mathcal{W}_\psi(\mathfrak{g}, f)$ the simple quotient of $\mathcal{W}_\psi(\mathfrak{g}, f)$.

If $f = f_{\text{prin}}$ is a principal nilpotent, write $\mathcal{W}^\psi(\mathfrak{g}, f) = \mathcal{W}^\psi(\mathfrak{g})$.

1. Notation

$\mathfrak{g}$ a simple, finite-dimensional Lie (super)algebra.

$\mathcal{W}^k(\mathfrak{g}, f)$ universal $\mathcal{W}$-algebra associated to $\mathfrak{g}$ and an even nilpotent $f \in \mathfrak{g}$.

Simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$.

For this talk: We will replace k with the **shifted level** $\psi = k + h^\vee$.

$\mathcal{W}^\psi(\mathfrak{g}, f)$ will always denote $\mathcal{W}^k(\mathfrak{g}, f)$ with $k = \psi - h^\vee$.

$\mathcal{W}_\psi(\mathfrak{g}, f)$ the simple quotient of $\mathcal{W}_\psi(\mathfrak{g}, f)$.

If $f = f_{\text{prin}}$ is a principal nilpotent, write $\mathcal{W}^\psi(\mathfrak{g}, f) = \mathcal{W}^\psi(\mathfrak{g})$.

2. Feigin-Frenkel duality

Thm: (Feigin, Frenkel, 1991) Let $\mathfrak{g}$ be a simple Lie algebra. Then

$$\mathcal{W}^{\psi}(\mathfrak{g}) \cong \mathcal{W}^{\psi'}({}^L\mathfrak{g}), \qquad r^\vee \psi\psi' = 1.$$

Here ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra, and $r^\vee$ is the lacity of $\mathfrak{g}$.

Similar result holds for $\mathfrak{g} = \mathfrak{osp}_{1|2n}$.

Thm: (Creutzig, Genra)

$$\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2n}) \cong \mathcal{W}^{\psi'}(\mathfrak{osp}_{1|2n}), \qquad 4\psi\psi' = 1.$$

2. Feigin-Frenkel duality

Thm: (Feigin, Frenkel, 1991) Let $\mathfrak{g}$ be a simple Lie algebra. Then

$$\mathcal{W}^{\psi}(\mathfrak{g}) \cong \mathcal{W}^{\psi'}({}^L\mathfrak{g}), \qquad r^{\vee}\psi\psi' = 1.$$

Here ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra, and $r^{\vee}$ is the lacity of $\mathfrak{g}$.

Similar result holds for $\mathfrak{g} = \mathfrak{osp}_{1|2n}$.

Thm: (Creutzig, Genra)

$$\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2n}) \cong \mathcal{W}^{\psi'}(\mathfrak{osp}_{1|2n}), \qquad 4\psi\psi' = 1.$$

2. Feigin-Frenkel duality

Thm: (Feigin, Frenkel, 1991) Let $\mathfrak{g}$ be a simple Lie algebra. Then

$$\mathcal{W}^{\psi}(\mathfrak{g}) \cong \mathcal{W}^{\psi'}({}^L\mathfrak{g}), \qquad r^\vee \psi\psi' = 1.$$

Here ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra, and $r^\vee$ is the lacity of $\mathfrak{g}$.

Similar result holds for $\mathfrak{g} = \mathfrak{osp}_{1|2n}$.

Thm: (Creutzig, Genra)

$$\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2n}) \cong \mathcal{W}^{\psi'}(\mathfrak{osp}_{1|2n}), \qquad 4\psi\psi' = 1.$$

3. Coset construction of principal $\mathcal{W}$-algebras

Thm: (Arakawa, Creutzig, L., 2018) Let $\mathfrak{g}$ be simple and simply-laced. We have diagonal embedding

$$V^{k+1}(\mathfrak{g}) \hookrightarrow V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g}), \qquad u \mapsto u \otimes 1 + 1 \otimes u, \qquad u \in \mathfrak{g}.$$

Set

$$\mathcal{C}^k(\mathfrak{g}) = \text{Com}(V^{k+1}(\mathfrak{g}), V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g})).$$

We have an isomorphism of 1-parameter VOAs

$$C^k(\mathfrak{g}) \cong \mathcal{W}^\psi(\mathfrak{g}), \qquad \psi = \frac{k+h^\vee}{k+h^\vee+1}.$$

Coset realization for B (and C) is different.

Thm: (Creutzig, Genra, and Creutzig-L., 2021) We have an isomorphism of 1-parameter VOAs

$$\text{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{osp}_{1|2n})) \cong \mathcal{W}^\psi(\mathfrak{so}_{2n+1}), \qquad \psi = \frac{2k+2n+1}{2(1+k+n)}.$$

3. Coset construction of principal $\mathcal{W}$-algebras

Thm: (Arakawa, Creutzig, L., 2018) Let $\mathfrak{g}$ be simple and simply-laced. We have diagonal embedding

$$V^{k+1}(\mathfrak{g}) \hookrightarrow V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g}), \qquad u \mapsto u \otimes 1 + 1 \otimes u, \qquad u \in \mathfrak{g}.$$

Set

$$\mathcal{C}^k(\mathfrak{g}) = \text{Com}(V^{k+1}(\mathfrak{g}), V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g})).$$

We have an isomorphism of 1-parameter VOAs

$$C^k(\mathfrak{g}) \cong \mathcal{W}^\psi(\mathfrak{g}), \qquad \psi = \frac{k + h^\vee}{k + h^\vee + 1}.$$

Coset realization for B (and C) is different.

Thm: (Creutzig, Genra, and Creutzig-L., 2021) We have an isomorphism of 1-parameter VOAs

$$\text{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{osp}_{1|2n})) \cong \mathcal{W}^\psi(\mathfrak{so}_{2n+1}), \qquad \psi = \frac{2k + 2n + 1}{2(1 + k + n)}.$$

3. Coset construction of principal $\mathcal{W}$-algebras

Thm: (Arakawa, Creutzig, L., 2018) Let $\mathfrak{g}$ be simple and simply-laced. We have diagonal embedding

$$V^{k+1}(\mathfrak{g}) \hookrightarrow V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g}), \qquad u \mapsto u \otimes 1 + 1 \otimes u, \qquad u \in \mathfrak{g}.$$

Set

$$\mathcal{C}^k(\mathfrak{g}) = \text{Com}(V^{k+1}(\mathfrak{g}), V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g})).$$

We have an isomorphism of 1-parameter VOAs

$$C^k(\mathfrak{g}) \cong \mathcal{W}^{\psi}(\mathfrak{g}), \qquad \psi = \frac{k + h^{\vee}}{k + h^{\vee} + 1}.$$

Coset realization for B (and C) is different.

Thm: (Creutzig, Genra, and Creutzig-L., 2021) We have an isomorphism of 1-parameter VOAs

$$\text{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{osp}_{1|2n})) \cong \mathcal{W}^{\psi}(\mathfrak{so}_{2n+1}), \qquad \psi = \frac{2k + 2n + 1}{2(1 + k + n)}.$$

4. What are trialities of $\mathcal{W}$-algebras?

Let $f \in \mathfrak{g}$ be a nilpotent, and complete f to a copy $\{f, h, e\}$ of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ denote the centralizer of this $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Then $\mathcal{W}^{\psi}(\mathfrak{g}, f)$ has affine subVOA $V^{\psi'}(\mathfrak{a})$, for some level ψ'.

By the **affine coset**, we mean $\mathcal{C}^{\psi}(\mathfrak{g}, f) := \text{Com}(V^{\psi'}(\mathfrak{a}), \mathcal{W}^{\psi}(\mathfrak{g}, f))$.

Sometimes we also take invariants under some group of **outer automorphisms**.

Trialities are isomorphisms between three different affine cosets

$$\mathcal{C}^{\psi}(\mathfrak{g}, f) \cong \mathcal{C}^{\psi'}(\mathfrak{g}', f') \cong \mathcal{C}^{\psi''}(\mathfrak{g}'', f'').$$

These unify and generalize both Feigin-Frenkel duality and the coset realization of principal $\mathcal{W}$-algebras.

4. What are trialities of $\mathcal{W}$-algebras?

Let $f \in \mathfrak{g}$ be a nilpotent, and complete f to a copy $\{f, h, e\}$ of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ denote the centralizer of this $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Then $\mathcal{W}^{\psi}(\mathfrak{g}, f)$ has affine subVOA $V^{\psi'}(\mathfrak{a})$, for some level ψ'.

By the **affine coset**, we mean $\mathcal{C}^{\psi}(\mathfrak{g}, f) := \text{Com}(V^{\psi'}(\mathfrak{a}), \mathcal{W}^{\psi}(\mathfrak{g}, f))$.

Sometimes we also take invariants under some group of **outer automorphisms**.

Trialities are isomorphisms between three different affine cosets

$$\mathcal{C}^{\psi}(\mathfrak{g}, f) \cong \mathcal{C}^{\psi'}(\mathfrak{g}', f') \cong \mathcal{C}^{\psi''}(\mathfrak{g}'', f'').$$

These unify and generalize both Feigin-Frenkel duality and the coset realization of principal $\mathcal{W}$-algebras.

4. What are trialities of $\mathcal{W}$-algebras?

Let $f \in \mathfrak{g}$ be a nilpotent, and complete f to a copy $\{f, h, e\}$ of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ denote the centralizer of this $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Then $\mathcal{W}^{\psi}(\mathfrak{g}, f)$ has affine subVOA $V^{\psi'}(\mathfrak{a})$, for some level ψ'.

By the **affine coset**, we mean $\mathcal{C}^{\psi}(\mathfrak{g}, f) := \mathrm{Com}(V^{\psi'}(\mathfrak{a}), \mathcal{W}^{\psi}(\mathfrak{g}, f))$.

Sometimes we also take invariants under some group of **outer automorphisms**.

Trialities are isomorphisms between three different affine cosets

$$\mathcal{C}^{\psi}(\mathfrak{g}, f) \cong \mathcal{C}^{\psi'}(\mathfrak{g}', f') \cong \mathcal{C}^{\psi''}(\mathfrak{g}'', f'').$$

These unify and generalize both Feigin-Frenkel duality and the coset realization of principal $\mathcal{W}$-algebras.

4. What are trialities of $\mathcal{W}$-algebras?

Let $f \in \mathfrak{g}$ be a nilpotent, and complete f to a copy $\{f, h, e\}$ of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ denote the centralizer of this $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Then $\mathcal{W}^{\psi}(\mathfrak{g}, f)$ has affine subVOA $V^{\psi'}(\mathfrak{a})$, for some level ψ'.

By the **affine coset**, we mean $\mathcal{C}^{\psi}(\mathfrak{g}, f) := \mathrm{Com}(V^{\psi'}(\mathfrak{a}), \mathcal{W}^{\psi}(\mathfrak{g}, f))$.

Sometimes we also take invariants under some group of **outer automorphisms**.

Trialities are isomorphisms between three different affine cosets

$$\mathcal{C}^{\psi}(\mathfrak{g}, f) \cong \mathcal{C}^{\psi'}(\mathfrak{g}', f') \cong \mathcal{C}^{\psi''}(\mathfrak{g}'', f'').$$

These unify and generalize both Feigin-Frenkel duality and the coset realization of principal $\mathcal{W}$-algebras.

4. What are trialities of $\mathcal{W}$-algebras?

Let $f \in \mathfrak{g}$ be a nilpotent, and complete f to a copy $\{f, h, e\}$ of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ denote the centralizer of this $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Then $\mathcal{W}^{\psi}(\mathfrak{g}, f)$ has affine subVOA $V^{\psi'}(\mathfrak{a})$, for some level ψ'.

By the **affine coset**, we mean $\mathcal{C}^{\psi}(\mathfrak{g}, f) := \text{Com}(V^{\psi'}(\mathfrak{a}), \mathcal{W}^{\psi}(\mathfrak{g}, f))$.

Sometimes we also take invariants under some group of **outer automorphisms**.

Trialities are isomorphisms between three different affine cosets

$$\mathcal{C}^{\psi}(\mathfrak{g}, f) \cong \mathcal{C}^{\psi'}(\mathfrak{g}', f') \cong \mathcal{C}^{\psi''}(\mathfrak{g}'', f'').$$

These unify and generalize both Feigin-Frenkel duality and the coset realization of principal $\mathcal{W}$-algebras.

4. What are trialities of $\mathcal{W}$-algebras?

Let $f \in \mathfrak{g}$ be a nilpotent, and complete f to a copy $\{f, h, e\}$ of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ denote the centralizer of this $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Then $\mathcal{W}^{\psi}(\mathfrak{g}, f)$ has affine subVOA $V^{\psi'}(\mathfrak{a})$, for some level ψ'.

By the **affine coset**, we mean $\mathcal{C}^{\psi}(\mathfrak{g}, f) := \text{Com}(V^{\psi'}(\mathfrak{a}), \mathcal{W}^{\psi}(\mathfrak{g}, f))$.

Sometimes we also take invariants under some group of **outer automorphisms**.

Trialities are isomorphisms between three different affine cosets

$$\mathcal{C}^{\psi}(\mathfrak{g}, f) \cong \mathcal{C}^{\psi'}(\mathfrak{g}', f') \cong \mathcal{C}^{\psi''}(\mathfrak{g}'', f'').$$

These unify and generalize both Feigin-Frenkel duality and the coset realization of principal $\mathcal{W}$-algebras.

5. Hook-type $\mathcal{W}$-algebras in type A

Recall: For $n \geq 1$, write

$$\mathfrak{sl}_{n+m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \left(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\right) \oplus \left((\mathbb{C}^n)^* \otimes \mathbb{C}^m\right).$$

Let $f_n \in \mathfrak{sl}_{n+m}$ be the nilpotent which is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

f_n corresponds to the **hook-type partition** $n + 1 + \cdots + 1$.

Then $\psi = k + n + m$, and we define

$$\mathcal{W}^\psi(n, m) := \mathcal{W}^\psi(\mathfrak{sl}_{n+m}, f_n),$$

which has level $k = \psi - n - m$.

5. Hook-type $\mathcal{W}$-algebras in type A

Recall: For $n \geq 1$, write

$$\mathfrak{sl}_{n+m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \left(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\right) \oplus \left((\mathbb{C}^n)^* \otimes \mathbb{C}^m\right).$$

Let $f_n \in \mathfrak{sl}_{n+m}$ be the nilpotent which is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

f_n corresponds to the **hook-type partition** $n + 1 + \cdots + 1$.

Then $\psi = k + n + m$, and we define

$$\mathcal{W}^\psi(n, m) := \mathcal{W}^\psi(\mathfrak{sl}_{n+m}, f_n),$$

which has level $k = \psi - n - m$.

5. Hook-type $\mathcal{W}$-algebras in type A

Recall: For $n \geq 1$, write

$$\mathfrak{sl}_{n+m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \left(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\right) \oplus \left((\mathbb{C}^n)^* \otimes \mathbb{C}^m\right).$$

Let $f_n \in \mathfrak{sl}_{n+m}$ be the nilpotent which is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

f_n corresponds to the **hook-type partition** $n + 1 + \cdots + 1$.

Then $\psi = k + n + m$, and we define

$$\mathcal{W}^{\psi}(n, m) := \mathcal{W}^{\psi}(\mathfrak{sl}_{n+m}, f_n),$$

which has level $k = \psi - n - m$.

5. Hook-type $\mathcal{W}$-algebras in type A

Recall: For $n \geq 1$, write

$$\mathfrak{sl}_{n+m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \left(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\right) \oplus \left((\mathbb{C}^n)^* \otimes \mathbb{C}^m\right).$$

Let $f_n \in \mathfrak{sl}_{n+m}$ be the nilpotent which is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

f_n corresponds to the **hook-type partition** $n + 1 + \cdots + 1$.

Then $\psi = k + n + m$, and we define

$$\mathcal{W}^{\psi}(n, m) := \mathcal{W}^{\psi}(\mathfrak{sl}_{n+m}, f_n),$$

which has level $k = \psi - n - m$.

6. Features of $\mathcal{W}^{\psi}(n, m)$

For $m \geq 2$, $\mathcal{W}^{\psi}(n, m)$ has affine subalgebra

$$V^{\psi-m-1}(\mathfrak{gl}_m) = \mathcal{H} \otimes V^{\psi-m-1}(\mathfrak{sl}_m).$$

Additional **even** generators are in weights $2, 3, \dots, n$ together with $2m$ **even** fields in weight $\frac{n+1}{2}$ which transform under $\mathfrak{gl}_m$ as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$.

We define the case $\mathcal{W}^{\psi}(0, m)$ separately as follows.

1. For $m \geq 2$,

$$\mathcal{W}^{\psi}(0, m) = V^{\psi-m}(\mathfrak{sl}_m) \otimes \mathcal{S}(m),$$

   where $\mathcal{S}(m)$ is the rank m $\beta\gamma$-system.
2. $\mathcal{W}^{\psi}(0, 1) = \mathcal{S}(1)$.
3. $\mathcal{W}^{\psi}(0, 0) \cong \mathbb{C}$.

6. Features of $\mathcal{W}^{\psi}(n, m)$

For $m \geq 2$, $\mathcal{W}^{\psi}(n, m)$ has affine subalgebra

$$V^{\psi-m-1}(\mathfrak{gl}_m) = \mathcal{H} \otimes V^{\psi-m-1}(\mathfrak{sl}_m).$$

Additional **even** generators are in weights $2, 3, \dots, n$ together with $2m$ **even** fields in weight $\frac{n+1}{2}$ which transform under $\mathfrak{gl}_m$ as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$.

We define the case $\mathcal{W}^{\psi}(0, m)$ separately as follows.

1. For $m \geq 2$,

$$\mathcal{W}^{\psi}(0, m) = V^{\psi-m}(\mathfrak{sl}_m) \otimes \mathcal{S}(m),$$

   where $\mathcal{S}(m)$ is the rank m $\beta\gamma$-system.
2. $\mathcal{W}^{\psi}(0, 1) = \mathcal{S}(1)$.
3. $\mathcal{W}^{\psi}(0, 0) \cong \mathbb{C}$.

6. Features of $\mathcal{W}^{\psi}(n, m)$

For $m \geq 2$, $\mathcal{W}^{\psi}(n, m)$ has affine subalgebra

$$V^{\psi-m-1}(\mathfrak{gl}_m) = \mathcal{H} \otimes V^{\psi-m-1}(\mathfrak{sl}_m).$$

Additional **even** generators are in weights $2, 3, \dots, n$ together with $2m$ **even** fields in weight $\frac{n+1}{2}$ which transform under $\mathfrak{gl}_m$ as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$.

We define the case $\mathcal{W}^{\psi}(0, m)$ separately as follows.

1. For $m \geq 2$,

$$\mathcal{W}^{\psi}(0, m) = V^{\psi-m}(\mathfrak{sl}_m) \otimes \mathcal{S}(m),$$

   where $\mathcal{S}(m)$ is the rank m $\beta\gamma$-system.
2. $\mathcal{W}^{\psi}(0, 1) = \mathcal{S}(1)$.
3. $\mathcal{W}^{\psi}(0, 0) \cong \mathbb{C}$.

7. Hook-type $\mathcal{W}$-superalgebras of type A

For $n + m \geq 2$ and $n \neq m$, write

$$\mathfrak{sl}_{n|m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \Big(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\Big) \oplus \Big((\mathbb{C}^n)^* \otimes \mathbb{C}^m\Big).$$

Nilpotent $f_n \in \mathfrak{sl}_n$ is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

Define shifted level $\psi = k + n - m$, and let

$$\mathcal{V}^{\psi}(n, m) = \mathcal{W}^{\psi}(\mathfrak{sl}_{n|m}, f_n),$$

which has level $k = \psi - n + m$.

Case $n = m \geq 2$ slightly different: $\mathcal{V}^{\psi}(n, n) = \mathcal{W}^{\psi}(\mathfrak{psl}_{n|n}, f_n)$.

7. Hook-type $\mathcal{W}$-superalgebras of type A

For $n + m \geq 2$ and $n \neq m$, write

$$\mathfrak{sl}_{n|m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \left(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\right) \oplus \left((\mathbb{C}^n)^* \otimes \mathbb{C}^m\right).$$

Nilpotent $f_n \in \mathfrak{sl}_n$ is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

Define shifted level $\psi = k + n - m$, and let

$$\mathcal{V}^{\psi}(n, m) = \mathcal{W}^{\psi}(\mathfrak{sl}_{n|m}, f_n),$$

which has level $k = \psi - n + m$.

Case $n = m \geq 2$ slightly different: $\mathcal{V}^{\psi}(n, n) = \mathcal{W}^{\psi}(\mathfrak{psl}_{n|n}, f_n)$.

7. Hook-type $\mathcal{W}$-superalgebras of type A

For $n + m \geq 2$ and $n \neq m$, write

$$\mathfrak{sl}_{n|m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \Big(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\Big) \oplus \Big((\mathbb{C}^n)^* \otimes \mathbb{C}^m\Big).$$

Nilpotent $f_n \in \mathfrak{sl}_n$ is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

Define shifted level $\psi = k + n - m$, and let

$$\mathcal{V}^{\psi}(n, m) = \mathcal{W}^{\psi}(\mathfrak{sl}_{n|m}, f_n),$$

which has level $k = \psi - n + m$.

Case $n = m \geq 2$ slightly different: $\mathcal{V}^{\psi}(n, n) = \mathcal{W}^{\psi}(\mathfrak{psl}_{n|n}, f_n)$.

7. Hook-type $\mathcal{W}$-superalgebras of type A

For $n+m \geq 2$ and $n \neq m$, write

$$\mathfrak{sl}_{n|m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \left(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\right) \oplus \left((\mathbb{C}^n)^* \otimes \mathbb{C}^m\right).$$

Nilpotent $f_n \in \mathfrak{sl}_n$ is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

Define shifted level $\psi = k + n - m$, and let

$$\mathcal{V}^{\psi}(n,m) = \mathcal{W}^{\psi}(\mathfrak{sl}_{n|m}, f_n),$$

which has level $k = \psi - n + m$.

Case $n = m \geq 2$ slightly different: $\mathcal{V}^{\psi}(n,n) = \mathcal{W}^{\psi}(\mathfrak{psl}_{n|n}, f_n)$.

8. Hook-type $\mathcal{W}$-superalgebras of type A

For $m \geq 2$, $\mathcal{V}^{\psi}(n, m)$ has affine subalgebra

$$V^{-\psi-m+1}(\mathfrak{gl}_m), \qquad m \neq n,$$
$$V^{-\psi-n+1}(\mathfrak{sl}_n), \qquad m = n.$$

Additional **even** generators in weights $2, 3, \dots, n$, together with $2m$ **odd** fields in weight $\frac{n+1}{2}$ transforming under $\mathfrak{gl}_m$ as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$.

We define the cases $\mathcal{V}^{\psi}(0, m)$ and $\mathcal{V}^{\psi}(1, 1)$ separately as follows.

1. For $m \geq 2$,

$$\mathcal{V}^{\psi}(0, m) = V^{-\psi-m}(\mathfrak{sl}_m) \otimes \mathcal{F}(2m),$$

   where $\mathcal{F}(2m)$ is the rank $2m$ free fermion algebra.
2. $\mathcal{V}^{\psi}(1, 1) = \mathcal{A}(1)$, rank one symplectic fermion algebra.
3. $\mathcal{V}^{\psi}(0, 1) = \mathcal{F}(2)$.
4. $\mathcal{V}^{\psi}(0, 0) \cong \mathcal{V}^{\psi}(1, 0) \cong \mathbb{C}$.

8. Hook-type $\mathcal{W}$-superalgebras of type A

For $m \geq 2$, $\mathcal{V}^{\psi}(n, m)$ has affine subalgebra

$$V^{-\psi-m+1}(\mathfrak{gl}_m), \qquad m \neq n,$$
$$V^{-\psi-n+1}(\mathfrak{sl}_n), \qquad m = n.$$

Additional **even** generators in weights $2, 3, \dots, n$, together with $2m$ **odd** fields in weight $\frac{n+1}{2}$ transforming under $\mathfrak{gl}_m$ as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$.

We define the cases $\mathcal{V}^{\psi}(0, m)$ and $\mathcal{V}^{\psi}(1, 1)$ separately as follows.

1. For $m \geq 2$,

$$\mathcal{V}^{\psi}(0, m) = V^{-\psi-m}(\mathfrak{sl}_m) \otimes \mathcal{F}(2m),$$

   where $\mathcal{F}(2m)$ is the rank $2m$ free fermion algebra.
2. $\mathcal{V}^{\psi}(1, 1) = \mathcal{A}(1)$, rank one symplectic fermion algebra.
3. $\mathcal{V}^{\psi}(0, 1) = \mathcal{F}(2)$.
4. $\mathcal{V}^{\psi}(0, 0) \cong \mathcal{V}^{\psi}(1, 0) \cong \mathbb{C}$.

8. Hook-type $\mathcal{W}$-superalgebras of type A

For $m \geq 2$, $\mathcal{V}^{\psi}(n,m)$ has affine subalgebra

$$\begin{aligned} &V^{-\psi-m+1}(\mathfrak{gl}_m), \qquad m \neq n, \\ &V^{-\psi-n+1}(\mathfrak{sl}_n), \qquad m = n. \end{aligned}$$

Additional **even** generators in weights $2, 3, \dots, n$, together with $2m$ **odd** fields in weight $\frac{n+1}{2}$ transforming under $\mathfrak{gl}_m$ as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$.

We define the cases $\mathcal{V}^{\psi}(0,m)$ and $\mathcal{V}^{\psi}(1,1)$ separately as follows.

1. For $m \geq 2$,

   $$\mathcal{V}^{\psi}(0,m) = V^{-\psi-m}(\mathfrak{sl}_m) \otimes \mathcal{F}(2m),$$

   where $\mathcal{F}(2m)$ is the rank $2m$ free fermion algebra.
2. $\mathcal{V}^{\psi}(1,1) = \mathcal{A}(1)$, rank one symplectic fermion algebra.
3. $\mathcal{V}^{\psi}(0,1) = \mathcal{F}(2)$.
4. $\mathcal{V}^{\psi}(0,0) \cong \mathcal{V}^{\psi}(1,0) \cong \mathbb{C}$.

9. Trialities in type A

Consider the affine cosets

$$\mathcal{C}^{\psi}(n,m) = \mathrm{Com}(V^{\psi-m-1}(\mathfrak{gl}_m), \mathcal{W}^{\psi}(n,m)),$$

$$\mathcal{D}^{\psi}(n,m) = \mathrm{Com}(V^{-\psi-m+1}(\mathfrak{gl}_m), \mathcal{V}^{\psi}(n,m)), \qquad n \neq m,$$

$$\mathcal{D}^{\psi}(n,n) = \mathrm{Com}(V^{-\psi-n+1}(\mathfrak{sl}_n), \mathcal{V}^{\psi}(n,n))^{U(1)}.$$

Thm: (Creutzig-L., 2020) Let $n \geq m$ be non-negative integers. We have isomorphisms of 1-parameter VOAs

$$\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m) \cong \mathcal{D}^{\psi'}(m,n), \qquad \frac{1}{\psi} + \frac{1}{\psi'} = 1.$$

Originally conjectured in physics by Gaiotto and Rapčák (2017).

9. Trialities in type A

Consider the affine cosets

$$\mathcal{C}^{\psi}(n,m) = \mathrm{Com}(V^{\psi-m-1}(\mathfrak{gl}_m), \mathcal{W}^{\psi}(n,m)),$$

$$\mathcal{D}^{\psi}(n,m) = \mathrm{Com}(V^{-\psi-m+1}(\mathfrak{gl}_m), \mathcal{V}^{\psi}(n,m)), \qquad n \neq m,$$

$$\mathcal{D}^{\psi}(n,n) = \mathrm{Com}(V^{-\psi-n+1}(\mathfrak{sl}_n), \mathcal{V}^{\psi}(n,n))^{U(1)}.$$

Thm: (Creutzig-L., 2020) Let $n \geq m$ be non-negative integers. We have isomorphisms of 1-parameter VOAs

$$\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m) \cong \mathcal{D}^{\psi'}(m,n), \qquad \frac{1}{\psi} + \frac{1}{\psi'} = 1.$$

Originally conjectured in physics by Gaiotto and Rapčák (2017).

9. Trialities in type A

Consider the affine cosets

$$\mathcal{C}^{\psi}(n,m) = \mathrm{Com}(V^{\psi-m-1}(\mathfrak{gl}_m), \mathcal{W}^{\psi}(n,m)),$$

$$\mathcal{D}^{\psi}(n,m) = \mathrm{Com}(V^{-\psi-m+1}(\mathfrak{gl}_m), \mathcal{V}^{\psi}(n,m)), \qquad n \neq m,$$

$$\mathcal{D}^{\psi}(n,n) = \mathrm{Com}(V^{-\psi-n+1}(\mathfrak{sl}_n), \mathcal{V}^{\psi}(n,n))^{U(1)}.$$

Thm: (Creutzig-L., 2020) Let $n \geq m$ be non-negative integers. We have isomorphisms of 1-parameter VOAs

$$\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m) \cong \mathcal{D}^{\psi'}(m,n), \qquad \frac{1}{\psi} + \frac{1}{\psi'} = 1.$$

Originally conjectured in physics by Gaiotto and Rapčák (2017).

10. Some special cases

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{C}^{\psi^{-1}}(n,0)$ recovers **Feigin-Frenkel duality** in type A.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m)$ are of **Feigin-Frenkel type**.

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{D}^{\psi'}(0,n)$ recovers the coset realization of $\mathcal{W}^{\psi}(\mathfrak{sl}_n)$.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{D}^{\psi'}(m,n)$ are of **coset realization type**.

One more example:

$$\mathcal{D}^{\psi}(n,1) \cong \mathcal{C}^{\psi^{-1}}(n-1,1) \cong \mathcal{D}^{\psi'}(1,n),$$

recovers a duality conjectured by Feigin and Semikhatov and proved in a different way by Creutzig, Genra, and Nakatsuka.

10. Some special cases

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{C}^{\psi^{-1}}(n,0)$ recovers **Feigin-Frenkel duality** in type A.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m)$ are of **Feigin-Frenkel type**.

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{D}^{\psi'}(0,n)$ recovers the coset realization of $\mathcal{W}^{\psi}(\mathfrak{sl}_n)$.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{D}^{\psi'}(m,n)$ are of **coset realization type**.

One more example:

$$\mathcal{D}^{\psi}(n,1) \cong \mathcal{C}^{\psi^{-1}}(n-1,1) \cong \mathcal{D}^{\psi'}(1,n),$$

recovers a duality conjectured by Feigin and Semikhatov and proved in a different way by Creutzig, Genra, and Nakatsuka.

10. Some special cases

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{C}^{\psi^{-1}}(n,0)$ recovers **Feigin-Frenkel duality** in type A.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m)$ are of **Feigin-Frenkel type**.

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{D}^{\psi'}(0,n)$ recovers the coset realization of $\mathcal{W}^{\psi}(\mathfrak{sl}_n)$.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{D}^{\psi'}(m,n)$ are of **coset realization type**.

One more example:

$$\mathcal{D}^{\psi}(n,1) \cong \mathcal{C}^{\psi^{-1}}(n-1,1) \cong \mathcal{D}^{\psi'}(1,n),$$

recovers a duality conjectured by Feigin and Semikhatov and proved in a different way by Creutzig, Genra, and Nakatsuka.

10. Some special cases

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{C}^{\psi^{-1}}(n,0)$ recovers **Feigin-Frenkel duality** in type A.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m)$ are of **Feigin-Frenkel type**.

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{D}^{\psi'}(0,n)$ recovers the coset realization of $\mathcal{W}^{\psi}(\mathfrak{sl}_n)$.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{D}^{\psi'}(m,n)$ are of **coset realization type**.

One more example:

$$\mathcal{D}^{\psi}(n,1) \cong \mathcal{C}^{\psi^{-1}}(n-1,1) \cong \mathcal{D}^{\psi'}(1,n),$$

recovers a duality conjectured by Feigin and Semikhatov and proved in a different way by Creutzig, Genra, and Nakatsuka.

10. Some special cases

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{C}^{\psi^{-1}}(n,0)$ recovers **Feigin-Frenkel duality** in type A.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m)$ are of **Feigin-Frenkel type**.

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{D}^{\psi'}(0,n)$ recovers the coset realization of $\mathcal{W}^{\psi}(\mathfrak{sl}_n)$.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{D}^{\psi'}(m,n)$ are of **coset realization type**.

One more example:

$$\mathcal{D}^{\psi}(n,1) \cong \mathcal{C}^{\psi^{-1}}(n-1,1) \cong \mathcal{D}^{\psi'}(1,n),$$

recovers a duality conjectured by Feigin and Semikhatov and proved in a different way by Creutzig, Genra, and Nakatsuka.

11. Sketch of proof, cont'd

Step 1: In the $\psi \to \infty$ limit, both $\mathcal{C}^{\psi}(n,m)$ and $\mathcal{D}^{\psi}(n,m)$ become GL_m-orbifolds of certain **free field algebras**.

Using **classical invariant theory**, it is shown that

1. $\mathcal{C}^{\psi}(n,m)$ has generating type $\mathcal{W}(2,3,\dots,(m+1)(m+n+1)-1)$,
2. $\mathcal{D}^{\psi}(n,m)$ has generating type $\mathcal{W}(2,3,\dots,(m+1)(n+1)-1)$.

Step 2: Universal two-parameter $\mathcal{W}_{\infty}$-algebra $\mathcal{W}(c,\lambda)$ serves is a **classifying object** for VOAs of type $\mathcal{W}(2,3,\dots,N)$ for some N.

$\mathcal{W}(c,\lambda)$ is freely generated of type $\mathcal{W}(2,3,\dots)$, and is defined over the polynomial ring $\mathbb{C}[c,\lambda] \cong \mathcal{W}(c,\lambda)[0]$.

11. Sketch of proof, cont'd

Step 1: In the $\psi \to \infty$ limit, both $\mathcal{C}^\psi(n, m)$ and $\mathcal{D}^\psi(n, m)$ become GL_m-orbifolds of certain **free field algebras**.

Using **classical invariant theory**, it is shown that

1. $\mathcal{C}^\psi(n, m)$ has generating type $\mathcal{W}(2, 3, \dots, (m+1)(m+n+1)-1)$,
2. $\mathcal{D}^\psi(n, m)$ has generating type $\mathcal{W}(2, 3, \dots, (m+1)(n+1)-1)$.

Step 2: Universal two-parameter $\mathcal{W}_\infty$-algebra $\mathcal{W}(c, \lambda)$ serves is a **classifying object** for VOAs of type $\mathcal{W}(2, 3, \dots, N)$ for some N.

$\mathcal{W}(c, \lambda)$ is freely generated of type $\mathcal{W}(2, 3, \dots)$, and is defined over the polynomial ring $\mathbb{C}[c, \lambda] \cong \mathcal{W}(c, \lambda)[0]$.

11. Sketch of proof, cont'd

Step 1: In the $\psi \to \infty$ limit, both $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ become GL_m-orbifolds of certain **free field algebras**.

Using **classical invariant theory**, it is shown that

1. $\mathcal{C}^{\psi}(n, m)$ has generating type $\mathcal{W}(2, 3, \dots, (m+1)(m+n+1)-1)$,

2. $\mathcal{D}^{\psi}(n, m)$ has generating type $\mathcal{W}(2, 3, \dots, (m+1)(n+1)-1)$.

Step 2: Universal two-parameter $\mathcal{W}_{\infty}$-algebra $\mathcal{W}(c, \lambda)$ serves is a **classifying object** for VOAs of type $\mathcal{W}(2, 3, \dots, N)$ for some N.

$\mathcal{W}(c, \lambda)$ is freely generated of type $\mathcal{W}(2, 3, \dots)$, and is defined over the polynomial ring $\mathbb{C}[c, \lambda] \cong \mathcal{W}(c, \lambda)[0]$.

11. Sketch of proof, cont'd

Step 1: In the $\psi \to \infty$ limit, both $\mathcal{C}^{\psi}(n,m)$ and $\mathcal{D}^{\psi}(n,m)$ become GL_m-orbifolds of certain **free field algebras**.

Using **classical invariant theory**, it is shown that

1. $\mathcal{C}^{\psi}(n,m)$ has generating type $\mathcal{W}(2,3,\dots,(m+1)(m+n+1)-1)$,

2. $\mathcal{D}^{\psi}(n,m)$ has generating type $\mathcal{W}(2,3,\dots,(m+1)(n+1)-1)$.

Step 2: Universal two-parameter $\mathcal{W}_{\infty}$-algebra $\mathcal{W}(c,\lambda)$ serves is a **classifying object** for VOAs of type $\mathcal{W}(2,3,\dots,N)$ for some N.

$\mathcal{W}(c,\lambda)$ is freely generated of type $\mathcal{W}(2,3,\dots)$, and is defined over the polynomial ring $\mathbb{C}[c,\lambda] \cong \mathcal{W}(c,\lambda)[0]$.

12. One-parameter quotients of $\mathcal{W}(c,\lambda)$

Let $I \subseteq \mathbb{C}[c,\lambda]$ be a prime ideal.

$I \cdot \mathcal{W}(c,\lambda)$ the VOA ideal generated by I.

Quotient

$$\mathcal{W}^I(c,\lambda) = \mathcal{W}(c,\lambda)/(I \cdot \mathcal{W}(c,\lambda))$$

is a VOA over $R = \mathbb{C}[c,\lambda]/I$.

$\mathcal{W}^I(c,\lambda)$ is simple for a generic ideal I, but for certain special ideals I, $\mathcal{W}^I(c,\lambda)$ is not simple.

Let $\mathcal{W}_I(c,\lambda)$ be simple graded quotient of $\mathcal{W}^I(c,\lambda)$.

Thm: (L., 2017) All simple, one-parameter VOAs of type $\mathcal{W}(2,3,\dots,N)$ satisfying mild hypotheses, are of this form.

Variety $V(I) \subseteq \mathbb{C}^2$ is called the **truncation curve**.

12. One-parameter quotients of $\mathcal{W}(c,\lambda)$

Let $I \subseteq \mathbb{C}[c,\lambda]$ be a prime ideal.

$I \cdot \mathcal{W}(c,\lambda)$ the VOA ideal generated by I.

Quotient

$$\mathcal{W}^I(c,\lambda) = \mathcal{W}(c,\lambda)/(I \cdot \mathcal{W}(c,\lambda))$$

is a VOA over $R = \mathbb{C}[c,\lambda]/I$.

$\mathcal{W}^I(c,\lambda)$ is simple for a generic ideal I, but for certain special ideals I, $\mathcal{W}^I(c,\lambda)$ is not simple.

Let $\mathcal{W}_I(c,\lambda)$ be simple graded quotient of $\mathcal{W}^I(c,\lambda)$.

Thm: (L., 2017) All simple, one-parameter VOAs of type $\mathcal{W}(2,3,\dots,N)$ satisfying mild hypotheses, are of this form.

Variety $V(I) \subseteq \mathbb{C}^2$ is called the **truncation curve**.

12. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Let $I \subseteq \mathbb{C}[c, \lambda]$ be a prime ideal.

$I \cdot \mathcal{W}(c, \lambda)$ the VOA ideal generated by I.

Quotient

$$\mathcal{W}^I(c, \lambda) = \mathcal{W}(c, \lambda)/(I \cdot \mathcal{W}(c, \lambda))$$

is a VOA over $R = \mathbb{C}[c, \lambda]/I$.

$\mathcal{W}^I(c, \lambda)$ is simple for a generic ideal I, but for certain special ideals I, $\mathcal{W}^I(c, \lambda)$ is not simple.

Let $\mathcal{W}_I(c, \lambda)$ be simple graded quotient of $\mathcal{W}^I(c, \lambda)$.

Thm: (L., 2017) All simple, one-parameter VOAs of type $\mathcal{W}(2, 3, \dots, N)$ satisfying mild hypotheses, are of this form.

Variety $V(I) \subseteq \mathbb{C}^2$ is called the **truncation curve**.

12. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Let $I \subseteq \mathbb{C}[c, \lambda]$ be a prime ideal.

$I \cdot \mathcal{W}(c, \lambda)$ the VOA ideal generated by I.

Quotient

$$\mathcal{W}^I(c, \lambda) = \mathcal{W}(c, \lambda)/(I \cdot \mathcal{W}(c, \lambda))$$

is a VOA over $R = \mathbb{C}[c, \lambda]/I$.

$\mathcal{W}^I(c, \lambda)$ is simple for a generic ideal I, but for certain special ideals I, $\mathcal{W}^I(c, \lambda)$ is not simple.

Let $\mathcal{W}_I(c, \lambda)$ be simple graded quotient of $\mathcal{W}^I(c, \lambda)$.

Thm: (L., 2017) All simple, one-parameter VOAs of type $\mathcal{W}(2, 3, \dots, N)$ satisfying mild hypotheses, are of this form.

Variety $V(I) \subseteq \mathbb{C}^2$ is called the **truncation curve**.

12. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Let $I \subseteq \mathbb{C}[c, \lambda]$ be a prime ideal.

$I \cdot \mathcal{W}(c, \lambda)$ the VOA ideal generated by I.

Quotient

$$\mathcal{W}^I(c, \lambda) = \mathcal{W}(c, \lambda)/(I \cdot \mathcal{W}(c, \lambda))$$

is a VOA over $R = \mathbb{C}[c, \lambda]/I$.

$\mathcal{W}^I(c, \lambda)$ is simple for a generic ideal I, but for certain special ideals I, $\mathcal{W}^I(c, \lambda)$ is not simple.

Let $\mathcal{W}_I(c, \lambda)$ be simple graded quotient of $\mathcal{W}^I(c, \lambda)$.

Thm: (L., 2017) All simple, one-parameter VOAs of type $\mathcal{W}(2, 3, \dots, N)$ satisfying mild hypotheses, are of this form.

Variety $V(I) \subseteq \mathbb{C}^2$ is called the **truncation curve**.

12. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Let $I \subseteq \mathbb{C}[c, \lambda]$ be a prime ideal.

$I \cdot \mathcal{W}(c, \lambda)$ the VOA ideal generated by I.

Quotient

$$\mathcal{W}^I(c, \lambda) = \mathcal{W}(c, \lambda)/(I \cdot \mathcal{W}(c, \lambda))$$

is a VOA over $R = \mathbb{C}[c, \lambda]/I$.

$\mathcal{W}^I(c, \lambda)$ is simple for a generic ideal I, but for certain special ideals I, $\mathcal{W}^I(c, \lambda)$ is not simple.

Let $\mathcal{W}_I(c, \lambda)$ be simple graded quotient of $\mathcal{W}^I(c, \lambda)$.

Thm: (L., 2017) All simple, one-parameter VOAs of type $\mathcal{W}(2, 3, \dots, N)$ satisfying mild hypotheses, are of this form.

Variety $V(I) \subseteq \mathbb{C}^2$ is called the **truncation curve**.

12. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Let $I \subseteq \mathbb{C}[c, \lambda]$ be a prime ideal.

$I \cdot \mathcal{W}(c, \lambda)$ the VOA ideal generated by I.

Quotient

$$\mathcal{W}^I(c, \lambda) = \mathcal{W}(c, \lambda)/(I \cdot \mathcal{W}(c, \lambda))$$

is a VOA over $R = \mathbb{C}[c, \lambda]/I$.

$\mathcal{W}^I(c, \lambda)$ is simple for a generic ideal I, but for certain special ideals I, $\mathcal{W}^I(c, \lambda)$ is not simple.

Let $\mathcal{W}_I(c, \lambda)$ be simple graded quotient of $\mathcal{W}^I(c, \lambda)$.

Thm: (L., 2017) All simple, one-parameter VOAs of type $\mathcal{W}(2, 3, \dots, N)$ satisfying mild hypotheses, are of this form.

Variety $V(I) \subseteq \mathbb{C}^2$ is called the **truncation curve**.

13. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Then $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ are of the form $\mathcal{W}_I(c, \lambda)$ for some I.

Step 3: Explicit truncation curves for $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$.

$\mathcal{W}^{\psi}(n, m)$ is an extension $V^{\psi-m+1}(\mathfrak{gl}_m) \otimes \mathcal{W}_I(c, \lambda)$ for some I

Extension is generated by $2m$ fields in weight $\frac{n+1}{2}$ which transform as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$ under $\mathfrak{gl}_m$.

Existence of such an extension uniquely and explicitly determines I.

Same method works for $\mathcal{V}^{\psi}(n, m)$.

Triality theorem follows from explicit form of I.

13. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Then $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ are of the form $\mathcal{W}_I(c, \lambda)$ for some I.

Step 3: Explicit truncation curves for $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$.

$\mathcal{W}^{\psi}(n, m)$ is an extension $V^{\psi-m+1}(\mathfrak{gl}_m) \otimes \mathcal{W}_I(c, \lambda)$ for some I

Extension is generated by $2m$ fields in weight $\frac{n+1}{2}$ which transform as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$ under $\mathfrak{gl}_m$.

Existence of such an extension uniquely and explicitly determines I.

Same method works for $\mathcal{V}^{\psi}(n, m)$.

Triality theorem follows from explicit form of I.

13. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Then $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ are of the form $\mathcal{W}_I(c, \lambda)$ for some I.

Step 3: Explicit truncation curves for $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$.

$\mathcal{W}^{\psi}(n, m)$ is an extension $V^{\psi-m+1}(\mathfrak{gl}_m) \otimes \mathcal{W}_I(c, \lambda)$ for some I

Extension is generated by $2m$ fields in weight $\frac{n+1}{2}$ which transform as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$ under $\mathfrak{gl}_m$.

Existence of such an extension uniquely and explicitly determines I.

Same method works for $\mathcal{V}^{\psi}(n, m)$.

Triality theorem follows from explicit form of I.

13. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Then $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ are of the form $\mathcal{W}_I(c, \lambda)$ for some I.

Step 3: Explicit truncation curves for $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$.

$\mathcal{W}^{\psi}(n, m)$ is an extension $V^{\psi - m+1}(\mathfrak{gl}_m) \otimes \mathcal{W}_I(c, \lambda)$ for some I

Extension is generated by $2m$ fields in weight $\frac{n+1}{2}$ which transform as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$ under $\mathfrak{gl}_m$.

Existence of such an extension uniquely and explicitly determines I.

Same method works for $\mathcal{V}^{\psi}(n, m)$.

Triality theorem follows from explicit form of I.

13. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Then $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ are of the form $\mathcal{W}_I(c, \lambda)$ for some I.

Step 3: Explicit truncation curves for $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$.

$\mathcal{W}^{\psi}(n, m)$ is an extension $V^{\psi-m+1}(\mathfrak{gl}_m) \otimes \mathcal{W}_I(c, \lambda)$ for some I

Extension is generated by $2m$ fields in weight $\frac{n+1}{2}$ which transform as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$ under $\mathfrak{gl}_m$.

Existence of such an extension uniquely and explicitly determines I.

Same method works for $\mathcal{V}^{\psi}(n, m)$.

Triality theorem follows from explicit form of I.

13. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Then $\mathcal{C}^{\psi}(n,m)$ and $\mathcal{D}^{\psi}(n,m)$ are of the form $\mathcal{W}_I(c,\lambda)$ for some I.

Step 3: Explicit truncation curves for $\mathcal{C}^{\psi}(n,m)$ and $\mathcal{D}^{\psi}(n,m)$.

$\mathcal{W}^{\psi}(n,m)$ is an extension $V^{\psi-m+1}(\mathfrak{gl}_m) \otimes \mathcal{W}_I(c,\lambda)$ for some I

Extension is generated by $2m$ fields in weight $\frac{n+1}{2}$ which transform as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$ under $\mathfrak{gl}_m$.

Existence of such an extension uniquely and explicitly determines I.

Same method works for $\mathcal{V}^{\psi}(n,m)$.

Triality theorem follows from explicit form of I.

13. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Then $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ are of the form $\mathcal{W}_I(c, \lambda)$ for some I.

Step 3: Explicit truncation curves for $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$.

$\mathcal{W}^{\psi}(n, m)$ is an extension $V^{\psi - m + 1}(\mathfrak{gl}_m) \otimes \mathcal{W}_I(c, \lambda)$ for some I

Extension is generated by $2m$ fields in weight $\frac{n+1}{2}$ which transform as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$ under $\mathfrak{gl}_m$.

Existence of such an extension uniquely and explicitly determines I.

Same method works for $\mathcal{V}^{\psi}(n, m)$.

Triality theorem follows from explicit form of I.

14. Application: coincidences

Nontrivial pointwise isomorphisms among simple algebras $\mathcal{C}_\psi(n,m)$ correspond to **intersection points** of truncation curves.

Thm: (Creutzig, L. 2020) We have $\mathcal{C}_\psi(n,m) \cong \mathcal{W}_\phi(\mathfrak{sl}_s)$ where

1. $\psi = \dfrac{m+n+s}{n}, \qquad \phi = \dfrac{m+s}{m+n+s},$
2. $\psi = \dfrac{m+n}{n+s}, \qquad \phi = \dfrac{s-m}{s+n},$
3. $\psi = \dfrac{m+n-s}{n-s}, \qquad \phi = \dfrac{m+n-s}{n-s}.$

$C_\psi(1,1)$ is just **parafermion algebra** $N_k(\mathfrak{sl}_2)$ for $k=\psi-2$. (Dong, Lam, Yamada, Wang, and others).

Family (1) says for $k \in \mathbb{N}$,

$$N_k(\mathfrak{sl}_2) \cong \mathcal{W}_\phi(\mathfrak{sl}_s), \qquad \phi = \frac{1+s}{2+s}.$$

First proven by Arakawa, Lam, Yamada (2019).

14. Application: coincidences

Nontrivial pointwise isomorphisms among simple algebras $\mathcal{C}_\psi(n,m)$ correspond to **intersection points** of truncation curves.

Thm: (Creutzig, L. 2020) We have $\mathcal{C}_\psi(n,m) \cong \mathcal{W}_\phi(\mathfrak{sl}_s)$ where

1. $\psi = \dfrac{m+n+s}{n}, \qquad \phi = \dfrac{m+s}{m+n+s}$,
2. $\psi = \dfrac{m+n}{n+s}, \qquad \phi = \dfrac{s-m}{s+n}$,
3. $\psi = \dfrac{m+n-s}{n-s}, \qquad \phi = \dfrac{m+n-s}{n-s}$.

$C_\psi(1,1)$ is just **parafermion algebra** $N_k(\mathfrak{sl}_2)$ for $k = \psi - 2$. (Dong, Lam, Yamada, Wang, and others).

Family (1) says for $k \in \mathbb{N}$,

$$N_k(\mathfrak{sl}_2) \cong \mathcal{W}_\phi(\mathfrak{sl}_s), \qquad \phi = \frac{1+s}{2+s}.$$

First proven by Arakawa, Lam, Yamada (2019).

14. Application: coincidences

Nontrivial pointwise isomorphisms among simple algebras $\mathcal{C}_{\psi}(n,m)$ correspond to **intersection points** of truncation curves.

Thm: (Creutzig, L. 2020) We have $\mathcal{C}_{\psi}(n,m) \cong \mathcal{W}_{\phi}(\mathfrak{sl}_s)$ where

1. $\psi = \dfrac{m+n+s}{n}, \qquad \phi = \dfrac{m+s}{m+n+s},$
2. $\psi = \dfrac{m+n}{n+s}, \qquad \phi = \dfrac{s-m}{s+n},$
3. $\psi = \dfrac{m+n-s}{n-s}, \qquad \phi = \dfrac{m+n-s}{n-s}.$

$C_{\psi}(1,1)$ is just **parafermion algebra** $N_k(\mathfrak{sl}_2)$ for $k = \psi - 2$. (Dong, Lam, Yamada, Wang, and others).

Family (1) says for $k \in \mathbb{N}$,

$$N_k(\mathfrak{sl}_2) \cong \mathcal{W}_{\phi}(\mathfrak{sl}_s), \qquad \phi = \frac{1+s}{2+s}.$$

First proven by Arakawa, Lam, Yamada (2019).

14. Application: coincidences

Nontrivial pointwise isomorphisms among simple algebras $\mathcal{C}_\psi(n, m)$ correspond to **intersection points** of truncation curves.

Thm: (Creutzig, L. 2020) We have $\mathcal{C}_\psi(n, m) \cong \mathcal{W}_\phi(\mathfrak{sl}_s)$ where

1. $\psi = \dfrac{m+n+s}{n}, \qquad \phi = \dfrac{m+s}{m+n+s},$
2. $\psi = \dfrac{m+n}{n+s}, \qquad \phi = \dfrac{s-m}{s+n},$
3. $\psi = \dfrac{m+n-s}{n-s}, \qquad \phi = \dfrac{m+n-s}{n-s}.$

$C_\psi(1,1)$ is just **parafermion algebra** $N_k(\mathfrak{sl}_2)$ for $k = \psi - 2$. (Dong, Lam, Yamada, Wang, and others).

Family (1) says for $k \in \mathbb{N}$,

$$N_k(\mathfrak{sl}_2) \cong \mathcal{W}_\phi(\mathfrak{sl}_s), \qquad \phi = \frac{1+s}{2+s}.$$

First proven by Arakawa, Lam, Yamada (2019).

15. Analogue for orthosymplectic types

We consider 8 families Lie (super)algebras $\mathfrak{g}$ of type B, C, D or $\mathfrak{osp}_{s|2r}$, with following properties:

1. We have a decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \oplus \rho_{\mathfrak{a}} \otimes \rho_{\mathfrak{b}}$,
2. $\mathfrak{a}$ and $\mathfrak{b}$ are Lie sub(super)algebras of $\mathfrak{g}$,
3. $\mathfrak{b} = \mathfrak{so}_{2m+1}$ or $\mathfrak{sp}_{2m}$,
4. $\mathfrak{a} = \mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, or $\mathfrak{osp}_{1|2n}$.
5. $\rho_{\mathfrak{a}}$, $\rho_{\mathfrak{b}}$ transform as the standard representations of $\mathfrak{a}$, $\mathfrak{b}$, respectively.

Consider $\mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{b}})$, where $f_{\mathfrak{b}} \in \mathfrak{g}$ is the nilpotent which is **principal** in $\mathfrak{b}$ and **trivial** in $\mathfrak{a}$.

Affine cosets are quotients of universal 2-parameter algebra $\mathcal{W}^{\mathrm{ev}}(c, \lambda)$ of type $\mathcal{W}(2, 4, 6, \dots)$ constructed by Kanade-L. (2019).

Similar triality theorem is due to Creutzig, L. (2021), also conjectured by Gaiotto and Rapčák (2017).

15. Analogue for orthosymplectic types

We consider 8 families Lie (super)algebras $\mathfrak{g}$ of type B, C, D or $\mathfrak{osp}_{s|2r}$, with following properties:

1. We have a decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \oplus \rho_{\mathfrak{a}} \otimes \rho_{\mathfrak{b}}$,
2. $\mathfrak{a}$ and $\mathfrak{b}$ are Lie sub(super)algebras of $\mathfrak{g}$,
3. $\mathfrak{b} = \mathfrak{so}_{2m+1}$ or $\mathfrak{sp}_{2m}$,
4. $\mathfrak{a} = \mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, or $\mathfrak{osp}_{1|2n}$.
5. $\rho_{\mathfrak{a}}$, $\rho_{\mathfrak{b}}$ transform as the standard representations of $\mathfrak{a}$, $\mathfrak{b}$, respectively.

Consider $\mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{b}})$, where $f_{\mathfrak{b}} \in \mathfrak{g}$ is the nilpotent which is **principal** in $\mathfrak{b}$ and **trivial** in $\mathfrak{a}$.

Affine cosets are quotients of universal 2-parameter algebra $\mathcal{W}^{\mathrm{ev}}(c, \lambda)$ of type $\mathcal{W}(2, 4, 6, \dots)$ constructed by Kanade-L. (2019).

Similar triality theorem is due to Creutzig, L. (2021), also conjectured by Gaiotto and Rapčák (2017).

15. Analogue for orthosymplectic types

We consider 8 families Lie (super)algebras $\mathfrak{g}$ of type B, C, D or $\mathfrak{osp}_{s|2r}$, with following properties:

1. We have a decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \oplus \rho_{\mathfrak{a}} \otimes \rho_{\mathfrak{b}}$,
2. $\mathfrak{a}$ and $\mathfrak{b}$ are Lie sub(super)algebras of $\mathfrak{g}$,
3. $\mathfrak{b} = \mathfrak{so}_{2m+1}$ or $\mathfrak{sp}_{2m}$,
4. $\mathfrak{a} = \mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, or $\mathfrak{osp}_{1|2n}$.
5. $\rho_{\mathfrak{a}}$, $\rho_{\mathfrak{b}}$ transform as the standard representations of $\mathfrak{a}$, $\mathfrak{b}$, respectively.

Consider $\mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{b}})$, where $f_{\mathfrak{b}} \in \mathfrak{g}$ is the nilpotent which is **principal** in $\mathfrak{b}$ and **trivial** in $\mathfrak{a}$.

Affine cosets are quotients of universal 2-parameter algebra $\mathcal{W}^{\mathrm{ev}}(c, \lambda)$ of type $\mathcal{W}(2, 4, 6, \dots)$ constructed by Kanade-L. (2019).

Similar triality theorem is due to Creutzig, L. (2021), also conjectured by Gaiotto and Rapčák (2017).

15. Analogue for orthosymplectic types

We consider 8 families Lie (super)algebras $\mathfrak{g}$ of type B, C, D or $\mathfrak{osp}_{s|2r}$, with following properties:

1. We have a decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \oplus \rho_{\mathfrak{a}} \otimes \rho_{\mathfrak{b}}$,
2. $\mathfrak{a}$ and $\mathfrak{b}$ are Lie sub(super)algebras of $\mathfrak{g}$,
3. $\mathfrak{b} = \mathfrak{so}_{2m+1}$ or $\mathfrak{sp}_{2m}$,
4. $\mathfrak{a} = \mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, or $\mathfrak{osp}_{1|2n}$.
5. $\rho_{\mathfrak{a}}$, $\rho_{\mathfrak{b}}$ transform as the standard representations of $\mathfrak{a}$, $\mathfrak{b}$, respectively.

Consider $\mathcal{W}^{\psi}(\mathfrak{g}, f_{\mathfrak{b}})$, where $f_{\mathfrak{b}} \in \mathfrak{g}$ is the nilpotent which is **principal** in $\mathfrak{b}$ and **trivial** in $\mathfrak{a}$.

Affine cosets are quotients of universal 2-parameter algebra $\mathcal{W}^{\mathrm{ev}}(c, \lambda)$ of type $\mathcal{W}(2, 4, 6, \dots)$ constructed by Kanade-L. (2019).

Similar triality theorem is due to Creutzig, L. (2021), also conjectured by Gaiotto and Rapčák (2017).

16. How does this generalize?

Need families of VOAs that are quotients of a universal object.

Example: Rectangular $\mathcal{W}$-algebras in type A (Arakawa, Molev, 2017) and some variants.

Consider $\mathfrak{gl}_{nm}$ equipped with the nilpotent element f_{n^m} corresponding to the tableau with m blocks of height n.

$\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$ is called rectangular; it is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2})$.

Weight one fields generate $V^{n\psi - nm}(\mathfrak{gl}_m)$ and m^2 fields of weight d for $2 \leq d \leq n$ transform as adjoint $\mathfrak{gl}_m$-module.

Natural generalization of principal $\mathcal{W}$-algebra, which is case $m = 1$.

16. How does this generalize?

Need families of VOAs that are quotients of a universal object.

Example: Rectangular $\mathcal{W}$-algebras in type A (Arakawa, Molev, 2017) and some variants.

Consider $\mathfrak{gl}_{nm}$ equipped with the nilpotent element f_{n^m} corresponding to the tableau with m blocks of height n.

$\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$ is called rectangular; it is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2})$.

Weight one fields generate $V^{n\psi - nm}(\mathfrak{gl}_m)$ and m^2 fields of weight d for $2 \leq d \leq n$ transform as adjoint $\mathfrak{gl}_m$-module.

Natural generalization of principal $\mathcal{W}$-algebra, which is case $m = 1$.

16. How does this generalize?

Need families of VOAs that are quotients of a universal object.

Example: Rectangular $\mathcal{W}$-algebras in type A (Arakawa, Molev, 2017) and some variants.

Consider $\mathfrak{gl}_{nm}$ equipped with the nilpotent element f_{n^m} corresponding to the tableau with m blocks of height n.

$\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$ is called rectangular; it is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2})$.

Weight one fields generate $V^{n\psi - nm}(\mathfrak{gl}_m)$ and m^2 fields of weight d for $2 \leq d \leq n$ transform as adjoint $\mathfrak{gl}_m$-module.

Natural generalization of principal $\mathcal{W}$-algebra, which is case $m = 1$.

16. How does this generalize?

Need families of VOAs that are quotients of a universal object.

Example: Rectangular $\mathcal{W}$-algebras in type A (Arakawa, Molev, 2017) and some variants.

Consider $\mathfrak{gl}_{nm}$ equipped with the nilpotent element f_{n^m} corresponding to the tableau with m blocks of height n.

$\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$ is called rectangular; it is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2})$.

Weight one fields generate $V^{n\psi - nm}(\mathfrak{gl}_m)$ and m^2 fields of weight d for $2 \leq d \leq n$ transform as adjoint $\mathfrak{gl}_m$-module.

Natural generalization of principal $\mathcal{W}$-algebra, which is case $m = 1$.

16. How does this generalize?

Need families of VOAs that are quotients of a universal object.

Example: Rectangular $\mathcal{W}$-algebras in type A (Arakawa, Molev, 2017) and some variants.

Consider $\mathfrak{gl}_{nm}$ equipped with the nilpotent element f_{n^m} corresponding to the tableau with m blocks of height n.

$\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$ is called rectangular; it is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2})$.

Weight one fields generate $V^{n\psi - nm}(\mathfrak{gl}_m)$ and m^2 fields of weight d for $2 \leq d \leq n$ transform as adjoint $\mathfrak{gl}_m$-module.

Natural generalization of principal $\mathcal{W}$-algebra, which is case $m = 1$.

16. How does this generalize?

Need families of VOAs that are quotients of a universal object.

Example: Rectangular $\mathcal{W}$-algebras in type A (Arakawa, Molev, 2017) and some variants.

Consider $\mathfrak{gl}_{nm}$ equipped with the nilpotent element f_{n^m} corresponding to the tableau with m blocks of height n.

$\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$ is called rectangular; it is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2})$.

Weight one fields generate $V^{n\psi - nm}(\mathfrak{gl}_m)$ and m^2 fields of weight d for $2 \leq d \leq n$ transform as adjoint $\mathfrak{gl}_m$-module.

Natural generalization of principal $\mathcal{W}$-algebra, which is case $m = 1$.

17. Rectangular $\mathcal{W}$-algebras with tail

Now consider $\mathfrak{gl}_{nm+r}$ equipped with the nilpotent element f_{n^m} which is **rectangular** in $\mathfrak{gl}_{nm}$ and **trivial** in $\mathfrak{gl}_r$.

Embedding $V^{n\psi-nm-r}(\mathfrak{gl}_m) \otimes V^{\psi-r-m}(\mathfrak{gl}_r) \hookrightarrow \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m})$.

Using large level limit together with classical invariant theory, coset

$$\mathrm{Com}(V^{\psi-r-m}(\mathfrak{gl}_r), \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}))$$

has strong generating type
$\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2}, (n+1)^{m^2}, \dots, N^{m^2})$ for some N.

Similar statement holds for superalgebra $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm|r}, f_{n^m})$.

Expectation: For each $m \in \mathbb{N}$, there exists a 2-parameter VOA $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ which is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots)$, such that all of these VOAs arise as one-parameter quotients.

In the case $m = 1$, we should just recover $\mathcal{H} \otimes \mathcal{W}(c, \lambda)$.

17. Rectangular $\mathcal{W}$-algebras with tail

Now consider $\mathfrak{gl}_{nm+r}$ equipped with the nilpotent element f_{n^m} which is **rectangular** in $\mathfrak{gl}_{nm}$ and **trivial** in $\mathfrak{gl}_r$.

Embedding $V^{n\psi-nm-r}(\mathfrak{gl}_m) \otimes V^{\psi-r-m}(\mathfrak{gl}_r) \hookrightarrow \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m})$.

Using large level limit together with classical invariant theory, coset

$$\mathrm{Com}(V^{\psi-r-m}(\mathfrak{gl}_r), \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}))$$

has strong generating type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2}, (n+1)^{m^2}, \dots, N^{m^2})$ for some N.

Similar statement holds for superalgebra $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm|r}, f_{n^m})$.

Expectation: For each $m \in \mathbb{N}$, there exists a 2-parameter VOA $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ which is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots)$, such that all of these VOAs arise as one-parameter quotients.

In the case $m = 1$, we should just recover $\mathcal{H} \otimes \mathcal{W}(c, \lambda)$.

17. Rectangular $\mathcal{W}$-algebras with tail

Now consider $\mathfrak{gl}_{nm+r}$ equipped with the nilpotent element f_{n^m} which is **rectangular** in $\mathfrak{gl}_{nm}$ and **trivial** in $\mathfrak{gl}_r$.

Embedding $V^{n\psi-nm-r}(\mathfrak{gl}_m) \otimes V^{\psi-r-m}(\mathfrak{gl}_r) \hookrightarrow \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m})$.

Using large level limit together with classical invariant theory, coset

$$\mathrm{Com}(V^{\psi-r-m}(\mathfrak{gl}_r), \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}))$$

has strong generating type
$\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2}, (n+1)^{m^2}, \dots, N^{m^2})$ for some N.

Similar statement holds for superalgebra $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm|r}, f_{n^m})$.

Expectation: For each $m \in \mathbb{N}$, there exists a 2-parameter VOA $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ which is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots)$, such that all of these VOAs arise as one-parameter quotients.

In the case $m = 1$, we should just recover $\mathcal{H} \otimes \mathcal{W}(c, \lambda)$.

17. Rectangular $\mathcal{W}$-algebras with tail

Now consider $\mathfrak{gl}_{nm+r}$ equipped with the nilpotent element f_{n^m} which is **rectangular** in $\mathfrak{gl}_{nm}$ and **trivial** in $\mathfrak{gl}_r$.

Embedding $V^{n\psi-nm-r}(\mathfrak{gl}_m) \otimes V^{\psi-r-m}(\mathfrak{gl}_r) \hookrightarrow \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m})$.

Using large level limit together with classical invariant theory, coset

$$\mathrm{Com}(V^{\psi-r-m}(\mathfrak{gl}_r), \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}))$$

has strong generating type
$\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2}, (n+1)^{m^2}, \dots, N^{m^2})$ for some N.

Similar statement holds for superalgebra $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm|r}, f_{n^m})$.

Expectation: For each $m \in \mathbb{N}$, there exists a 2-parameter VOA $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ which is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots)$, such that all of these VOAs arise as one-parameter quotients.

In the case $m = 1$, we should just recover $\mathcal{H} \otimes \mathcal{W}(c, \lambda)$.

17. Rectangular $\mathcal{W}$-algebras with tail

Now consider $\mathfrak{gl}_{nm+r}$ equipped with the nilpotent element f_{n^m} which is **rectangular** in $\mathfrak{gl}_{nm}$ and **trivial** in $\mathfrak{gl}_r$.

Embedding $V^{n\psi-nm-r}(\mathfrak{gl}_m) \otimes V^{\psi-r-m}(\mathfrak{gl}_r) \hookrightarrow \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m})$.

Using large level limit together with classical invariant theory, coset

$$\mathrm{Com}(V^{\psi-r-m}(\mathfrak{gl}_r), \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}))$$

has strong generating type
$\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2}, (n+1)^{m^2}, \dots, N^{m^2})$ for some N.

Similar statement holds for superalgebra $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm|r}, f_{n^m})$.

Expectation: For each $m \in \mathbb{N}$, there exists a 2-parameter VOA $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ which is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots)$, such that all of these VOAs arise as one-parameter quotients.

In the case $m = 1$, we should just recover $\mathcal{H} \otimes \mathcal{W}(c, \lambda)$.

17. Rectangular $\mathcal{W}$-algebras with tail

Now consider $\mathfrak{gl}_{nm+r}$ equipped with the nilpotent element f_{n^m} which is **rectangular** in $\mathfrak{gl}_{nm}$ and **trivial** in $\mathfrak{gl}_r$.

Embedding $V^{n\psi-nm-r}(\mathfrak{gl}_m) \otimes V^{\psi-r-m}(\mathfrak{gl}_r) \hookrightarrow \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m})$.

Using large level limit together with classical invariant theory, coset

$$\mathrm{Com}(V^{\psi-r-m}(\mathfrak{gl}_r), \mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}))$$

has strong generating type
$\mathcal{W}(1^{m^2}, 2^{m^2}, \dots, n^{m^2}, (n+1)^{m^2}, \dots, N^{m^2})$ for some N.

Similar statement holds for superalgebra $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm|r}, f_{n^m})$.

Expectation: For each $m \in \mathbb{N}$, there exists a 2-parameter VOA $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ which is freely generated of type $\mathcal{W}(1^{m^2}, 2^{m^2}, \dots)$, such that all of these VOAs arise as one-parameter quotients.

In the case $m = 1$, we should just recover $\mathcal{H} \otimes \mathcal{W}(c, \lambda)$.

18. Rectangular $\mathcal{W}$-algebras with tail

To construct $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ from first principles is difficult.

One model for $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ is obtained by taking $n = 1$.

Then for each $r, m \in \mathbb{N}$, $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}) = V^{\ell}(\mathfrak{gl}_{m+r})$ for $\ell = \psi - m - r$, and above coset is $\mathrm{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$.

It is possible to replace r with a complex parameter to construct a VOA with two parameters r and ℓ.

This should be isomorphic to $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ after a suitable change of parameters.

Note: For each fixed r, $\mathrm{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$ is an extension of a Heisenberg algebra tensored with m commuting quotients of $\mathcal{W}(c, \lambda)$.

Therefore we expect $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ to be such an extension.

18. Rectangular $\mathcal{W}$-algebras with tail

To construct $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ from first principles is difficult.

One model for $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ is obtained by taking $n=1$.

Then for each $r, m \in \mathbb{N}$, $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}) = V^{\ell}(\mathfrak{gl}_{m+r})$ for $\ell = \psi - m - r$, and above coset is $\mathrm{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$.

It is possible to replace r with a complex parameter to construct a VOA with two parameters r and ℓ.

This should be isomorphic to $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ after a suitable change of parameters.

Note: For each fixed r, $\mathrm{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$ is an extension of a Heisenberg algebra tensored with m commuting quotients of $\mathcal{W}(c,\lambda)$.

Therefore we expect $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ to be such an extension.

18. Rectangular $\mathcal{W}$-algebras with tail

To construct $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ from first principles is difficult.

One model for $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ is obtained by taking $n=1$.

Then for each $r, m \in \mathbb{N}$, $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}) = V^{\ell}(\mathfrak{gl}_{m+r})$ for $\ell = \psi - m - r$, and above coset is $\text{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$.

It is possible to replace r with a complex parameter to construct a VOA with two parameters r and ℓ.

This should be isomorphic to $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ after a suitable change of parameters.

Note: For each fixed r, $\text{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$ is an extension of a Heisenberg algebra tensored with m commuting quotients of $\mathcal{W}(c,\lambda)$.

Therefore we expect $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ to be such an extension.

18. Rectangular $\mathcal{W}$-algebras with tail

To construct $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ from first principles is difficult.

One model for $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ is obtained by taking $n=1$.

Then for each $r,m\in\mathbb{N}$, $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}) = V^{\ell}(\mathfrak{gl}_{m+r})$ for $\ell=\psi-m-r$, and above coset is $\mathrm{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$.

It is possible to replace r with a complex parameter to construct a VOA with two parameters r and ℓ.

This should be isomorphic to $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ after a suitable change of parameters.

Note: For each fixed r, $\mathrm{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$ is an extension of a Heisenberg algebra tensored with m commuting quotients of $\mathcal{W}(c,\lambda)$.

Therefore we expect $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ to be such an extension.

18. Rectangular $\mathcal{W}$-algebras with tail

To construct $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ from first principles is difficult.

One model for $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ is obtained by taking $n=1$.

Then for each $r,m \in \mathbb{N}$, $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}) = V^{\ell}(\mathfrak{gl}_{m+r})$ for $\ell = \psi - m - r$, and above coset is $\text{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$.

It is possible to replace r with a complex parameter to construct a VOA with two parameters r and ℓ.

This should be isomorphic to $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ after a suitable change of parameters.

Note: For each fixed r, $\text{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$ is an extension of a Heisenberg algebra tensored with m commuting quotients of $\mathcal{W}(c,\lambda)$.

Therefore we expect $\mathcal{W}^{\mathfrak{gl}_m}(c,\lambda)$ to be such an extension.

18. Rectangular $\mathcal{W}$-algebras with tail

To construct $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ from first principles is difficult.

One model for $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ is obtained by taking $n = 1$.

Then for each $r, m \in \mathbb{N}$, $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}) = V^{\ell}(\mathfrak{gl}_{m+r})$ for $\ell = \psi - m - r$, and above coset is $\text{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$.

It is possible to replace r with a complex parameter to construct a VOA with two parameters r and ℓ.

This should be isomorphic to $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ after a suitable change of parameters.

Note: For each fixed r, $\text{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$ is an extension of a Heisenberg algebra tensored with m commuting quotients of $\mathcal{W}(c, \lambda)$.

Therefore we expect $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ to be such an extension.

18. Rectangular $\mathcal{W}$-algebras with tail

To construct $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ from first principles is difficult.

One model for $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ is obtained by taking $n = 1$.

Then for each $r, m \in \mathbb{N}$, $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m}) = V^{\ell}(\mathfrak{gl}_{m+r})$ for $\ell = \psi - m - r$, and above coset is $\text{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$.

It is possible to replace r with a complex parameter to construct a VOA with two parameters r and ℓ.

This should be isomorphic to $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ after a suitable change of parameters.

Note: For each fixed r, $\text{Com}(V^{\ell}(\mathfrak{gl}_r), V^{\ell}(\mathfrak{gl}_{m+r}))$ is an extension of a Heisenberg algebra tensored with m commuting quotients of $\mathcal{W}(c, \lambda)$.

Therefore we expect $\mathcal{W}^{\mathfrak{gl}_m}(c, \lambda)$ to be such an extension.

19. Other types

We expect similar 2-parameter VOAs $\mathcal{W}^{\mathfrak{so}_n}(c,\lambda)$ of type

$\mathcal{W}(1^{d_1}, 2^{d_2}, 3^{d_1}, 4^{d_2}, \dots)$, $\quad d_1 = \dim \wedge^2(\mathbb{C}^n)$, $\quad d_2 = \dim \mathrm{Sym}^2(\mathbb{C}^n)$.

(Currently under construction with Flor Orosz Hunziker).

Expected to be extensions of tensor products of quotients of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$.

This is because $\mathrm{Com}(V^k(\mathfrak{so}_n), V^k(\mathfrak{so}_{n+1}))$ is a quotient of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ for all $n \geq 1$.

In type C the story is different because $\mathrm{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{sp}_{2n+2}))$ has a copy of $V^k(\mathfrak{sp}_2)$, and has strong generating type $\mathcal{W}(1^3, 2, 3^3, 4, 5^3, \dots)$.

Can replace n with a complex parameter to construct a 2-parameter VOA which is freely generated of type $\mathcal{W}(1^3, 2, 3^3, 4, 5^3, \dots)$.

19. Other types

We expect similar 2-parameter VOAs $\mathcal{W}^{\mathfrak{so}_n}(c, \lambda)$ of type

$\mathcal{W}(1^{d_1}, 2^{d_2}, 3^{d_1}, 4^{d_2}, \dots), \quad d_1 = \dim \wedge^2(\mathbb{C}^n), \quad d_2 = \dim \mathrm{Sym}^2(\mathbb{C}^n).$

(Currently under construction with Flor Orosz Hunziker).

Expected to be extensions of tensor products of quotients of $\mathcal{W}^{\mathrm{ev}}(c, \lambda)$.

This is because $\mathrm{Com}(V^k(\mathfrak{so}_n), V^k(\mathfrak{so}_{n+1}))$ is a quotient of $\mathcal{W}^{\mathrm{ev}}(c, \lambda)$ for all $n \geq 1$.

In type C the story is different because $\mathrm{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{sp}_{2n+2}))$ has a copy of $V^k(\mathfrak{sp}_2)$, and has strong generating type $\mathcal{W}(1^3, 2, 3^3, 4, 5^3, \dots)$.

Can replace n with a complex parameter to construct a 2-parameter VOA which is freely generated of type $\mathcal{W}(1^3, 2, 3^3, 4, 5^3, \dots)$.

19. Other types

We expect similar 2-parameter VOAs $\mathcal{W}^{\mathfrak{so}_n}(c,\lambda)$ of type

$\mathcal{W}(1^{d_1}, 2^{d_2}, 3^{d_1}, 4^{d_2}, \dots)$, $\quad d_1 = \dim \wedge^2(\mathbb{C}^n)$, $\quad d_2 = \dim \mathrm{Sym}^2(\mathbb{C}^n)$.

(Currently under construction with Flor Orosz Hunziker).

Expected to be extensions of tensor products of quotients of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$.

This is because $\mathrm{Com}(V^k(\mathfrak{so}_n), V^k(\mathfrak{so}_{n+1}))$ is a quotient of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ for all $n \geq 1$.

In type C the story is different because $\mathrm{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{sp}_{2n+2}))$ has a copy of $V^k(\mathfrak{sp}_2)$, and has strong generating type $\mathcal{W}(1^3, 2, 3^3, 4, 5^3, \dots)$.

Can replace n with a complex parameter to construct a 2-parameter VOA which is freely generated of type $\mathcal{W}(1^3, 2, 3^3, 4, 5^3, \dots)$.

19. Other types

We expect similar 2-parameter VOAs $\mathcal{W}^{\mathfrak{so}_n}(c,\lambda)$ of type

$$\mathcal{W}(1^{d_1}, 2^{d_2}, 3^{d_1}, 4^{d_2}, \dots), \quad d_1 = \dim \wedge^2(\mathbb{C}^n), \quad d_2 = \dim \mathrm{Sym}^2(\mathbb{C}^n).$$

(Currently under construction with Flor Orosz Hunziker).

Expected to be extensions of tensor products of quotients of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$.

This is because $\mathrm{Com}(V^k(\mathfrak{so}_n), V^k(\mathfrak{so}_{n+1}))$ is a quotient of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ for all $n \geq 1$.

In type C the story is different because $\mathrm{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{sp}_{2n+2}))$ has a copy of $V^k(\mathfrak{sp}_2)$, and has strong generating type $\mathcal{W}(1^3, 2, 3^3, 4, 5^3, \dots)$.

Can replace n with a complex parameter to construct a 2-parameter VOA which is freely generated of type $\mathcal{W}(1^3, 2, 3^3, 4, 5^3, \dots)$.

19. Other types

We expect similar 2-parameter VOAs $\mathcal{W}^{\mathfrak{so}_n}(c,\lambda)$ of type

$$\mathcal{W}(1^{d_1}, 2^{d_2}, 3^{d_1}, 4^{d_2}, \dots), \quad d_1 = \dim \wedge^2(\mathbb{C}^n), \quad d_2 = \dim \mathrm{Sym}^2(\mathbb{C}^n).$$

(Currently under construction with Flor Orosz Hunziker).

Expected to be extensions of tensor products of quotients of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$.

This is because $\mathrm{Com}(V^k(\mathfrak{so}_n), V^k(\mathfrak{so}_{n+1}))$ is a quotient of $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$ for all $n \geq 1$.

In type C the story is different because $\mathrm{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{sp}_{2n+2}))$ has a copy of $V^k(\mathfrak{sp}_2)$, and has strong generating type $\mathcal{W}(1^3, 2, 3^3, 4, 5^3, \dots)$.

Can replace n with a complex parameter to construct a 2-parameter VOA which is freely generated of type $\mathcal{W}(1^3, 2, 3^3, 4, 5^3, \dots)$.

20. The universal object $\mathcal{W}^{\mathfrak{sp}_2}$

Vladimir Kovalchuk is (nearly) finished proving that this is a universal object, which we denote by $\mathcal{W}^{\mathfrak{sp}_2}$.

Similar to, but more involved, than construction of $\mathcal{W}(c, \lambda)$. Expect it has exactly 2 free parameters c, k.

In addition to generators X, Y, H of $V^k(\mathfrak{sp}_2)$, we have:

1. Fields $L, W^4, W^6, \dots$ which are $\mathfrak{sp}_2$-trivial,
2. Fields $X^{2i+1}, Y^{2i+1}, h^{2i+1}$ for all $i \geq 1$, which transform as adjoint $\mathfrak{sp}_2$-module.

We assume:

1. $W^{2i+2} = W^4_{(1)} W^{2i}$ for $i \geq 2$,
2. $Z^{2i+3} = W^4_{(1)} Z^{2i+1}$ for all $i \geq 1$ and $Z = X, Y, H$.

Write down most general OPE algebra that is compatible with $\mathfrak{sp}_2$-symmetry, with undermined structure constants.

Structure constants are determined by imposing Jacobi identities.

20. The universal object $\mathcal{W}^{\mathfrak{sp}_2}$

Vladimir Kovalchuk is (nearly) finished proving that this is a universal object, which we denote by $\mathcal{W}^{\mathfrak{sp}_2}$.

Similar to, but more involved, than construction of $\mathcal{W}(c, \lambda)$. Expect it has exactly 2 free parameters c, k.

In addition to generators X, Y, H of $V^k(\mathfrak{sp}_2)$, we have:

1. Fields $L, W^4, W^6, \dots$ which are $\mathfrak{sp}_2$-trivial,
2. Fields $X^{2i+1}, Y^{2i+1}, h^{2i+1}$ for all $i \geq 1$, which transform as adjoint $\mathfrak{sp}_2$-module.

We assume:

1. $W^{2i+2} = W^4_{(1)} W^{2i}$ for $i \geq 2$,
2. $Z^{2i+3} = W^4_{(1)} Z^{2i+1}$ for all $i \geq 1$ and $Z = X, Y, H$.

Write down most general OPE algebra that is compatible with $\mathfrak{sp}_2$-symmetry, with undermined structure constants.

Structure constants are determined by imposing Jacobi identities.

20. The universal object $\mathcal{W}^{\mathfrak{sp}_2}$

Vladimir Kovalchuk is (nearly) finished proving that this is a universal object, which we denote by $\mathcal{W}^{\mathfrak{sp}_2}$.

Similar to, but more involved, than construction of $\mathcal{W}(c, \lambda)$. Expect it has exactly 2 free parameters c, k.

In addition to generators X, Y, H of $V^k(\mathfrak{sp}_2)$, we have:

1. Fields $L, W^4, W^6, \dots$ which are $\mathfrak{sp}_2$-trivial,
2. Fields $X^{2i+1}, Y^{2i+1}, h^{2i+1}$ for all $i \geq 1$, which transform as adjoint $\mathfrak{sp}_2$-module.

We assume:

1. $W^{2i+2} = W^4_{(1)} W^{2i}$ for $i \geq 2$,
2. $Z^{2i+3} = W^4_{(1)} Z^{2i+1}$ for all $i \geq 1$ and $Z = X, Y, H$.

Write down most general OPE algebra that is compatible with $\mathfrak{sp}_2$-symmetry, with undermined structure constants.

Structure constants are determined by imposing Jacobi identities.

20. The universal object $\mathcal{W}^{\mathfrak{sp}_2}$

Vladimir Kovalchuk is (nearly) finished proving that this is a universal object, which we denote by $\mathcal{W}^{\mathfrak{sp}_2}$.

Similar to, but more involved, than construction of $\mathcal{W}(c, \lambda)$. Expect it has exactly 2 free parameters c, k.

In addition to generators X, Y, H of $V^k(\mathfrak{sp}_2)$, we have:

1. Fields $L, W^4, W^6, \dots$ which are $\mathfrak{sp}_2$-trivial,
2. Fields $X^{2i+1}, Y^{2i+1}, h^{2i+1}$ for all $i \geq 1$, which transform as adjoint $\mathfrak{sp}_2$-module.

We assume:

1. $W^{2i+2} = W^4_{(1)} W^{2i}$ for $i \geq 2$,
2. $Z^{2i+3} = W^4_{(1)} Z^{2i+1}$ for all $i \geq 1$ and $Z = X, Y, H$.

Write down most general OPE algebra that is compatible with $\mathfrak{sp}_2$-symmetry, with undermined structure constants.

Structure constants are determined by imposing Jacobi identities.

20. The universal object $\mathcal{W}^{\mathfrak{sp}_2}$

Vladimir Kovalchuk is (nearly) finished proving that this is a universal object, which we denote by $\mathcal{W}^{\mathfrak{sp}_2}$.

Similar to, but more involved, than construction of $\mathcal{W}(c, \lambda)$. Expect it has exactly 2 free parameters c, k.

In addition to generators X, Y, H of $V^k(\mathfrak{sp}_2)$, we have:

1. Fields $L, W^4, W^6, \dots$ which are $\mathfrak{sp}_2$-trivial,
2. Fields $X^{2i+1}, Y^{2i+1}, h^{2i+1}$ for all $i \geq 1$, which transform as adjoint $\mathfrak{sp}_2$-module.

We assume:

1. $W^{2i+2} = W^4_{(1)} W^{2i}$ for $i \geq 2$,
2. $Z^{2i+3} = W^4_{(1)} Z^{2i+1}$ for all $i \geq 1$ and $Z = X, Y, H$.

Write down most general OPE algebra that is compatible with $\mathfrak{sp}_2$-symmetry, with undermined structure constants.

Structure constants are determined by imposing Jacobi identities.

20. The universal object $\mathcal{W}^{\mathfrak{sp}_2}$

Vladimir Kovalchuk is (nearly) finished proving that this is a universal object, which we denote by $\mathcal{W}^{\mathfrak{sp}_2}$.

Similar to, but more involved, than construction of $\mathcal{W}(c, \lambda)$. Expect it has exactly 2 free parameters c, k.

In addition to generators X, Y, H of $V^k(\mathfrak{sp}_2)$, we have:

1. Fields $L, W^4, W^6, \dots$ which are $\mathfrak{sp}_2$-trivial,
2. Fields $X^{2i+1}, Y^{2i+1}, h^{2i+1}$ for all $i \geq 1$, which transform as adjoint $\mathfrak{sp}_2$-module.

We assume:

1. $W^{2i+2} = W^4_{(1)} W^{2i}$ for $i \geq 2$,
2. $Z^{2i+3} = W^4_{(1)} Z^{2i+1}$ for all $i \geq 1$ and $Z = X, Y, H$.

Write down most general OPE algebra that is compatible with $\mathfrak{sp}_2$-symmetry, with undermined structure constants.

Structure constants are determined by imposing Jacobi identities.

21. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$

Let $\mathfrak{g} = \mathfrak{so}_{4n}$ which has a subalgebra $\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2$ and decomposes as

$$\mathfrak{so}_{4n} \cong \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2 \oplus (\rho_{\omega_2} \otimes \mathbb{C}^3).$$

Note: As a $\mathfrak{sp}_{2n}$-module,

$$\wedge^2(\mathbb{C}^{2n}) \cong \mathbb{C} \oplus \rho_{\omega_2},$$

so dim $\rho_{\omega_2} = n(2n-1) - 1$.

Let $f_{\mathfrak{sp}_{2n}}$ be the nilpotent which is principal in $\mathfrak{sp}_{2n}$.

$\mathcal{W}^{\psi}(\mathfrak{so}_{4n}, f_{\mathfrak{sp}_{2n}})$ is analogous to $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$, where

1. Principal part $\mathfrak{gl}_n$ is replaced with $\mathfrak{sp}_{2n}$.
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$.

$\mathcal{W}^{\psi}(\mathfrak{so}_{4n}, f_{\mathfrak{sp}_{2n}})$ is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n).$$

21. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$

Let $\mathfrak{g} = \mathfrak{so}_{4n}$ which has a subalgebra $\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2$ and decomposes as

$$\mathfrak{so}_{4n} \cong \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2 \oplus (\rho_{\omega_2} \otimes \mathbb{C}^3).$$

Note: As a $\mathfrak{sp}_{2n}$-module,

$$\wedge^2(\mathbb{C}^{2n}) \cong \mathbb{C} \oplus \rho_{\omega_2},$$

so dim $\rho_{\omega_2} = n(2n-1) - 1$.

Let $f_{\mathfrak{sp}_{2n}}$ be the nilpotent which is principal in $\mathfrak{sp}_{2n}$.

$\mathcal{W}^{\psi}(\mathfrak{so}_{4n}, f_{\mathfrak{sp}_{2n}})$ is analogous to $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$, where

1. Principal part $\mathfrak{gl}_n$ is replaced with $\mathfrak{sp}_{2n}$.
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$.

$\mathcal{W}^{\psi}(\mathfrak{so}_{4n}, f_{\mathfrak{sp}_{2n}})$ is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n).$$

21. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$

Let $\mathfrak{g} = \mathfrak{so}_{4n}$ which has a subalgebra $\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2$ and decomposes as

$$\mathfrak{so}_{4n} \cong \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2 \oplus (\rho_{\omega_2} \otimes \mathbb{C}^3).$$

Note: As a $\mathfrak{sp}_{2n}$-module,

$$\wedge^2(\mathbb{C}^{2n}) \cong \mathbb{C} \oplus \rho_{\omega_2},$$

so dim $\rho_{\omega_2} = n(2n-1) - 1$.

Let $f_{\mathfrak{sp}_{2n}}$ be the nilpotent which is principal in $\mathfrak{sp}_{2n}$.

$\mathcal{W}^{\psi}(\mathfrak{so}_{4n}, f_{\mathfrak{sp}_{2n}})$ is analogous to $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$, where

1. Principal part $\mathfrak{gl}_n$ is replaced with $\mathfrak{sp}_{2n}$.
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$.

$\mathcal{W}^{\psi}(\mathfrak{so}_{4n}, f_{\mathfrak{sp}_{2n}})$ is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n).$$

21. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$

Let $\mathfrak{g} = \mathfrak{so}_{4n}$ which has a subalgebra $\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2$ and decomposes as

$$\mathfrak{so}_{4n} \cong \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2 \oplus (\rho_{\omega_2} \otimes \mathbb{C}^3).$$

Note: As a $\mathfrak{sp}_{2n}$-module,

$$\wedge^2(\mathbb{C}^{2n}) \cong \mathbb{C} \oplus \rho_{\omega_2},$$

so $\dim \rho_{\omega_2} = n(2n-1) - 1$.

Let $f_{\mathfrak{sp}_{2n}}$ be the nilpotent which is principal in $\mathfrak{sp}_{2n}$.

$\mathcal{W}^{\psi}(\mathfrak{so}_{4n}, f_{\mathfrak{sp}_{2n}})$ is analogous to $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$, where

1. Principal part $\mathfrak{gl}_n$ is replaced with $\mathfrak{sp}_{2n}$.
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$.

$\mathcal{W}^{\psi}(\mathfrak{so}_{4n}, f_{\mathfrak{sp}_{2n}})$ is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n).$$

21. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$

Let $\mathfrak{g} = \mathfrak{so}_{4n}$ which has a subalgebra $\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2$ and decomposes as

$$\mathfrak{so}_{4n} \cong \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_2 \oplus (\rho_{\omega_2} \otimes \mathbb{C}^3).$$

Note: As a $\mathfrak{sp}_{2n}$-module,

$$\wedge^2(\mathbb{C}^{2n}) \cong \mathbb{C} \oplus \rho_{\omega_2},$$

so dim $\rho_{\omega_2} = n(2n-1) - 1$.

Let $f_{\mathfrak{sp}_{2n}}$ be the nilpotent which is principal in $\mathfrak{sp}_{2n}$.

$\mathcal{W}^{\psi}(\mathfrak{so}_{4n}, f_{\mathfrak{sp}_{2n}})$ is analogous to $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$, where

1. Principal part $\mathfrak{gl}_n$ is replaced with $\mathfrak{sp}_{2n}$.
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$.

$\mathcal{W}^{\psi}(\mathfrak{so}_{4n}, f_{\mathfrak{sp}_{2n}})$ is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n).$$

22. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$

Let $\mathfrak{g} = \mathfrak{sp}_{2(2n+1)}$ which has a subalgebra $\mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2$ and decomposes as

$$\mathfrak{sp}_{2(2n+1)} \cong \mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2 \oplus (\rho_{2\omega_1} \otimes \mathbb{C}^3).$$

Note: As a $\mathfrak{so}_{2n+1}$-module,

$$\mathrm{Sym}^2(\mathbb{C}^{2n+1}) \cong \mathbb{C} \oplus \rho_{2\omega_1},$$

so $\dim \rho_{2\omega_1} = (2n+1)(n+1) - 1$.

Let $f_{\mathfrak{so}_{2n+1}}$ be the nilpotent which is principal in $\mathfrak{so}_{2n+1}$.

$\mathcal{W}^{\psi}(\mathfrak{sp}_{2(2n+1)}, f_{\mathfrak{so}_{2n+1}})$ is analogous to $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$, where

1. Principal part $\mathfrak{gl}_n$ is replaced with $\mathfrak{so}_{2n+1}$.
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$.

$\mathcal{W}^{\psi}(\mathfrak{sp}_{2(2n+1)}, f_{\mathfrak{so}_{2n+1}})$ is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n).$$

22. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$

Let $\mathfrak{g} = \mathfrak{sp}_{2(2n+1)}$ which has a subalgebra $\mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2$ and decomposes as

$$\mathfrak{sp}_{2(2n+1)} \cong \mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2 \oplus (\rho_{2\omega_1} \otimes \mathbb{C}^3).$$

Note: As a $\mathfrak{so}_{2n+1}$-module,

$$\mathrm{Sym}^2(\mathbb{C}^{2n+1}) \cong \mathbb{C} \oplus \rho_{2\omega_1},$$

so dim $\rho_{2\omega_1} = (2n+1)(n+1) - 1$.

Let $f_{\mathfrak{so}_{2n+1}}$ be the nilpotent which is principal in $\mathfrak{so}_{2n+1}$.

$\mathcal{W}^{\psi}(\mathfrak{sp}_{2(2n+1)}, f_{\mathfrak{so}_{2n+1}})$ is analogous to $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$, where

1. Principal part $\mathfrak{gl}_n$ is replaced with $\mathfrak{so}_{2n+1}$.
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$.

$\mathcal{W}^{\psi}(\mathfrak{sp}_{2(2n+1)}, f_{\mathfrak{so}_{2n+1}})$ is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n).$$

22. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$

Let $\mathfrak{g} = \mathfrak{sp}_{2(2n+1)}$ which has a subalgebra $\mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2$ and decomposes as

$$\mathfrak{sp}_{2(2n+1)} \cong \mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2 \oplus (\rho_{2\omega_1} \otimes \mathbb{C}^3).$$

Note: As a $\mathfrak{so}_{2n+1}$-module,

$$\mathrm{Sym}^2(\mathbb{C}^{2n+1}) \cong \mathbb{C} \oplus \rho_{2\omega_1},$$

so dim $\rho_{2\omega_1} = (2n+1)(n+1) - 1$.

Let $f_{\mathfrak{so}_{2n+1}}$ be the nilpotent which is principal in $\mathfrak{so}_{2n+1}$.

$\mathcal{W}^{\psi}(\mathfrak{sp}_{2(2n+1)}, f_{\mathfrak{so}_{2n+1}})$ is analogous to $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$, where

1. Principal part $\mathfrak{gl}_n$ is replaced with $\mathfrak{so}_{2n+1}$.
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$.

$\mathcal{W}^{\psi}(\mathfrak{sp}_{2(2n+1)}, f_{\mathfrak{so}_{2n+1}})$ is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n).$$

22. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$

Let $\mathfrak{g} = \mathfrak{sp}_{2(2n+1)}$ which has a subalgebra $\mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2$ and decomposes as

$$\mathfrak{sp}_{2(2n+1)} \cong \mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2 \oplus (\rho_{2\omega_1} \otimes \mathbb{C}^3).$$

Note: As a $\mathfrak{so}_{2n+1}$-module,

$$\mathrm{Sym}^2(\mathbb{C}^{2n+1}) \cong \mathbb{C} \oplus \rho_{2\omega_1},$$

so dim $\rho_{2\omega_1} = (2n+1)(n+1) - 1$.

Let $f_{\mathfrak{so}_{2n+1}}$ be the nilpotent which is principal in $\mathfrak{so}_{2n+1}$.

$\mathcal{W}^{\psi}(\mathfrak{sp}_{2(2n+1)}, f_{\mathfrak{so}_{2n+1}})$ is analogous to $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$, where

1. Principal part $\mathfrak{gl}_n$ is replaced with $\mathfrak{so}_{2n+1}$.
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$.

$\mathcal{W}^{\psi}(\mathfrak{sp}_{2(2n+1)}, f_{\mathfrak{so}_{2n+1}})$ is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n).$$

22. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$

Let $\mathfrak{g} = \mathfrak{sp}_{2(2n+1)}$ which has a subalgebra $\mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2$ and decomposes as

$$\mathfrak{sp}_{2(2n+1)} \cong \mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2 \oplus (\rho_{2\omega_1} \otimes \mathbb{C}^3).$$

Note: As a $\mathfrak{so}_{2n+1}$-module,

$$\mathrm{Sym}^2(\mathbb{C}^{2n+1}) \cong \mathbb{C} \oplus \rho_{2\omega_1},$$

so dim $\rho_{2\omega_1} = (2n+1)(n+1) - 1$.

Let $f_{\mathfrak{so}_{2n+1}}$ be the nilpotent which is principal in $\mathfrak{so}_{2n+1}$.

$\mathcal{W}^{\psi}(\mathfrak{sp}_{2(2n+1)}, f_{\mathfrak{so}_{2n+1}})$ is analogous to $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm}, f_{n^m})$, where

1. Principal part $\mathfrak{gl}_n$ is replaced with $\mathfrak{so}_{2n+1}$.
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$.

$\mathcal{W}^{\psi}(\mathfrak{sp}_{2(2n+1)}, f_{\mathfrak{so}_{2n+1}})$ is freely generated of type

$$\mathcal{W}(1^3, 2, 3^3, 4, \dots, (2n-1)^3, 2n).$$

23. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$, cont'd

More generally, we have 8 families of $\mathcal{W}$-(super)algebras that are analogues of "rectangular with tail" algebras $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m})$:

1. Principal part $\mathfrak{gl}_n$ is replaced with either $\mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n+1}$,
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$,
3. Tail part $\mathfrak{gl}_r$ is replaced with $\mathfrak{g}$, either $\mathfrak{so}_{2r}, \mathfrak{so}_{2r+1}, \mathfrak{sp}_{2r}, \mathfrak{osp}_{1|2r}$.

Example: $\mathfrak{g} = \mathfrak{so}_{4n+2r}$, where $f_{\mathrm{sp}_{2n}}$ is as above inside $\mathfrak{so}_{4n}$. Then $\mathcal{W}^{\psi}(\mathfrak{so}_{4n+2r}, f_{\mathrm{sp}_{2n}})$ has affine subVOA

$$V^{\ell_1}(\mathfrak{sp}_2) \otimes V^{\ell_2}(\mathfrak{so}_{2r}), \qquad \ell_1 = n\psi - r - 2n, \qquad \ell_2 = \psi - 2r - 2.$$

Then $\mathrm{Com}(V^{\ell_2}(\mathfrak{so}_{2r}), \mathcal{W}^{\psi}(\mathfrak{so}_{4n+2r}, f_{\mathrm{sp}_{2n}}))^{\mathbb{Z}_2}$ is a quotient of $\mathcal{W}^{\mathfrak{sp}_2}$.

23. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$, cont'd

More generally, we have 8 families of $\mathcal{W}$-(super)algebras that are analogues of "rectangular with tail" algebras $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m})$:

1. Principal part $\mathfrak{gl}_n$ is replaced with either $\mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n+1}$,
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$,
3. Tail part $\mathfrak{gl}_r$ is replaced with $\mathfrak{g}$, either $\mathfrak{so}_{2r}, \mathfrak{so}_{2r+1}, \mathfrak{sp}_{2r}, \mathfrak{osp}_{1|2r}$.

Example: $\mathfrak{g} = \mathfrak{so}_{4n+2r}$, where $f_{\mathrm{sp}_{2n}}$ is as above inside $\mathfrak{so}_{4n}$. Then $\mathcal{W}^{\psi}(\mathfrak{so}_{4n+2r}, f_{\mathrm{sp}_{2n}})$ has affine subVOA

$$V^{\ell_1}(\mathfrak{sp}_2) \otimes V^{\ell_2}(\mathfrak{so}_{2r}), \qquad \ell_1 = n\psi - r - 2n, \qquad \ell_2 = \psi - 2r - 2.$$

Then $\mathrm{Com}(V^{\ell_2}(\mathfrak{so}_{2r}), \mathcal{W}^{\psi}(\mathfrak{so}_{4n+2r}, f_{\mathrm{sp}_{2n}}))^{\mathbb{Z}_2}$ is a quotient of $\mathcal{W}^{\mathfrak{sp}_2}$.

23. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$, cont'd

More generally, we have 8 families of $\mathcal{W}$-(super)algebras that are analogues of "rectangular with tail" algebras $\mathcal{W}^{\psi}(\mathfrak{gl}_{nm+r}, f_{n^m})$:

1. Principal part $\mathfrak{gl}_n$ is replaced with either $\mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n+1}$,
2. Rectangular part $\mathfrak{gl}_m$ is replaced with $\mathfrak{sp}_2$,
3. Tail part $\mathfrak{gl}_r$ is replaced with $\mathfrak{g}$, either $\mathfrak{so}_{2r}, \mathfrak{so}_{2r+1}, \mathfrak{sp}_{2r}, \mathfrak{osp}_{1|2r}$.

Example: $\mathfrak{g} = \mathfrak{so}_{4n+2r}$, where $f_{\mathrm{sp}_{2n}}$ is as above inside $\mathfrak{so}_{4n}$. Then $\mathcal{W}^{\psi}(\mathfrak{so}_{4n+2r}, f_{\mathrm{sp}_{2n}})$ has affine subVOA

$$V^{\ell_1}(\mathfrak{sp}_2) \otimes V^{\ell_2}(\mathfrak{so}_{2r}), \qquad \ell_1 = n\psi - r - 2n, \qquad \ell_2 = \psi - 2r - 2.$$

Then $\mathrm{Com}(V^{\ell_2}(\mathfrak{so}_{2r}), \mathcal{W}^{\psi}(\mathfrak{so}_{4n+2r}, f_{\mathrm{sp}_{2n}}))^{\mathbb{Z}_2}$ is a quotient of $\mathcal{W}^{\mathfrak{sp}_2}$.

24. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$, cont'd

Note: These 8 families are the type C analogues of the Gaiotto-Rapčák Y-algebras.

Kovalchuk has computed their truncation curves.

Intersection points are all rational, leading to interesting coincidences.

Strong uniqueness theorem: all $\mathcal{W}$-algebras in these families are uniquely determined up to isomorphism by:

1. Structure of $\mathcal{W}^{\mathfrak{sp}_2}$
2. Action of "tail" Lie (super)algebra on extension fields.

Observation: There are no trialities: all VOAs in these 8 families are distinct as 1-parameter VOAs.

Unlike the $\mathcal{W}_\infty$-algebras, these 8 families do **not** exhaust all the 1-parameter quotients of $\mathcal{W}^{\mathfrak{sp}_2}$.

24. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$, cont'd

Note: These 8 families are the type C analogues of the Gaiotto-Rapčák Y-algebras.

Kovalchuk has computed their truncation curves.

Intersection points are all rational, leading to interesting coincidences.

Strong uniqueness theorem: all $\mathcal{W}$-algebras in these families are uniquely determined up to isomorphism by:

1. Structure of $\mathcal{W}^{\mathfrak{sp}_2}$
2. Action of "tail" Lie (super)algebra on extension fields.

Observation: There are no trialities: all VOAs in these 8 families are distinct as 1-parameter VOAs.

Unlike the $\mathcal{W}_\infty$-algebras, these 8 families do **not** exhaust all the 1-parameter quotients of $\mathcal{W}^{\mathfrak{sp}_2}$.

24. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$, cont'd

Note: These 8 families are the type C analogues of the Gaiotto-Rapčák Y-algebras.

Kovalchuk has computed their truncation curves.

Intersection points are all rational, leading to interesting coincidences.

Strong uniqueness theorem: all $\mathcal{W}$-algebras in these families are uniquely determined up to isomorphism by:

1. Structure of $\mathcal{W}^{\mathfrak{sp}_2}$
2. Action of "tail" Lie (super)algebra on extension fields.

Observation: There are no trialities: all VOAs in these 8 families are distinct as 1-parameter VOAs.

Unlike the $\mathcal{W}_\infty$-algebras, these 8 families do **not** exhaust all the 1-parameter quotients of $\mathcal{W}^{\mathfrak{sp}_2}$.

24. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$, cont'd

Note: These 8 families are the type C analogues of the Gaiotto-Rapčák Y-algebras.

Kovalchuk has computed their truncation curves.

Intersection points are all rational, leading to interesting coincidences.

Strong uniqueness theorem: all $\mathcal{W}$-algebras in these families are uniquely determined up to isomorphism by:

1. Structure of $\mathcal{W}^{\mathfrak{sp}_2}$
2. Action of "tail" Lie (super)algebra on extension fields.

Observation: There are no trialities: all VOAs in these 8 families are distinct as 1-parameter VOAs.

Unlike the $\mathcal{W}_\infty$-algebras, these 8 families do **not** exhaust all the 1-parameter quotients of $\mathcal{W}^{\mathfrak{sp}_2}$.

24. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$, cont'd

Note: These 8 families are the type C analogues of the Gaiotto-Rapčák Y-algebras.

Kovalchuk has computed their truncation curves.

Intersection points are all rational, leading to interesting coincidences.

Strong uniqueness theorem: all $\mathcal{W}$-algebras in these families are uniquely determined up to isomorphism by:

1. Structure of $\mathcal{W}^{\mathfrak{sp}_2}$
2. Action of "tail" Lie (super)algebra on extension fields.

Observation: There are no trialities: all VOAs in these 8 families are distinct as 1-parameter VOAs.

Unlike the $\mathcal{W}_\infty$-algebras, these 8 families do **not** exhaust all the 1-parameter quotients of $\mathcal{W}^{\mathfrak{sp}_2}$.

24. Some quotients of $\mathcal{W}^{\mathfrak{sp}_2}$, cont'd

Note: These 8 families are the type C analogues of the Gaiotto-Rapčák Y-algebras.

Kovalchuk has computed their truncation curves.

Intersection points are all rational, leading to interesting coincidences.

Strong uniqueness theorem: all $\mathcal{W}$-algebras in these families are uniquely determined up to isomorphism by:

1. Structure of $\mathcal{W}^{\mathfrak{sp}_2}$
2. Action of "tail" Lie (super)algebra on extension fields.

Observation: There are no trialities: all VOAs in these 8 families are distinct as 1-parameter VOAs.

Unlike the $\mathcal{W}_\infty$-algebras, these 8 families do **not** exhaust all the 1-parameter quotients of $\mathcal{W}^{\mathfrak{sp}_2}$.

25. Motivation: diagonal cosets in type A

For $\mathfrak{g} = \mathfrak{sl}_n$, we have homomorphism

$$V^{k+1}(\mathfrak{sl}_n) \to V^k(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n).$$

Fermion algebra $F(2n)$ is an extension of $L_1(\mathfrak{sl}_n)$, can replace above by

$$V^{k+1}(\mathfrak{sl}_n) \to V^k(\mathfrak{sl}_n) \otimes F(2n).$$

Thm: (Kac, Wakimoto, 1989) If k is admissible, this descends to

$$L_{k+1}(\mathfrak{sl}_n) \to L_k(\mathfrak{sl}_n) \otimes F(2n).$$

Thm: (Arakawa, Creutzig, L. 2018) If k is admissible,

$$\mathrm{Com}(L_{k+1}(\mathfrak{sl}_n), L_k(\mathfrak{sl}_n) \otimes F(2n)) \cong \mathcal{W}_\phi(\mathfrak{sl}_n), \quad \phi = \frac{k+n}{k+n+1}$$

which is lisse and rational.

25. Motivation: diagonal cosets in type A

For $\mathfrak{g} = \mathfrak{sl}_n$, we have homomorphism

$$V^{k+1}(\mathfrak{sl}_n) \to V^k(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n).$$

Fermion algebra $F(2n)$ is an extension of $L_1(\mathfrak{sl}_n)$, can replace above by

$$V^{k+1}(\mathfrak{sl}_n) \to V^k(\mathfrak{sl}_n) \otimes F(2n).$$

Thm: (Kac, Wakimoto, 1989) If k is admissible, this descends to

$$L_{k+1}(\mathfrak{sl}_n) \to L_k(\mathfrak{sl}_n) \otimes F(2n).$$

Thm: (Arakawa, Creutzig, L. 2018) If k is admissible,

$$\mathrm{Com}(L_{k+1}(\mathfrak{sl}_n), L_k(\mathfrak{sl}_n) \otimes F(2n)) \cong \mathcal{W}_\phi(\mathfrak{sl}_n), \quad \phi = \frac{k+n}{k+n+1}$$

which is lisse and rational.

25. Motivation: diagonal cosets in type A

For $\mathfrak{g} = \mathfrak{sl}_n$, we have homomorphism

$$V^{k+1}(\mathfrak{sl}_n) \to V^k(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n).$$

Fermion algebra $F(2n)$ is an extension of $L_1(\mathfrak{sl}_n)$, can replace above by

$$V^{k+1}(\mathfrak{sl}_n) \to V^k(\mathfrak{sl}_n) \otimes F(2n).$$

Thm: (Kac, Wakimoto, 1989) If k is admissible, this descends to

$$L_{k+1}(\mathfrak{sl}_n) \to L_k(\mathfrak{sl}_n) \otimes F(2n).$$

Thm: (Arakawa, Creutzig, L. 2018) If k is admissible,

$$\mathrm{Com}(L_{k+1}(\mathfrak{sl}_n), L_k(\mathfrak{sl}_n) \otimes F(2n)) \cong \mathcal{W}_\phi(\mathfrak{sl}_n), \quad \phi = \frac{k+n}{k+n+1}$$

which is lisse and rational.

26. Diagonal cosets in type A

In type B, we can similarly consider the embedding

$$V^{k+1}(\mathfrak{so}_{2n+1}) \to V^k(\mathfrak{so}_{2n+1}) \otimes F(2n+1),$$

which descends to

$$L_{k+1}(\mathfrak{so}_{2n+1}) \to L_k(\mathfrak{so}_{2n+1}) \otimes F(2n+1),$$

when k is admissible.

Thm: (Creutzig, Genra, and Creutzig-L. 2021). For k admissible,

$$\mathrm{Com}(L_{k+1}(\mathfrak{so}_{2n+1}), L_k(\mathfrak{so}_{2n+1}) \otimes F(2n+1)) \cong \mathcal{W}_{\psi'}(\mathfrak{osp}_{1|2n}),$$

where $k = -2\psi - 2n + 1$ and $\frac{1}{\psi} + \frac{1}{\psi'} = 2$.

Conj: These algebras are lisse and rational.

26. Diagonal cosets in type A

In type B, we can similarly consider the embedding

$$V^{k+1}(\mathfrak{so}_{2n+1}) \to V^k(\mathfrak{so}_{2n+1}) \otimes F(2n+1),$$

which descends to

$$L_{k+1}(\mathfrak{so}_{2n+1}) \to L_k(\mathfrak{so}_{2n+1}) \otimes F(2n+1),$$

when k is admissible.

Thm: (Creutzig, Genra, and Creutzig-L. 2021). For k admissible,

$$\mathrm{Com}(L_{k+1}(\mathfrak{so}_{2n+1}), L_k(\mathfrak{so}_{2n+1}) \otimes F(2n+1)) \cong \mathcal{W}_{\psi'}(\mathfrak{osp}_{1|2n}),$$

where $k = -2\psi - 2n + 1$ and $\frac{1}{\psi} + \frac{1}{\psi'} = 2$.

Conj: These algebras are lisse and rational.

26. Diagonal cosets in type A

In type B, we can similarly consider the embedding

$$V^{k+1}(\mathfrak{so}_{2n+1}) \to V^k(\mathfrak{so}_{2n+1}) \otimes F(2n+1),$$

which descends to

$$L_{k+1}(\mathfrak{so}_{2n+1}) \to L_k(\mathfrak{so}_{2n+1}) \otimes F(2n+1),$$

when k is admissible.

Thm: (Creutzig, Genra, and Creutzig-L. 2021). For k admissible,

$$\mathrm{Com}(L_{k+1}(\mathfrak{so}_{2n+1}), L_k(\mathfrak{so}_{2n+1}) \otimes F(2n+1)) \cong \mathcal{W}_{\psi'}(\mathfrak{osp}_{1|2n}),$$

where $k = -2\psi - 2n + 1$ and $\frac{1}{\psi} + \frac{1}{\psi'} = 2$.

Conj: These algebras are lisse and rational.

27. Diagonal cosets in type C

We have conformal embedding

$$L_1(\mathfrak{sp}_{2k}) \otimes L_k(\mathfrak{sp}_2) \to F(4k),$$

where $F(4k)$ is the algebra of $4k$ free fermions.

The images of $L_1(\mathfrak{sp}_{2k})$ and $L_k(\mathfrak{sp}_2)$ form a dual pair inside $F(4k)$.

For $k \in \mathbb{N}$ and $\ell \in \mathbb{C}$, consider the diagonal coset

$$\mathcal{C}_k^\ell = \mathrm{Com}(V^{\ell-1}(\mathfrak{sp}_{2k}), V^\ell(\mathfrak{sp}_{2k}) \otimes F(4k)),$$

which contains subalgebra $L_k(\mathfrak{sp}_2)$.

$\mathcal{C}_k^\ell$ has the same strong generating type as $\mathcal{F}(4k)^{\mathrm{Sp}_{2k}}$, which is of type $\mathcal{W}(1^3, 2, 3^3, 4, \cdots)$ by classical invariant theory.

So $\mathcal{C}_k^\ell$ is a 1-parameter quotient of $\mathcal{W}^{\mathfrak{sp}_2}$ containing $L_k(\mathfrak{sp}_2)$.

27. Diagonal cosets in type C

We have conformal embedding

$$L_1(\mathfrak{sp}_{2k}) \otimes L_k(\mathfrak{sp}_2) \to F(4k),$$

where $F(4k)$ is the algebra of $4k$ free fermions.

The images of $L_1(\mathfrak{sp}_{2k})$ and $L_k(\mathfrak{sp}_2)$ form a dual pair inside $F(4k)$.

For $k \in \mathbb{N}$ and $\ell \in \mathbb{C}$, consider the diagonal coset

$$\mathcal{C}^{\ell}_k = \mathrm{Com}(V^{\ell-1}(\mathfrak{sp}_{2k}), V^{\ell}(\mathfrak{sp}_{2k}) \otimes F(4k)),$$

which contains subalgebra $L_k(\mathfrak{sp}_2)$.

$\mathcal{C}^{\ell}_k$ has the same strong generating type as $\mathcal{F}(4k)^{\mathrm{Sp}_{2k}}$, which is of type $\mathcal{W}(1^3, 2, 3^3, 4, \cdots)$ by classical invariant theory.

So $\mathcal{C}^{\ell}_k$ is a 1-parameter quotient of $\mathcal{W}^{\mathfrak{sp}_2}$ containing $L_k(\mathfrak{sp}_2)$.

27. Diagonal cosets in type C

We have conformal embedding

$$L_1(\mathfrak{sp}_{2k}) \otimes L_k(\mathfrak{sp}_2) \to F(4k),$$

where $F(4k)$ is the algebra of $4k$ free fermions.

The images of $L_1(\mathfrak{sp}_{2k})$ and $L_k(\mathfrak{sp}_2)$ form a dual pair inside $F(4k)$.

For $k \in \mathbb{N}$ and $\ell \in \mathbb{C}$, consider the diagonal coset

$$\mathcal{C}^\ell_k = \mathrm{Com}(V^{\ell-1}(\mathfrak{sp}_{2k}), V^\ell(\mathfrak{sp}_{2k}) \otimes F(4k)),$$

which contains subalgebra $L_k(\mathfrak{sp}_2)$.

$\mathcal{C}^\ell_k$ has the same strong generating type as $\mathcal{F}(4k)^{\mathrm{Sp}_{2k}}$, which is of type $\mathcal{W}(1^3, 2, 3^3, 4, \cdots)$ by classical invariant theory.

So $\mathcal{C}^\ell_k$ is a 1-parameter quotient of $\mathcal{W}^{\mathfrak{sp}_2}$ containing $L_k(\mathfrak{sp}_2)$.

27. Diagonal cosets in type C

We have conformal embedding

$$L_1(\mathfrak{sp}_{2k}) \otimes L_k(\mathfrak{sp}_2) \to F(4k),$$

where $F(4k)$ is the algebra of $4k$ free fermions.

The images of $L_1(\mathfrak{sp}_{2k})$ and $L_k(\mathfrak{sp}_2)$ form a dual pair inside $F(4k)$.

For $k \in \mathbb{N}$ and $\ell \in \mathbb{C}$, consider the diagonal coset

$$\mathcal{C}_k^\ell = \mathrm{Com}(V^{\ell-1}(\mathfrak{sp}_{2k}), V^\ell(\mathfrak{sp}_{2k}) \otimes F(4k)),$$

which contains subalgebra $L_k(\mathfrak{sp}_2)$.

$\mathcal{C}_k^\ell$ has the same strong generating type as $\mathcal{F}(4k)^{\mathrm{Sp}_{2k}}$, which is of type $\mathcal{W}(1^3, 2, 3^3, 4, \cdots)$ by classical invariant theory.

So $\mathcal{C}_k^\ell$ is a 1-parameter quotient of $\mathcal{W}^{\mathfrak{sp}_2}$ containing $L_k(\mathfrak{sp}_2)$.

27. Diagonal cosets in type C

We have conformal embedding

$$L_1(\mathfrak{sp}_{2k}) \otimes L_k(\mathfrak{sp}_2) \to F(4k),$$

where $F(4k)$ is the algebra of $4k$ free fermions.

The images of $L_1(\mathfrak{sp}_{2k})$ and $L_k(\mathfrak{sp}_2)$ form a dual pair inside $F(4k)$.

For $k \in \mathbb{N}$ and $\ell \in \mathbb{C}$, consider the diagonal coset

$$\mathcal{C}^{\ell}_k = \mathrm{Com}(V^{\ell-1}(\mathfrak{sp}_{2k}), V^{\ell}(\mathfrak{sp}_{2k}) \otimes F(4k)),$$

which contains subalgebra $L_k(\mathfrak{sp}_2)$.

$\mathcal{C}^{\ell}_k$ has the same strong generating type as $\mathcal{F}(4k)^{\mathrm{Sp}_{2k}}$, which is of type $\mathcal{W}(1^3, 2, 3^3, 4, \cdots)$ by classical invariant theory.

So $\mathcal{C}^{\ell}_k$ is a 1-parameter quotient of $\mathcal{W}^{\mathfrak{sp}_2}$ containing $L_k(\mathfrak{sp}_2)$.

28. Diagonal cosets in type C

Thm: $\mathcal{C}^{\ell}_{k}$ is an extension of the product of universal $\mathcal{W}$-algebras $\mathcal{W}^{\phi_1}(\mathfrak{sp}_{2k}) \otimes \mathcal{W}^{\phi_2}(\mathfrak{sp}_{2k})$ where

$$\phi_1 = -(k+1) + \frac{1+\ell+k}{1+2\ell+2k}, \qquad \phi_2 = -(k+1) + \frac{\ell+k}{1+2\ell+2k}.$$

If $\ell - 1$ is admissible for $\mathfrak{sp}_{2k}$, we have an embedding

$$L_{\ell}(\mathfrak{sp}_{2k}) \to L_{\ell-1}(\mathfrak{sp}_{2k}) \otimes F(4k).$$

Moreover, for admissible $\ell - 1$, we have

$$\mathrm{Com}(L_{\ell}(\mathfrak{sp}_{2k}), L_{\ell-1}(\mathfrak{sp}_{2k}) \otimes F(4k)) \cong \mathcal{C}_{k,\ell},$$

where $\mathcal{C}_{k,\ell}$ is the simple quotient of $\mathcal{C}^{\ell}_{k}$.

Conj: For all $k \in \mathbb{N}$ if $\ell - 1$ is admissible and ϕ_1, ϕ_2 are nondegenerate admissible, $\mathcal{C}_{k,\ell}$ is lisse and rational. (True when $\ell \in \mathbb{N}$).

28. Diagonal cosets in type C

Thm: $\mathcal{C}_k^\ell$ is an extension of the product of universal $\mathcal{W}$-algebras $\mathcal{W}^{\phi_1}(\mathfrak{sp}_{2k}) \otimes \mathcal{W}^{\phi_2}(\mathfrak{sp}_{2k})$ where

$$\phi_1 = -(k+1) + \frac{1+\ell+k}{1+2\ell+2k}, \qquad \phi_2 = -(k+1) + \frac{\ell+k}{1+2\ell+2k}.$$

If $\ell - 1$ is admissible for $\mathfrak{sp}_{2k}$, we have an embedding

$$L_\ell(\mathfrak{sp}_{2k}) \to L_{\ell-1}(\mathfrak{sp}_{2k}) \otimes F(4k).$$

Moreover, for admissible $\ell - 1$, we have

$$\mathrm{Com}(L_\ell(\mathfrak{sp}_{2k}), L_{\ell-1}(\mathfrak{sp}_{2k}) \otimes F(4k)) \cong \mathcal{C}_{k,\ell},$$

where $\mathcal{C}_{k,\ell}$ is the simple quotient of $\mathcal{C}_k^\ell$.

Conj: For all $k \in \mathbb{N}$ if $\ell - 1$ is admissible and ϕ_1, ϕ_2 are nondegenerate admissible, $\mathcal{C}_{k,\ell}$ is lisse and rational. (True when $\ell \in \mathbb{N}$).

28. Diagonal cosets in type C

Thm: $\mathcal{C}^{\ell}_{k}$ is an extension of the product of universal $\mathcal{W}$-algebras $\mathcal{W}^{\phi_1}(\mathfrak{sp}_{2k}) \otimes \mathcal{W}^{\phi_2}(\mathfrak{sp}_{2k})$ where

$$\phi_1 = -(k+1) + \frac{1+\ell+k}{1+2\ell+2k}, \qquad \phi_2 = -(k+1) + \frac{\ell+k}{1+2\ell+2k}.$$

If $\ell - 1$ is admissible for $\mathfrak{sp}_{2k}$, we have an embedding

$$L_{\ell}(\mathfrak{sp}_{2k}) \to L_{\ell-1}(\mathfrak{sp}_{2k}) \otimes F(4k).$$

Moreover, for admissible $\ell - 1$, we have

$$\mathrm{Com}(L_{\ell}(\mathfrak{sp}_{2k}), L_{\ell-1}(\mathfrak{sp}_{2k}) \otimes F(4k)) \cong \mathcal{C}_{k,\ell},$$

where $\mathcal{C}_{k,\ell}$ is the simple quotient of $\mathcal{C}^{\ell}_{k}$.

Conj: For all $k \in \mathbb{N}$ if $\ell - 1$ is admissible and ϕ_1, ϕ_2 are nondegenerate admissible, $\mathcal{C}_{k,\ell}$ is lisse and rational. (True when $\ell \in \mathbb{N}$).

28. Diagonal cosets in type C

Thm: $\mathcal{C}^{\ell}_{k}$ is an extension of the product of universal $\mathcal{W}$-algebras $\mathcal{W}^{\phi_1}(\mathfrak{sp}_{2k}) \otimes \mathcal{W}^{\phi_2}(\mathfrak{sp}_{2k})$ where

$$\phi_1 = -(k+1) + \frac{1+\ell+k}{1+2\ell+2k}, \qquad \phi_2 = -(k+1) + \frac{\ell+k}{1+2\ell+2k}.$$

If $\ell - 1$ is admissible for $\mathfrak{sp}_{2k}$, we have an embedding

$$L_{\ell}(\mathfrak{sp}_{2k}) \to L_{\ell-1}(\mathfrak{sp}_{2k}) \otimes F(4k).$$

Moreover, for admissible $\ell - 1$, we have

$$\mathrm{Com}(L_{\ell}(\mathfrak{sp}_{2k}), L_{\ell-1}(\mathfrak{sp}_{2k}) \otimes F(4k)) \cong \mathcal{C}_{k,\ell},$$

where $\mathcal{C}_{k,\ell}$ is the simple quotient of $\mathcal{C}^{\ell}_{k}$.

Conj: For all $k \in \mathbb{N}$ if $\ell - 1$ is admissible and ϕ_1, ϕ_2 are nondegenerate admissible, $\mathcal{C}_{k,\ell}$ is lisse and rational. (True when $\ell \in \mathbb{N}$).

28. Diagonal cosets in type C

Thm: $\mathcal{C}^{\ell}_k$ is an extension of the product of universal $\mathcal{W}$-algebras $\mathcal{W}^{\phi_1}(\mathfrak{sp}_{2k}) \otimes \mathcal{W}^{\phi_2}(\mathfrak{sp}_{2k})$ where

$$\phi_1 = -(k+1) + \frac{1+\ell+k}{1+2\ell+2k}, \qquad \phi_2 = -(k+1) + \frac{\ell+k}{1+2\ell+2k}.$$

If $\ell - 1$ is admissible for $\mathfrak{sp}_{2k}$, we have an embedding

$$L_{\ell}(\mathfrak{sp}_{2k}) \to L_{\ell-1}(\mathfrak{sp}_{2k}) \otimes F(4k).$$

Moreover, for admissible $\ell - 1$, we have

$$\text{Com}(L_{\ell}(\mathfrak{sp}_{2k}), L_{\ell-1}(\mathfrak{sp}_{2k}) \otimes F(4k)) \cong \mathcal{C}_{k,\ell},$$

where $\mathcal{C}_{k,\ell}$ is the simple quotient of $\mathcal{C}^{\ell}_k$.

Conj: For all $k \in \mathbb{N}$ if $\ell - 1$ is admissible and ϕ_1, ϕ_2 are nondegenerate admissible, $\mathcal{C}_{k,\ell}$ is lisse and rational. (True when $\ell \in \mathbb{N}$).

29. An application

Recall: $\mathfrak{sp}_{2(2n+1)}$ has a subalgebra $\mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2$, and

$$\mathfrak{sp}_{2(2n+1)} \cong \mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2 \oplus (\rho_{2\omega_1} \otimes \mathbb{C}^3).$$

Consider $\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$, where $f_{\mathfrak{so}_{2n+1}}$ a principal nilpotent in

$$\mathfrak{so}_{2n+1} \subseteq \mathfrak{sp}_{2(2n+1)} \subseteq \mathfrak{sp}_{2(2n+1)+2m} \subseteq \mathfrak{osp}_{1|2(2n+1)+2m}.$$

For $a \in \mathbb{N}$, let

$$\psi = \frac{3+2a+2m}{2}, \qquad k = a+n+2an+2mn.$$

Affine subalgebra of $\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$ is

$$V^k(\mathfrak{sp}_2) \otimes V^a(\mathfrak{osp}_{1|2m}).$$

29. An application

Recall: $\mathfrak{sp}_{2(2n+1)}$ has a subalgebra $\mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2$, and

$$\mathfrak{sp}_{2(2n+1)} \cong \mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2 \oplus (\rho_{2\omega_1} \otimes \mathbb{C}^3).$$

Consider $\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$, where $f_{\mathfrak{so}_{2n+1}}$ a principal nilpotent in

$$\mathfrak{so}_{2n+1} \subseteq \mathfrak{sp}_{2(2n+1)} \subseteq \mathfrak{sp}_{2(2n+1)+2m} \subseteq \mathfrak{osp}_{1|2(2n+1)+2m}.$$

For $a \in \mathbb{N}$, let

$$\psi = \frac{3+2a+2m}{2}, \qquad k = a+n+2an+2mn.$$

Affine subalgebra of $\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$ is

$$V^k(\mathfrak{sp}_2) \otimes V^a(\mathfrak{osp}_{1|2m}).$$

29. An application

Recall: $\mathfrak{sp}_{2(2n+1)}$ has a subalgebra $\mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2$, and

$$\mathfrak{sp}_{2(2n+1)} \cong \mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2 \oplus (\rho_{2\omega_1} \otimes \mathbb{C}^3).$$

Consider $\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$, where $f_{\mathfrak{so}_{2n+1}}$ a principal nilpotent in

$$\mathfrak{so}_{2n+1} \subseteq \mathfrak{sp}_{2(2n+1)} \subseteq \mathfrak{sp}_{2(2n+1)+2m} \subseteq \mathfrak{osp}_{1|2(2n+1)+2m}.$$

For $a \in \mathbb{N}$, let

$$\psi = \frac{3+2a+2m}{2}, \qquad k = a+n+2an+2mn.$$

Affine subalgebra of $\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$ is

$$V^k(\mathfrak{sp}_2) \otimes V^a(\mathfrak{osp}_{1|2m}).$$

29. An application

Recall: $\mathfrak{sp}_{2(2n+1)}$ has a subalgebra $\mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2$, and

$$\mathfrak{sp}_{2(2n+1)} \cong \mathfrak{so}_{2n+1} \oplus \mathfrak{sp}_2 \oplus (\rho_{2\omega_1} \otimes \mathbb{C}^3).$$

Consider $\mathcal{W}^\psi(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$, where $f_{\mathfrak{so}_{2n+1}}$ a principal nilpotent in

$$\mathfrak{so}_{2n+1} \subseteq \mathfrak{sp}_{2(2n+1)} \subseteq \mathfrak{sp}_{2(2n+1)+2m} \subseteq \mathfrak{osp}_{1|2(2n+1)+2m}.$$

For $a \in \mathbb{N}$, let

$$\psi = \frac{3+2a+2m}{2}, \qquad k = a + n + 2an + 2mn.$$

Affine subalgebra of $\mathcal{W}^\psi(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$ is

$$V^k(\mathfrak{sp}_2) \otimes V^a(\mathfrak{osp}_{1|2m}).$$

30. An application

This is one of the 8 families mentioned earlier, and

$$\mathrm{Com}(V^a(\mathfrak{osp}_{1|2m}), \mathcal{W}^{\psi}(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})^{\mathbb{Z}_2}$$

is a 1-parameter quotient of $\mathcal{W}^{\mathfrak{sp}_2}$.

Assuming the uniqueness of $\mathcal{W}^{\mathfrak{sp}_2}$ as a 2-parameter VOA, such quotients are classified by their truncation curves.

Conj: For all $a, k \in \mathbb{N}$, $\mathcal{W}_{\psi}(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$ has affine subalgebra

$$L_k(\mathfrak{sp}_2) \otimes L_a(\mathfrak{osp}_{1|2m}).$$

30. An application

This is one of the 8 families mentioned earlier, and

$$\mathrm{Com}(V^a(\mathfrak{osp}_{1|2m}), \mathcal{W}^{\psi}(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})^{\mathbb{Z}_2}$$

is a 1-parameter quotient of $\mathcal{W}^{\mathfrak{sp}_2}$.

Assuming the uniqueness of $\mathcal{W}^{\mathfrak{sp}_2}$ as a 2-parameter VOA, such quotients are classified by their truncation curves.

Conj: For all $a, k \in \mathbb{N}$, $\mathcal{W}_{\psi}(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$ has affine subalgebra

$$L_k(\mathfrak{sp}_2) \otimes L_a(\mathfrak{osp}_{1|2m}).$$

30. An application

This is one of the 8 families mentioned earlier, and

$$\mathrm{Com}(V^a(\mathfrak{osp}_{1|2m}), \mathcal{W}^\psi(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})^{\mathbb{Z}_2}$$

is a 1-parameter quotient of $\mathcal{W}^{\mathfrak{sp}_2}$.

Assuming the uniqueness of $\mathcal{W}^{\mathfrak{sp}_2}$ as a 2-parameter VOA, such quotients are classified by their truncation curves.

Conj: For all $a, k \in \mathbb{N}$, $\mathcal{W}_\psi(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$ has affine subalgebra

$$L_k(\mathfrak{sp}_2) \otimes L_a(\mathfrak{osp}_{1|2m}).$$

31. An application

Kovalchuk's formulas for truncation curves imply

$$\mathrm{Com}(L_a(\mathfrak{osp}_{1|2m}), \mathcal{W}_\psi(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})^{\mathbb{Z}_2} \cong \mathcal{C}_{k,r},$$

with $r = -(k+1) + \frac{3+2a+2m}{2}$.

$\mathcal{C}_{k,r}$ should be an extension of $\mathcal{W}_{\phi_1}(\mathfrak{sp}_{2k}) \otimes \mathcal{W}_{\phi_2}(\mathfrak{sp}_{2k})$ with

$$\phi_1 = \frac{3+2k+2m-2n-4an-4mn}{4(1+k+m-n-2an-2mn)},$$

$$\phi_2 = \frac{1+2k+m(2-4n)-2n-4an}{4(1+k+m-n-2an-2mn)}.$$

This suggest that $\mathcal{W}_\psi(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$ should be lisse and rational for a, n sufficiently large.

31. An application

Kovalchuk's formulas for truncation curves imply

$$\mathrm{Com}(L_a(\mathfrak{osp}_{1|2m}), \mathcal{W}_\psi(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})^{\mathbb{Z}_2} \cong \mathcal{C}_{k,r},$$

with $r = -(k+1) + \frac{3+2a+2m}{2}$.

$\mathcal{C}_{k,r}$ should be an extension of $\mathcal{W}_{\phi_1}(\mathfrak{sp}_{2k}) \otimes \mathcal{W}_{\phi_2}(\mathfrak{sp}_{2k})$ with

$$\phi_1 = \frac{3+2k+2m-2n-4an-4mn}{4(1+k+m-n-2an-2mn)},$$

$$\phi_2 = \frac{1+2k+m(2-4n)-2n-4an}{4(1+k+m-n-2an-2mn)}.$$

This suggest that $\mathcal{W}_\psi(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$ should be lisse and rational for a, n sufficiently large.

31. An application

Kovalchuk's formulas for truncation curves imply

$$\mathrm{Com}(L_a(\mathfrak{osp}_{1|2m}), \mathcal{W}_\psi(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})^{\mathbb{Z}_2} \cong \mathcal{C}_{k,r},$$

with $r = -(k+1) + \frac{3+2a+2m}{2}$.

$\mathcal{C}_{k,r}$ should be an extension of $\mathcal{W}_{\phi_1}(\mathfrak{sp}_{2k}) \otimes \mathcal{W}_{\phi_2}(\mathfrak{sp}_{2k})$ with

$$\phi_1 = \frac{3+2k+2m-2n-4an-4mn}{4(1+k+m-n-2an-2mn)},$$

$$\phi_2 = \frac{1+2k+m(2-4n)-2n-4an}{4(1+k+m-n-2an-2mn)}.$$

This suggest that $\mathcal{W}_\psi(\mathfrak{osp}_{1|2(2n+1)+2m}, f_{\mathfrak{so}_{2n+1}})$ should be lisse and rational for a, n sufficiently large.