Quasi-lisse vertex (super)algebras

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Based on papers: Quasi-lisse vertex (super)algebras, Hao Li

Spectral flow, twisted modules and MLDE of quasi-lisse vertex algebras
Bohan Li, Hao Li, Yan Wenbin (arxiv.org/pdf/2304.09681)

Overview

- Motivations
- Main results
- Applications
- Future works

Motivations

- Vertex algebras and their representations are the rigorous mathematical formulation of two dimensional conformal field theory of certain type.
- [Zhu96] One can associate every vertex algebra V with a Poisson algebra R_V called the C_2-algebra. Genus-1 1-point correlation functions satisfy some modular differential equations if $\dim R_V < \infty$.
- [Ara12] Arakawa first studied R_V from the geometric perspective by thinking of it as the coordinate ring of the associated variety of V, X_V. He showed that the C_2-confiniteness condition is equivalent with the singular support of the corresponding vertex algebra is trivial. [AK18] Arakawa and Kawasetsu discovered that one can still derive the MLDE if we enlarge X_V to have finitely many symplectic leaves. The corresponding vertex algebra is called the quasi-lisse vertex algebra.

Motivations

- In [BLL+15], Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees constructed a remarkable map:

$$\chi : \begin{Bmatrix} \text{Four dimensional } \mathcal{N} = 2 \\ \text{superconformal field theory} \\ \text{(4d SCFT)} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{Two dimensional} \\ \text{chiral algebra(VOA)} \end{Bmatrix}$$

It was conjectured by Beem and Rastelli that the Higgs branch of 4d SCFT should coincide with the associated variety of the corresponding 2d CFT. One can identify the 4d Higgs branch with 3d Coulomb branch via 3D mirror symmetry. If the mirror theory is quiver gauge theory, Coulomb branch is defined by Braverman, Finkelberg and Nakajima which is expected to have finitely many symplectic leaves. There is a demand to study these type vertex operator algebras and the category of their modules.

Motivations

- Recently Robert McRae [McR21] proved that for some proper C_2-cofinite vertex operator algebra, its tensor category of grading-restricted generalized V-modules is rigid. He used the method of Huang-Moore-Seiberg by expanding 2-point genus-1 correlation functions to derive some identities of 1-point correlation functions. In order to show the rigidity of the category of the modules for quasi-lisse vertex algebras, the convergence of their genus-1 n-point functions is crucial.

Motivating example

Quasi-lisse vertex operator algebras can be viewed as a vertex algebraic generalization of vacuum representations of affine Lie algebras at admissible level, since Arakawa showed that their associated varieties are the closure of certain nilpotent orbits.

Example

Simple affine vertex algebra $L_{-\frac{4}{3}}(\mathfrak{sl}_2)$ has three irreducible modules in category $\mathcal{O}$, i.e., $\{\lambda_1 := L(-\frac{4}{3}\Lambda_0), \lambda_2 := L(-\frac{2}{3}\Lambda_0 - \frac{2}{3}\Lambda_1), \lambda_3 := L(-\frac{4}{3}\Lambda_1)\}$.
The character

$$\mathrm{tr}_{L_{\lambda_i}} q^{L_{(0)} - \frac{c}{24}}$$

is not well-defined when $i = 2, 3$.
We need consider

$$\chi_\lambda(\tau) = \mathrm{tr}_{L_\lambda}(e^{\frac{1}{2}\pi i h_{(0)}} q^{L_{(0)} - \frac{c}{24}})$$

where h is the standard Cartan element of $\mathfrak{sl}_2$. Or we can consider the character of the $\mathbb{Z}/2\mathbb{Z}$-twisted module, $\sigma^{-\frac{1}{2}}(\chi_{\lambda_i}(\tau))$, which is well-defined.

Motivating example

Example

From this example one can see that in order to study quasi-lisse vertex algebras, it is helpful to take the twsited modules into consideration. It is also worth mentioning that $L_{-\frac{4}{3}}(\mathfrak{sl}_2)$ can be constructed from Argyres-Douglas (A_1, D_3) theory and the $\sigma^{-\frac{1}{2}}(\chi_{\lambda_i}(\tau))$ coincides with lens space index of (A_1, D_3) theory which is studied by Fluder and Song [FS18].

Part I

- The new finiteness condition for the convergence of genus-0 and genus-1 correlation functions
- Quasi-lisse vertex (super)algebras (definitions and properties)
- Corollary and Examples

Definition of vertex operator algebras

Definition

A vertex superalgebra contains the following data: a vector space of states V, the vacuum vector $\mathbf{1} \in V_{\bar{0}}$, derivation T, and the state-field correspondence map

$$a \longmapsto Y(a,x) = \sum_{n\in\mathbb{Z}} a_{(n)} x^{-n-1},$$

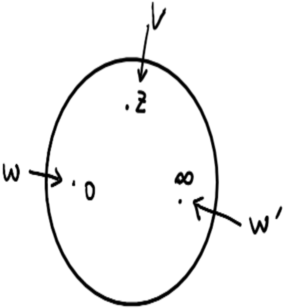
satisfying translation coinvariance, vacuum property and the Jacobi identity:

$$x_0^{-1}\delta(\frac{x_1 - x_2}{x_0})Y(u,x_1)Y(v,x_2) - (-1)^{|u||v|}x_0^{-1}\delta(\frac{-x_2 + x_1}{x_0})Y(v,x_2)Y(u,x_1)$$
$$= x_2^{-1}\delta(\frac{x_1 - x_0}{x_2})Y(Y(u,x_0)v),x_2).$$

We call V a vertex operator superalgebra if it has a conformal vector ω.

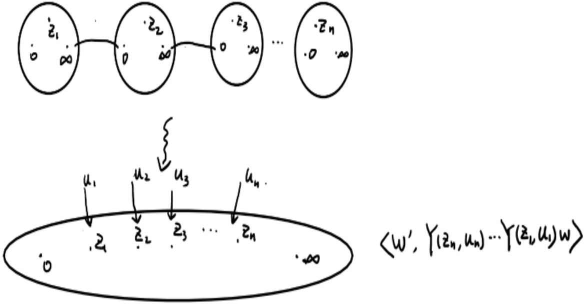
Analytic intuition

Let $v, w \in V$, $w \in V'$.



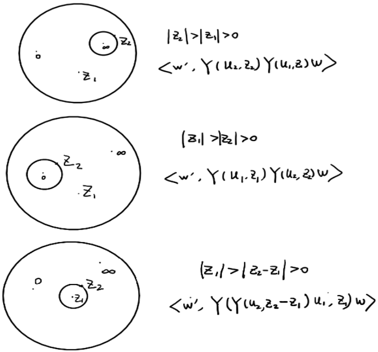
$$\langle w', Y(v,z)w \rangle$$

Analytic intuition



Analytic intuition

The expansions of the 4-point function in the different regions in the configuration space:



The Jacobi identity is equivalent to the residue theorem in complex analysis.

Twisted modules of vertex algebras

Definition

Let g be an automophism of V of order T. A weak g-twisted V-module is a $\mathbb{C}$-linear space M equipped with a linear map

$$V \to \mathrm{End}(V)[[z^{\frac{1}{T}}, z^{-\frac{1}{T}}]],$$
$$v \mapsto Y_M(v,x) = \sum_{n\in\mathbb{Q}} v_{(n)}x^{-n-1}$$

satisfying lower truncation property, vacuum property, formal monodromy condition and

$$x_0^{-1}\delta(\frac{x_1-x_2}{x_0})Y_M(v,x_1)Y_M(w,x_2)$$
$$-(-1)^{|v||w|}x_0^{-1}\delta(\frac{x_0-x_1}{-x_0})Y_M(v,x_2)Y_M(u,x_1)$$
$$=x_2^{-1}(\frac{x_1-x_0}{x_2})^{-\frac{r}{T}}\delta(\frac{x_1-x_0}{x_2})Y_M(Y(u,x_0)v,x_2).$$

Intertwining operators

Let (W_1, Y_1) be a weak module of vertex operator superalgebra V, and (W_j, Y_j), $j = 2, 3$ be weak g-twisted V-modules, where the order of g is T.

Definition

An intertwining operator of type $\binom{W_3}{W_1\ W_2}$ is a linear map

$$\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3\{x\}$$

satisfying lower truncation property, $L_{(0)}$-grading condition, $L_{(-1)}$-derivative condition and the Jacobi identity for $u \in V^k$

$$x_0^{-1}\delta(\frac{x_1 - x_2}{x_0})Y_3(u, x_1)\mathcal{Y}(w^1, x_2) - (-1)^{|u||w^1|}x_0^{-1}\delta(\frac{-x_2 + x_1}{x_0})\mathcal{Y}(w^1, x_2)Y_2(u, x_1) = x_2^{-1}\delta(\frac{x_1 - x_0}{x_2})(\frac{x_1 - x_0}{x_2})^{-\frac{k}{T}}\mathcal{Y}(Y_1(u, x_0)w^1, x_2).$$

The general idea

Given (weak) V-modules W_i, we have a natural map ψ

$$W_1 \otimes \cdots \otimes W_n \mapsto \text{n-point correlation functions.}$$

The main idea is to use Jacobi identity to derive the elements J in the kernel of this map such that $W_1 \otimes \cdots \otimes W_n / J$ is finitely generated over some Noetherian ring.

The more elements in the kernel we find, the weaker convergence condition we can obtain. This is our main task.

Genus-0 correlation functions

Huang [Hua05] previously obtained the following elements in the kernel:

$$\mathcal{A}(u, w_0, w_1, w_2, w_3) = -w_0 \otimes u_{(-1)} w_1 \otimes w_2 \otimes w_3 + \cdots,$$

$$\mathcal{B}(u, w_0, w_1, w_2, w_3) = -w_0 \otimes w_1 \otimes u_{(-1)} w_2 \otimes w_3 + \cdots$$

$$\mathcal{C}(u, w_0, w_1, w_2, w_3) = u^*_{(-1)} w_0 \otimes w_1 \otimes w_2 \otimes w_3 + \cdots$$

$$\mathcal{D}(u, w_0, w_1, w_2, w_3) = u_{(-1)} w_0 \otimes w_1 \otimes w_2 \otimes w_3 + \cdots$$

Genus-0 correlation functions

Here, using commutator formula twice, we get something more:

$$\begin{aligned}&\mathcal{E}(u, w_0, w_1, w_2, w_3)\\ &\quad = u^*_{(0)}w_0 \otimes w_1 \otimes w_2 \otimes w_3 - w_0 \otimes w_1 \otimes w_2 \otimes u_{(0)}w_3\\ &\quad - w_0 \otimes u_{(0)}w_1 \otimes w_2 \otimes w_3 - w_0 \otimes w_1 \otimes u_{(0)}w_2 \otimes w_3\end{aligned}$$

Let V be of CFT type and self-dual. If the weight of u is 1, using

$$\langle u^*_{(m)}w_1, w_2\rangle = -\sum_{i=0}^{\infty} \frac{1}{i!}\langle w_1, (L^i_{(1)}u)_{(-m-i)}w_2\rangle,$$

where $w_1 \in W'$ and $w_2 \in W$, we have $u^*_{(0)} = -u_{(0)}$.

Genus-0 correlation functions

Assume V is strongly generated by weight 1 vectors. By using proper filtration, we can construct a surjective map

$$\frac{R_{W_0} \otimes R_{W_1} \otimes R_{W_2} \otimes R_{W_3}}{\{R_V, R_{W_0} \otimes R_{W_1} \otimes R_{W_2} \otimes R_{W_3}\}} \twoheadrightarrow W_0 \otimes \cdots \otimes W_3 / gr(J).$$

Theorem (Li, Hao)

Let V be of CFT type, self-dual and strongly generated by weight 1 vectors. Let W_i for $i = 0, 1, 2, 3$ be weak $\mathbb{R}$-graded V-modules. Suppose $dim R_{W_0} \otimes_{R_V} \cdots \otimes_{R_V} R_{W_3}/\{R_V, R_{W_0} \otimes_{R_V} \cdots \otimes_{R_V} R_{W_3}\} < \infty$. Then for any $w_i \in W_i$ $(i = 0, 1, 2, 3)$, there exist

$$a_k(z_1, z_2), b_l(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}, (z_1 - z_2)^{-1}]$$

for $k = 1, ..., m$ and $l = 1, ..., n$ such that for any discretely $\mathbb{R}$-graded weak V-modules W_4, W_5 and W_6, any intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5$ and $\mathcal{Y}_6$ of types

$$\binom{W_0'}{W_1\, W_4}, \binom{W_4}{W_2\, W_3}, \binom{W_5}{W_1\, W_2}, \binom{W_0'}{W_5\, W_3}, \binom{W_0'}{W_2\, W_6} \text{ and } \binom{W_6}{W_1\, W_3},$$

respectively, the series

Genus-0 correlation functions

Theorem (Li, Hao)

$$\langle w_0, \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle,$$
$$\langle w_0, \mathcal{Y}_4(\mathcal{Y}_3(w_1, z_1 - z_2)w_2, z_2)w_3\rangle,$$

and

$$\langle w_0, \mathcal{Y}_5(w_2, z_2)\mathcal{Y}_6(w_1, z_1)w_3\rangle,$$

are convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$ *and* $|z_2| > |z_1| > 0$, *respectively, and can be analytically extended to multivalued functions in the region*

$$\{(z_1, z_2) \in \mathbb{C}^2 | z_1, z_2 \neq 0, z_1 \neq z_2\}.$$

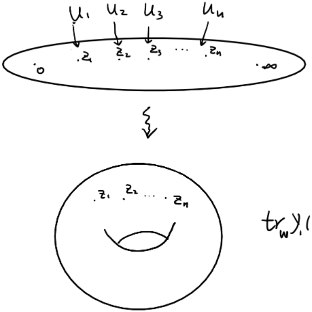
Genus-1 correlation functions

There are two local coordinates for a torus. One is described in terms of annuli $\mathbb{C} \setminus \{0\} / \{w \sim wq^n\}$ where q is a complex number. The other one is in terms of parallelogram $\mathbb{C}/\{m\tau + n\}$, $(\mathrm{im}\tau > 0,\ m, n \in \mathbb{Z})$. They are related by coordinate transformation $\phi(x) = e^{2\pi ix} - 1$.
On the torus, one need consider the geometrically modified intertwining operator

$$\mathcal{Y}(\mathcal{U}(x)v, x)$$

where $\mathcal{U}(x) = (2\pi ix)^{L_{(0)}} e^{-L^{+}(A)} \in \mathrm{End}\, M\{x\}$.

Genus-1 correlation functions



$$\langle w', \mathcal{Y}_1(u_n, z_n) \cdots \mathcal{Y}_n(u_1, z_1) w \rangle$$

$$\mathrm{tr}_W \mathcal{Y}_1(\mathcal{U}(z_n) u_n, z_n) \cdots \mathcal{Y}_n(\mathcal{U}(z_1) u_1, z_1)\, q^{L_0 - \frac{c}{24}}$$

Genus-1 correlation functions

Let h be a current satisfying

$$L(n)h = \delta_{n,0}h, \quad h_{(n)}h = k\delta_{n,1}\mathbf{1},$$

for all $n \geq 0$ and for $k \in \mathbb{C}$ such that $h_{(0)}$ acts semisimply on $\tilde{W}_n$. We assume that $\mathrm{tr}_{\tilde{W}_n} q_s^{h_{(0)}} q^{L_{(0)}}$, where $q_s = e^{2\pi i s}$, is a well-defined (q_s, q)-series.

Genus-1 correlation functions

Let g be of order T and $h_{(0)}(u) = \lambda^{-1}u$ for $\mathbb{Z}_2$-homogeneous element $u \in V^r$. For any $\mathbb{Z}_2$-homogeneous elements, $u_i \in W_i$, $i = 1, ..., n$, we denote $e^{2\pi i \frac{r}{T}}$ by μ and $|u|(|u_1| + \cdots + |u_n|)$ by t. Let $\mathcal{Y}_i$, $i, 2, ..., n$ be intertwining operator of type $\binom{\tilde{W}_{i-1}}{W_i\ \tilde{W}_i}$, respectively. Let $\theta = e^{2\pi i s \lambda^{-1}}$, $\phi = \mu$, and $\phi(h) = e^{2\pi i s h_{(0)}}$. Assume the genus-0 correlation functions are convergent, and commutativity and associativity hold.

Theorem (Li, Hao)

For any integer j satisfying $1 \leq j \leq n$ and any $l \in \mathbb{Z}_+$, in $\mathbb{G}_{|q_{z_1}|>\cdots>|q_{z_n}|>0}$, when $u \in V^0$, we have

$$\sum_{i=1}^{n} tr_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(z_1)u_1, z_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(z_{i-1})u_{i-1}, z_{i-1}) \mathcal{Y}_i(\mathcal{U}(z_i)u_{(0)}u_i, z_i) \cdot$$

$$\cdot \mathcal{Y}_{i+1}(\mathcal{U}(z_{i+1})u_{i+1}, z_{i+1}) \cdots \mathcal{Y}_n(\mathcal{U}(z_n)u_n, z_n)\phi(h)q^{L_{(0)}} = 0,$$

Genus-1 correlation functions

Theorem

when $\phi = 1$ *and* $(-1)^t\theta \neq 1$,

$$tr_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})u_1, q_{z_1})\cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})u_{j-1}, q_{z_{j-1}})\cdot\mathcal{Y}_j(\mathcal{U}(q_{z_j})u_{(-l)}u_j, q_{z_j})\cdot$$
$$\cdot\mathcal{Y}_{j+1}(\mathcal{U}(q_{z_{j+1}})u_{j+1}, q_{z_{j+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})u_n, q_{z_n})\phi(h)q^{L_{(0)}}$$
$$= \sum_{0\leq n-l}((-1)^{l-1}\binom{n-1}{n-l}\tilde{G}_n\begin{bmatrix}(-1)^t\theta\\ \phi\end{bmatrix}(q) - \delta_{n,l}2\pi i\frac{1}{1-\theta^{-1}})$$
$$tr_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})u_1, q_{z_1})\cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})u_{j-1}, q_{z_{j-1}})$$
$$\cdot\mathcal{Y}_j(\mathcal{U}(q_{z_j})u_{(n-l)}u_j, q_{z_j})\mathcal{Y}_{j+1}(\mathcal{U}(q_{z_{j+1}})u_{j+1}, q_{z_{j+1}})\cdot$$
$$\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})u_n, q_{z_n})\phi(h)q^{L_{(0)}} +$$

Genus-1 correlation functions

Theorem

$$\sum_{i\neq j}\sum_{m\geq 0}((-1)^{m+1}\binom{m+l-1}{l-1}\tilde{P}_{m+l}\begin{bmatrix}(-1)^t\theta\\ \phi\end{bmatrix}(z_i-z_j,q)-\delta_{0,m}2\pi i\frac{1}{1-\theta^{-1}})$$
$$tr_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})u_1,q_{z_1})\cdots\mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})u_{i-1},q_{z_{i-1}})\mathcal{Y}_i(\mathcal{U}(q_{z_i})u_{(m)}u_i,q_{z_i})\cdot$$
$$\cdot Y_{i+1}(\mathcal{U}(q_{z_{i+1}})u_{i+1},q_{z_{i+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})u_n,q_{z_n})\phi(h)q^{L_{(0)}}$$
$$+\delta_{l,1}\delta_{(-1)^t\theta^{-1},1}tr_{\tilde{W}_n}o(\mathcal{U}(1)u)\mathcal{Y}_1(\mathcal{U}(q_{z_1})u_1,q_{z_1})\cdots Y_n(\mathcal{U}(q_{z_n})u_n,q_{z_n})\phi(h)q^{L_{(0)}}$$

otherwise, the LHS of the identity remains the same and one omits the terms $\delta_{k,l}2\pi i\frac{1}{1-\theta^{-1}}$ *and* $\delta_{0,m}2\pi i\frac{1}{1-\theta^{-1}}$ *on RHS.*

Genus-1 correlation functions

We get more elements in the kernel of ψ

- $$u_1 \otimes \cdots \otimes u_{(-2)} u_i \otimes \cdots \otimes u_n + \cdots ,$$
  when $\theta = \phi = 1$ and $t = 0$,
- $$\sum_{i=1}^{n} u_1 \otimes \cdots u_{(0)} u_i \cdots \otimes u_n$$
  when $\phi = 1$,
- $$u_1 \otimes \cdots \otimes u_{(-2)} u_i \otimes \cdots \otimes u_n + \cdots$$
- $$u_1 \otimes \cdots \otimes u_{(-1)} u_j \otimes \cdots \otimes u_n + \cdots$$
  $$= u_1 \otimes \cdots \otimes u_{(-1)} u_s \otimes \cdots \otimes u_n + \cdots$$
  when $\phi = 1$ and $(-1)^t \theta \neq 1$,

- $$u_1 \otimes \cdots \otimes u_{(-2)} u_i \otimes \cdots \otimes u_n + \cdots$$

  when $\phi \neq 1$, and

  $$\begin{aligned} & u_1 \otimes \cdots \otimes u_{(-1)} u_j \otimes \cdots \otimes u_n + \cdots \\ = \; & u_1 \otimes \cdots \otimes u_{(-1)} u_s \otimes \cdots \otimes u_n + \cdots \end{aligned}$$

  for $1 \leq s, t \leq n$ when $\phi \neq 1$ or $(-1)^t \theta = 1$.

Using certain filtration, one gets a surjective map

$$R_{W_1} \otimes_{R_V} \cdots \otimes_{R_V} R_{W_n} / \{(R_V)^{\mathfrak{g}}, R_{W_1} \otimes_{R_V} \cdots \otimes_{R_V} R_{W_n}\} \twoheadrightarrow T/gr(J).$$

Genus-1 correlation functions

$T/gr(J)$ is a module over the ring $R' = N_s(T, T_1) \otimes \mathbb{C}[\tilde{P}_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z_i - z_j; q), \tilde{P}_2 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z_i - z_j; q), \tilde{\wp}_2(z_i - z_j; q), \tilde{\wp}_3(z_i - z_j; q)]_{i,j=1,\cdots,n, i\neq j}$, where $N_s(T, T_1) := \mathbb{C}[\tilde{G}_{2k}(q) | k \geq 2] \otimes \mathbb{C}[\tilde{G}_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (q) | k \geq 0]$. By obtaining the following identity

$$P_2 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau) - P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau)^2 - 2G_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z, \tau) = 3G_2 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) + G_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau)^2,$$

we can show that

Lemma

R' is Noetherian.

Genus-1 correlation functions

Theorem (Li, Hao)

If $\dim(R_{W_1} \otimes_{R_V} \cdots \otimes_{R_V} R_{W_n}/\{(R_V)^g, R_{W_1} \otimes_{R_V} \cdots \otimes_{R_V} R_{W_n}\}) < \infty$ *and* $\mathcal{Y}_i$, $i = 1, \cdots, n$, *are intertwining operators of type* $\binom{\tilde{W}_{i-1}}{W_i \;\; \tilde{W}_i}$, *where* $\tilde{W}_0 = \tilde{W}_n$. *Let* h *be an automorphism of* V.

- *For any homogeneous elements* $w_i \in W_i$, $i = 1, ..., n$, *there exist* $a_{p,i}(z_1, ..., z_n; q; h) \in R'_p$, $b_{p,i}(z_1, ..., z_n; q; h) \in R'_{2p}$, $c_{p,i}(z_1, ..., z_n; q; h) \in R'_l \otimes (\mathbb{C}[z_1, ..., z_n])_m$ $(l + m = p)$ *for* $p = 1, ..., m$ *and* $i = 1, ..., n$ *such that* $F_{\mathcal{Y}_1,...,\mathcal{Y}_n}(w_1, ..., w_n; z_1, ..., z_n; q; h)$ *satisfies the following system of differential equations:*

$$\frac{\partial^m \psi}{\partial z_i^m} + \sum_{p=1}^{m} a_{p,i}(z_1, ..., z_n; q; h) \frac{\partial^{m-p} \psi}{\partial z_i^{m-p}} = 0,$$

Genus-1 correlation functions

Theorem

- $$\prod_{k=1}^{m}\mathcal{O}_i^h\Big(\sum_{i=1}^{n} wtw_i + 2(m-k)\Big)\psi + \sum_{p=1}^{m} b_{p,i}(z_1,\dots,z_n;q;h)\prod_{k=1}^{m-p}\mathcal{O}_i^h\Big(\sum_{i=1}^{n} wtw_i + 2(m-p-k)\Big)\psi = 0,$$
- $$\Big(q_s\frac{\partial}{\partial q_s}\Big)^m\psi + \sum_{p=1}^{m} c_{p,i}(z_1,\dots,z_n;q;h)\Big(q_s\frac{\partial}{\partial q_s}\Big)^{m-p}\psi = 0,$$

$i = 1,\dots,n$, in the regions
$1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q| > 0, 0 < |q_s| < 1$ *where for any* $\alpha \in \mathbb{C}$.

Genus-1 correlation functions

Theorem

- *In the region* $\{(z_1, ..., z_n, \tau, s) | 1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0, 0 < |q_s| < 1\}$, $F_{\mathcal{Y}_1,...,\mathcal{Y}_n}(w_1, ..., w_n; z_1, ..., z_n; q; h)$ *is absolutely convergent and can be analytically extended to a multivalued analytic function in the region given by* $\tau \in \mathbb{H}$, $0 < |q_s| < 1$, $z_i \neq z_j + k\tau + l$ *for* $i, j = 1, ..., n$, $i \neq j$, $k, l \in \mathbb{Z}$.

Quasi-lisse vertex (super)algebras

Let X be an affine Poisson variety, i.e., the global sections $\mathcal{O}(X)$ is a Poisson algebra. Given any element $f \in \mathcal{O}(X)$, one can define a Hamiltonion vector field $\mathcal{E}_f := \mathrm{ad} f(\cdot) = \{f, \cdot\}$ on X. So we have a natural map

$$\alpha : \mathcal{O}(X) \to \mathrm{Vect}(X),$$

given by $\alpha(f) = \mathcal{E}_f$, where $\mathrm{Vect}(X)$ are vector fields on X.

Definition

A maximally locally closed connected subvariety $Z \subset X$ is called a symplectic leave if $\alpha_x : \mathcal{O}(X) \to T_x(X)$ is surjective for all $x \in Z$.

Definition

A finitely strongly generated vertex (super)algebra V is called lisse if $\mathrm{dimSpec}(R_V) = 0$. It is called quasi-lisse if the poisson (super)variety $(\tilde{X}_V)_{red}$ has finitely many symplectic leaves.

Varieties with finitely many symplectic leaves

Example

- $\mathbb{C}^{2n}$, cotangent bundle of any smooth manifold, the coadjoint orbit $\mathbb{O} \in \mathfrak{g}^*$, etc.
- [ES18] Given a normal variety X, if it admits a symplectic resolution $\rho : \tilde{X} \to X$, i,e., $\tilde{X}$ admits a global nondegenerate closed two-form, X has finitely many symplectic leaves. Examples include Nakajima quiver varieties (Nilcone $\mathcal{N} = \coprod \mathbb{O}_i$ and Slodowy slices in a complex semisimple Lie algebra, Kleinian singularities, $\mathbb{C}^2/S_n$, etc.).
- (Conjecture) The associated varieties of the vertex operator algebras coming from $4D$ theory have finitely many symplectic leaves, including genus zero Moore-Tachikawa symplectic varieties.

Varieties with finitely many symplectic leaves

Theorem (Etingof, Schedler)

[ES10] Let X be an affine Poisson variety. Let G be a finite group of Poisson automorphisms of X. If X has finitely many symmplectic leaves, then

- $N/\{\mathcal{O}_X, N\}$ *is finite dimensional for any coherent sheaf of Poisson modules N over $\mathcal{O}_X$.*
- $\mathcal{O}_X/\left\{\mathcal{O}_{X/G}, \mathcal{O}_X\right\}$ *is finite dimensional. In particular, the G-invariants,*

$$(\mathcal{O}_X(X))^G/\left\{(\mathcal{O}_X(X))^G, (\mathcal{O}_X(X))^G\right\}$$

  is finite-dimensional.

Applications

Theorem (Li, Hao)

The genus 1 n-point functions for the highest weight modules of $L_k(\mathfrak{sl}_2)$ at admissible level and their contragredient modules satisfy the partial differential modular equations.

Use the induction and the fact R_V has the quotient relation

$$e^p(ef - h^2)$$

for some p.

Applications

Theorem (Li, Hao)

Let V be a vertex superalgebra and W be its h-stable weak g-twisted module with $o(g) = T$ and $o(e^{2\pi i h_{(0)}}) = T'$ satisfying the condition

$$dim(R_W/\{(R_V)^g, R_W\}) < \infty.$$

For any homogeneous element $w \in W$, there exist

$$b_p(z; q; h) \in N_s(T, T_1)_{2p}, \quad c_p(z; q; h) \in N_s(T, T_1)_l \otimes (\mathbb{C}[z])_m \ \ (l+m = p),$$

for $p = 1, ..., m$ such that $F_{\mathcal{Y}}(w; z; q; h)$ satisfies the following differential equation

$$\partial^m F_{\mathcal{Y}}(w; z; q; h) + \sum_{p=1}^{m} b_p(z; q_\tau; h) \partial^{m-p} F_{\mathcal{Y}}(w; z; q_\tau; h) = 0$$

Applications

Theorem

$$(q_s \frac{\partial}{\partial q_s})^m F_{\mathcal{Y}}(w; z; q_\tau; h) + \sum_{p=1}^{m} c_p(z; q_\tau; h)(q_s \frac{\partial}{\partial q_s})^{m-p} F_{\mathcal{Y}}(w; z; q_\tau; h) = 0$$

in the regions $1 > |q_z| > |q_\tau| > 0, 0 < |q_s| < 1.$

Applications

Corollary

If V is a quasi-lisse vertex superalgebra, then the supercharacter of its simple g-twisted module satisfies the twisted modular linear differential equation.

Let $\bar{\Theta}_{r,s} = \theta_2^{4r}\theta_3(\tau)^{4s} + \theta_2(\tau)^{4s}\theta_3(\tau)^{4r}$, and
$D_q^{(k)}\chi(q) := \partial_{(2k-2)} \circ \cdots \circ \partial_{(2)} \circ \partial_{(0)}\chi(q)$.

Example

There are two independent characters of $e^{\frac{1}{2}\pi i h_{(0)}}$-twisted $L_{-\frac{4}{3}}(\mathfrak{sl}_2)$-modules, they satisfy the following twisted MLDE:

$$(D_{(q)}^{(2)} - \frac{1}{96}\bar{\Theta}_{1,1}(\tau))\mathrm{ch}[\sigma^{\frac{1}{2}\bar{\Lambda}^\vee}](q) = 0.$$

Applications(Li, Bohan, Li, Hao, Yan, Wenbin)

Type $A_2^{(1)}$ at the boundary admissible level $k = -\frac{3}{2}$.

$$\left(D_q^{(2)} - \frac{5}{576}\bar{\Theta}_{0,2}(\tau) - \frac{11}{576}\bar{\Theta}_{1,1}(\tau)\right) \mathrm{ch}\left[\sigma^{\frac{1}{2}\bar{\Lambda}_1^\vee}\left(\mathcal{L}\left(-\frac{3}{2}\Lambda_0\right)\right)\right](q) = 0.$$

(Digress: the character of the twisted modules $\sigma^{\frac{1}{3}\bar{\Lambda}_1^\vee + \frac{1}{3}\bar{\Lambda}_2^\vee}(\mathcal{L}(-\frac{\rho}{2}))$ coincides with the lens space index of (A_1, D_4) AD theory.)

Type D_4 at nonadmissible level $k = -2$

$$\left(D_q^{(2)} + \frac{1}{144}\bar{\Theta}_{0,2}(\tau) - \frac{37}{288}\bar{\Theta}_{1,1}(\tau)\right) \mathrm{ch}[\sigma^{\frac{1}{2}\bar{\Lambda}_2^\vee}(\mathcal{L}_{\mathfrak{d}_4}(-\Lambda_2))](q) = 0,$$

(Digress: According to the 4D-2D correspondence, Yiwen Pan and his collabrators give a conjectural full character of vacuum module of the $\mathcal{L}_{\mathfrak{d}_4}(-2\Lambda_0)$ in terms of $\eta(\tau)$, $\theta_1(z,\tau)$ and $E_2\left(\frac{1}{z}\right)$. By using spectral flow along the direction $\Lambda_1, \Lambda_3, \Lambda_4$, we could obtain the characters of modules $\mathcal{L}_{\mathfrak{d}_4}(-2\Lambda_1)$, $\mathcal{L}_{\mathfrak{d}_4}(-2\Lambda_3)$, $\mathcal{L}_{\mathfrak{d}_4}(-2\Lambda_4)$.)

Application

Conjecture (LLY)

Let V be a simple affine vertex algebras associated with the Deligne-Cvitanovic series,

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$$

at level $-h^\vee/6 - 1$. The normalized $\mathbb{Z}_2$-twisted characters $\chi^{\mathrm{twi}}(q)$ are solutions of a second-order $\Gamma^0(2)$-MLDE with suitable coefficients a_1 and a_2 but without a $D_q^{(1)}$ term,

$$\left(D_q^{(2)} + a_1\bar{\Theta}_{0,2} + a_2\bar{\Theta}_{1,1}\right)\chi^{\mathrm{twi}}(q) = 0$$

Part II

- Twisted Zhu's algebras and twisted Zhu's bimodules for $L_k(sl_2)$.

- Fusion rules

Zhu's algebra

Theorem (Li, Hao)

Let V be a quasi-lisse vertex operator superalgebra. Then it has finitely many simple ordinary g-twisted modules.

Example

The $\hat{\sigma}(= e^{\frac{1}{2}\pi i h_{(0)}})$-twisted Zhu's algebra of $L_{-\frac{4}{3}}(\mathfrak{sl}_2)$ is $C[x]/\langle x(x-\frac{2}{3})(x+\frac{2}{3})\rangle$. There are three simple twisted modules with Dynkin label $0, \pm\frac{2}{3}$.

Dong-Li-Mason [DLM97] calculated the the Zhu's bimodule using the shifted the conformal vector. Let M^1, M^2, M^3 be g_1, g_2, g_3-twisted modules, respectively. Zhu Yiyi defined the twisted Zhu's algebras A_{g_2}-bimodule $A_{g_1 g_2, g_2}(M^1)$. See [Zhu22].

Zhu's algebra

Theorem

The map $V \to V, a \mapsto \Delta(1)a$, *induces an algebra isomorphism*

$$A^{dlm}(V) \cong A_{\hat{\sigma}}(V).$$

The map $M^1 \to M^1, m \mapsto \Delta(1)m$ *induces an* $A_{\hat{\sigma},\hat{\sigma}}(V)(\cong A^{dlm}(V))$*-bimodule isomorphism*

$$A^{dlm}(M^1) \cong A_{\hat{\sigma},\hat{\sigma}}(M^1).$$

Theorem (LLY)

Let $\ell = -2 + \frac{p}{q}$ *be the admissible level where* p *and* q *coprime positive integers with* $p \geq 2$. *Then* $\mathbb{Z}_2$*-twisted Zhu's algebra for* $L_\ell(\mathfrak{sl}_2)$ *is* $\mathbb{C}[x]/\langle \prod_{r=0}^{p-2} \prod_{s=0}^{q-1} (x + \frac{1}{2}\ell - r + st) \rangle$, *where* $0 \leq r \leq p-2$, $0 \leq s \leq q-1$.

Zhu's bimodules

Theorem (LLY)

$A_{\hat{\sigma},\hat{\sigma}}(L_{\mathfrak{sl}_2}(\ell,j))$ is isomorphic to the quotient space of $\mathbb{C}[x,y]$ modulo the subspace

$$\mathbb{C}[x,y]y^n + \mathbb{C}[x]g_{j,0}(x,y) + \mathbb{C}[x]g_{j,1}(x,y) + \cdots + \mathbb{C}[x]g_{j,n-1}(x,y)$$

where $g_{j,i} = y^i \prod_{r=0}^{p-n-1} \prod_{s=0}^{q-k} (x + \frac{1}{2}\ell - r - i + st)$.

Zhu's bimodules

Theorem (LLY)

$A(L(\ell,0))$-bimodule $A((L(\ell,j))^*)$ is isomorphic to the quotient space of $\mathbb{C}[x,z]$ modulo the subspace

$$\mathbb{C}[x,z]z^n + \mathbb{C}[x]f'_{j,0}(x,z) + \mathbb{C}[x]f'_{j,1}(x,z) + \cdots + \mathbb{C}[x]f'_{j,n-1}(x,z)$$

where $f'_{j,i}(x,z) = z^i \prod_{r=0}^{p-n-1} \prod_{s=0}^{q-k}(x+r+i-st)$. It is isomorphic to $A_{\hat{\sigma}',\hat{\sigma}'}(L(\ell,j)^*)$ via $\Delta_{\frac{1}{2}}(1)$, where $\hat{\sigma}' = e^{-\frac{\pi i h_{(0)}}{2}}$.

Let admissible weights of the vertex affine algebra $L_k(\mathfrak{sl}_2)$ be $j_i = n_i - (k_i - 1)t$ $(i = 1, 2)$. We work with the category of all highest weight modules and their contragredient modules and the $\mathbb{Z}/2\mathbb{Z}$-twisted modules of $L_k(\mathfrak{sl}_2)$.

Fusion rules

Using the conjectural twisted version of Frenkel-Zhu's theorem, we get

Theorem (LLY)

$$L(\ell, j_1) \times \sigma^{-\frac{1}{2}}(L(\ell, j_2)) = \sum_{i=\max\{0, n_1+n_2-p\}}^{\min\{n_1-1, n_2-1\}} \sigma^{-\frac{1}{2}}(L(\ell, j_1 + j_2 - 2i)),$$

$$(L(\ell, j_1))^* \times \sigma^{\frac{1}{2}}((L(\ell, j_2))^*) = \sum_{i=max\{0, n_1+n_2-p\}}^{\min\{n_1-1, n_2-1\}} \sigma^{\frac{1}{2}}((L(\ell, j_1 + j_2 - 2i))^*)$$

for $0 \le k_2 - 1 \le q - k_1$.

$$(L(\ell, j_1))^* \times \sigma^{\frac{1}{2}}(L(\ell, j_2)) = \begin{cases} \sigma^{\frac{1}{2}}(L(\ell, -j_1 + j_2)), & \text{if } n_2 - n_1 \ge 0; \\ \sigma^{\frac{1}{2}}((L(\ell, j_1 - j_2))^*), & \text{if } n_2 - n_1 < 0. \end{cases}$$

$$L(\ell, j_2) \times \sigma^{-\frac{1}{2}}((L(\ell, j_1))^*) = \begin{cases} \sigma^{-\frac{1}{2}}(L(\ell, -j_1 + j_2)), & \text{if } n_2 - n_1 \ge 0; \\ \sigma^{-\frac{1}{2}}((L(\ell, j_1 - j_2))^*), & \text{if } n_2 - n_1 < 0. \end{cases}$$

Some questions and future works

- Do all modules of affine vertex algebras at admissible level in category $\mathcal{O}$ satisfy the new finiteness condition?
- Do we have modular tensor category structure for affine vertex algebras at admissible level beyond ordinary modules or do we have G-crossed tensor category structure for their g-twisted modules?
- What is the proper finiteness condition for the affine vertex superalgebras or its twisted modules? What is its relation with the characters of the affine VOSA (mock modular forms)?
- Find more examples of quasi-lisse vertex superalgebras. (**Show that Moore-Tachikawa varieties have symplectic singularities.**)
- Study the structure of modules of the non-simply laced affine VOA under the spectral flow. How can we obtain the ordinary twisted modules from weak modules?

Twsited arc space

Let X be a separated superscheme of finite type over $\mathbb{C}$. Let $g : X \to X$ be a linear automorphism of order m.
For m-th root of unity ξ_m we have an automorphism φ_λ of $\mathbb{C}[\![t]\!]$ determined by $t \mapsto \xi_m t$. Since

$$\mathrm{Hom}_{\mathbf{Ssch}}(D, X) \cong \mathrm{Hom}_{\mathbf{Rings}}(\Gamma(X, \mathcal{O}_X), \mathbb{C}[\![x]\!]),$$

φ_λ induces a natural automorphism ϕ_λ of superarcs in X by composing

$$\gamma \in \mathrm{Hom}\mathbf{Rings}(\Gamma(X, \mathcal{O}_X), \mathbb{C}[\![x]\!])$$

with φ_λ.

Definition

The g-twisted arc superspaces of X, X^g_∞, is the fixed sub-superscheme of X_∞ under the automorphism $g^{-1} \circ \phi_\lambda$.

Examples

$L(\tilde{X})$-module $L_k(X)$ would produce a module of $L(\tilde{X})^\mu$, $U(L(\tilde{X})^\mu)v_k$. In general, $U(L(\tilde{X})^\mu)v_k$ is not isomorphic to the irreducible vacuum module of $L(\tilde{X}^{(2)})$ or $L(\tilde{X}^{(3)})$.

If $L_k(X)$ for $k \in \mathbb{Z}_+$ is classically free, it would imply that

$$J_\infty^\sigma(\mathbb{C}[\mathfrak{g}]/\langle U(\mathfrak{g})e_\theta^{k+1}\rangle) \cong gr^F(U(L(\tilde{X})^\mu)v_k)$$

where $\mathfrak{g}$ is of type X, σ is an automorphism of $\mathfrak{g}$ induced from Dynkin diagram automorphism, and μ is the twisted graph automorphism.

Thank you!

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