Automorphism groups of cyclic orbifolds of lattice VOAs

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Main question

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One can construct a vertex operator algebra V_L from L.

Let $\{h_1, \ldots, h_\ell\}$ be an orthonormal basis of $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$.
As a vector space,

$$V_L = M(1) \otimes \mathbb{C}\{L\}$$

where

$$M(1) \cong \mathbb{C}[h_i(-n_i) \mid i = 1, \ldots, \ell, n_i \in \mathbb{Z}_{>0}]$$

and

$$\mathbb{C}\{L\} = \mathrm{Span}_{\mathbb{C}}\{e^\alpha | \alpha \in L\}$$

is a twisted group algebra of L such that $e^\alpha e^\beta = (-1)^{\langle \alpha, \beta \rangle} e^\beta e^\alpha$.

Note: $O(L)$ acts projectively on V_L.

Let $g \in O(L)$ be a fixed point free isometry of L ($gx = x$ implies $x = 0$).

Then g can be lifted to an automorphism $\hat{g}$ of V_L.
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Important Fact: If $L_2 = \{x \in L | \langle x, x \rangle = 2\} = \emptyset$ and g is fixed point free, then $\mathrm{Aut}\,(V_L^{\hat{g}})$ is finite.

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From now on, we assume $L_2 = \emptyset$.

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For $\varphi \in \mathrm{Aut}\,(\hat{L})$, define $\iota(\varphi) \in \mathrm{Aut}\,(L)$ by $\varphi(e^{\alpha}) \in \{\pm e^{\iota(\varphi)(\alpha)}\}, \alpha \in L$.

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Set $O(\hat{L}) = \{\varphi \in \mathrm{Aut}\,(\hat{L}) \mid \iota(\varphi) \in O(L)\}$.

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Set $O(\hat{L}) = \{\varphi \in \mathrm{Aut}\,(\hat{L}) \mid \iota(\varphi) \in O(L)\}$.
We can identify $O(\hat{L})$ as a subgroup of $\mathrm{Aut}\,(V_L)$ and there is an exact sequence of [FLM88, Proposition 5.4.1]

$$1 \to \mathrm{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \to O(\hat{L}) \xrightarrow{\varphi} O(L) \to 1.$$

Note that $\mathrm{Hom}(L, \mathbb{Z}_2) = \{\exp(2\pi\sqrt{-1}\alpha_{(0)}) \mid \alpha \in (L^*/2)/L^*\}$ in $\mathrm{Aut}\,(V_L)$.

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$$N(V_L) \cap O(\hat{L}) = \mathrm{Hom}(L, \mathbb{Z}/2\mathbb{Z}) \quad \text{and} \quad \mathrm{Aut}\,(V_L)/N(V_L) \cong O(L).$$

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In particular, we have an exact sequence

$$1 \to N(V_L) \to \mathrm{Aut}\,(V_L) \xrightarrow{\varphi} O(L) \to 1.$$

$N_{\mathrm{Aut}\,(V_L)}(\langle \hat{g} \rangle)$

Theorem

Let L be an even positive definite lattice with $L(2) = \emptyset$. Let g be a fixed point free isometry of L and $\hat{g}$ a lift of g in $O(\hat{L})$. Then we have the following exact sequences.

$$1 \longrightarrow \mathrm{Hom}(L/(1-g)L, \mathbb{C}^*) \longrightarrow N_{\mathrm{Aut}\,(V_L)}(\langle \hat{g} \rangle) \xrightarrow{\varphi} N_{O(L)}(\langle g \rangle) \longrightarrow 1;$$

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It is clear that $N_{\mathrm{Aut}\,(V_L)}(\langle \hat{g} \rangle)$ acts on $V_L^{\hat{g}}$ and there is a group homomorphism $f : N_{\mathrm{Aut}\,(V_L)}(\langle \hat{g} \rangle)/\langle \hat{g} \rangle \longrightarrow \mathrm{Aut}\,(V_L^{\hat{g}})$.

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Definition

An automorphism $h \in \mathrm{Aut}\,(V_L^{\hat{g}})$ is said to be an extra automorphism if it is not in the image of f.

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Let V be a VOA and $\tau \in \mathrm{Aut}(V)$. For any V-module $M = (M, Y_M)$, the τ-conjugate $(M \circ \tau, Y_{M\circ\tau}(\cdot, z))$ of M is defined as follows:

$$M \circ \tau = M \quad \text{as a vector space;}$$

$$Y_{M\circ\tau}(a, z) = Y_M(\tau a, z) \quad \text{for any } a \in V.$$

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That means $\mathrm{Aut}\,(V)$ acts on the set $\mathrm{Irr}(V)$ of irreducible modules of V.

Theorem (Shimakura)

Let $V_L(j) = \{v \in V_L \mid g(v) = e^{2\pi\sqrt{-1}j/n}v\}$, $n = |g|$ *and* $0 \le j \le n-1$. *Let* $\tau \in \mathrm{Aut}\,(V_L^{\hat{g}})$. *Then* τ *lifts to an automorphism of* V_L *iff*

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Remark: τ is extra if and only if $\{V_L(j) \circ \tau \mid 0 \le j \le n-1\} \neq \{V_L(j) \mid 0 \le j \le n-1\}$.

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Untwisted type: submodules of V_L-modules.

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Assume that $(1-g)\lambda \in L$, that is, $g(\lambda + L) = \lambda + L$.
Then $V_{\lambda+L}$ is $\hat{g}$-invariant and $\hat{g}$ acts on $V_{\lambda+L}$. For $0 \le i \le p-1$, we denote $V_{\lambda+L}(i) = \{v \in V_{\lambda+L} \mid \hat{g}(v) = \exp(2\pi\sqrt{-1}i/p)v\}$, which is an irreducible $V_L^{\hat{g}}$-module.

Twisted type: submodules of twisted V_L-modules.

Let $1 \leq s \leq p-1$. Recall from [Le85, DL96] that the irreducible $\hat{g}^s$-twisted module $V_L^T[\hat{g}^s]$ is given by

$$V^{T_\chi}[\hat{g}^s] = M(1)[g^s] \otimes T,$$

where $M(1)[\hat{g}^s]$ is the "$\hat{g}^s$-twisted" free bosonic space and T is an irreducible module for a certain "$\hat{g}^s$-twisted" central extension of L.

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and try to study their relations using the automorphism group of the bigger VOA.

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Theorem ([Sh04, Proposition 3.16])

Let L be an even lattice such that $L(2) = \emptyset$.
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Let $C < \mathbb{Z}_2^n$ be doubly even and let $\mathcal{B} = \{\alpha_i \mid i \in \{1, \ldots, n\}\} < \mathbb{R}^n$ s.t. $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$. The lattice

$$L_B(C) = \sum_{c \in C} \mathbb{Z}\frac{1}{2}\alpha_c + \sum_{i,j \in \{1,\ldots,n\}} \mathbb{Z}(\alpha_i + \alpha_j)$$

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Then the action of h_{A_n} on $sl_{n+1}(\mathbb{C})$ is given by the conjugation of P,i.e,

$$h_{A_n} : A \to P^{-1}AP \qquad \text{for } A \in sl(n+1, \mathbb{C}),$$

and

$$B^{-1}PB = \mathrm{diag}(\omega, \omega^2, ..., 1)$$

where

$$P = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{n+1}} \begin{pmatrix} \omega & \omega^2 & \cdots & \omega^n & 1 \\ \omega^2 & \omega^4 & \cdots & \omega^{2n} & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \omega^n & \omega^{2n} & \ddots & \omega^{n^2} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

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$$P = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{n+1}} \begin{pmatrix} \omega & \omega^2 & \cdots & \omega^n & 1 \\ \omega^2 & \omega^4 & \cdots & \omega^{2n} & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \omega^n & \omega^{2n} & \ddots & \omega^{n^2} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Define a map $\sigma_{A_n} : sl(n+1, \mathbb{C}) \to sl(n+1, \mathbb{C})$ by $\sigma_{A_n}(A) = B^{-1}AB$.

Extra automorphisms (generalization of FLM triality map)

Let A_n be a root lattice of type A_n. (Coxeter number = determinant)

Let h_{A_n} be an $(n+1)$-cycle in $Weyl(A_n) \cong Sym_{n+1}$.
Then the action of h_{A_n} on $sl_{n+1}(\mathbb{C})$ is given by the conjugation of P,i.e,

$$h_{A_n} : A \to P^{-1}AP \qquad \text{for } A \in sl(n+1, \mathbb{C}),$$

and

$$B^{-1}PB = \mathrm{diag}(\omega, \omega^2, ..., 1)$$

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$$P = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{n+1}} \begin{pmatrix} \omega & \omega^2 & \cdots & \omega^n & 1 \\ \omega^2 & \omega^4 & \cdots & \omega^{2n} & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \omega^n & \omega^{2n} & \ddots & \omega^{n^2} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

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Let $\rho_{A_n} = \frac{1}{2}(n-1, n-2, \dots, -(n-2), -(n-1))$ be the Weyl vector.

Define $\eta_{A_n} = \exp(\frac{1}{n+1}(2\pi i \rho_{A_n}(0))$.

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Lemma

We have $\sigma_{A_n} h_{A_n} \sigma_{A_n}^{-1} = \eta_{A_n}$ *and* $\sigma_{A_n} \eta_{A_n} \sigma_{A_n}^{-1} = h_{A_n}^{-1}$ *on* $sl_{n+1}(\mathbb{C})$.

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$$R = A_{k_1} \oplus \cdots \oplus A_{k_j}$$

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Since they are inner automorphisms, we can extend them to V_L by using the same exponential expressions.

Theorem

We have $\sigma(V_X^h) = V_X^h$ and σ induces an automorphism of V_X^h.

Definition

A lattice X is said to be obtained by Construction B if $X = L(\hat{\rho}) = \{\alpha \in L \mid \langle \alpha, \hat{\rho} \rangle \in \mathbb{Z}\}$, where L is an even overlattice of $R = A_{k_1} \oplus \cdots \oplus A_{k_j}$ and $\hat{\rho} = \sum_{i=1}^{j} \frac{1}{(k_i+1)} \rho_{A_{k_i}}$.

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Theorem

Let $R = \bigoplus_{i=1}^{t} A_{k_i-1}$ be a root lattice. Suppose $L = L_B(C)$ is constructed by Construction B associated with a subgroup C of $\mathcal{D}(R)$. If $\hat{g}$ is a lift of the fixed-point free isometry of L induced by a Coxeter element of R. Then $V_L^{\hat{g}}$ has an extra automorphism.

Proposition

Let U be a rootless even unimodular lattice. Let $h \in O(U)$ and $\hat{h} \in O(\hat{U})$ a standard lift of h. Assume that (1) $V_U^{\mathrm{orb}(\hat{h})} \cong V_U$ and (2) the conjugacy class of $\langle \hat{h} \rangle$ in $\mathrm{Aut}\,(V_U)$ is uniquely determined by $|\hat{h}|$ and the VOA structure of $V_U^{\hat{h}}$.
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Many isometries of the Leech lattice satisfy the above conditions, e.g., $-3A$.

The converse also holds when g has prime order under some assumptions.

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Let $L_2 = \emptyset$ and $g \in O(L)$ is fixed point free. Suppose $|g| = p$ is a prime and $V_L^{\hat{g}}$ has an extra automorphism.

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- *L is a coinvariant sublattice of the Leech lattice.* $(L \cong \Lambda_g,\ g = 2A, -2A, 3B, 3C, 5B, 5C, 7B, 11A$ *or* $23A.)$

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Then

$$V_L(1) \circ \sigma \not\cong V_L(r) \quad \text{for all } 1 \leq r \leq p-1. \tag{1}$$

Then either **Case (I):**

$$V_L(1) \circ \sigma \cong V_{\lambda+L}(r)$$

for some $0 \leq r \leq p-1$ and $\lambda \in \mathcal{D}(L) \setminus \{L\}$ with $(1-g)\lambda \in L$; or

Case (II):

$$V_L(1) \circ \sigma \cong V_L^T[\hat{g}^s](r)$$

for some $0 \leq r \leq p-1$, $1 \leq s \leq p-1$.

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$$|(\lambda+L)(2)| = \frac{pm}{p-1}. \tag{2}$$

By the similar argument, we also have

$$|(q\lambda+L)(2)| = \frac{pm}{p-1} \quad \text{for any } 1\le q\le p-1.$$

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Set $N=\mathrm{Span}_{\mathbb{Z}}\{\lambda, L\}$. Then by $L(2)=\emptyset$, we have

$$|N(2)| = \sum_{i=1}^{p-1} |(i\lambda+L)(2)| = pm.$$

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- If $(v|g^i(v)) = 1$ for $1 \leq i \leq |g| - 1$, then $(1 - g^i)(v) \in L(2)$, which contradicts that $L(2) = \emptyset$.

Therefore, $(v|g^i(v)) = 0, -1, -2$.

Suppose $(v|g^i(v)) = -2$ for some i. Then $(v|g^{p-i}(v)) = -2$.

$\sum_{j=0}^{p-1}\langle v, g^j v\rangle = 0$ implies $\langle v, g^j v\rangle = 0$ for all $i \neq j$
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Take $w \in N_2 \setminus \mathrm{Span}\{g^i v\}$. Then $\langle w, g^i v\rangle = 0$.

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Otherwise, $\langle w, g^i v\rangle = -1$ but $\sum_{j=0}^{|g|-1} g^j v = 0$; there is an r s.t. $\langle w, g^r v\rangle = 1$.

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Moreover, we get restrictions about L^*/L.

These restrictions $(+L(2) = \emptyset)$ are sufficient to prove that L is contained in Leech lattice.

Conjecture

Suppose $L_2 = \emptyset$ *and* $g \in O(L)$ *is fixed point free.*
If $V_L^{\hat{g}}$ *has an extra automorphism, then either*

- L *can be obtained by Construction B or*
- L *is a coinvariant sublattice of the Leech lattice.*

A counterexample

Let A_2 be a root lattice of type A_2. Let $\rho = (1, 0, -1)$ be a Weyl vector of A_2 and h a Coxeter element of A_2.

Set $X = \{x \in A_2 | (x, \rho) = 0 \mod 3\}$ and $L = X \perp \Lambda$.

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Therefore, we need some indecomposable conditions.

Another example

Assume that g^i is fixed point free on L for any $1 \leq i \leq |g| - 1$. We call such a $g \in O(L)$ a completely fixed point free isometry of L.

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Theorem

Let L be an even with $L_2 = \emptyset$ and let $g \in O(L)$ be completely fixed point free. Suppose $V_L^{\hat{g}}$ has extra automorphisms. Then either
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Let L be an even with $L_2 = \emptyset$ and let $g \in O(L)$ be completely fixed point free. Suppose $V_L^{\hat{g}}$ has extra automorphisms. Then either
(1) the order of g is a prime or

(2) L is isometric to the Leech lattice or some coinvariant sublattices of the Leech lattice.

Sketch of the proof

g is completely fixed point free of order n; the minimal polynomial of g on L is the n-th cyclotomic polynomial $\Phi_n(x)$ and
the characteristic polynomial of g on L is $\Phi_n(x)^{\ell/\varphi(n)}$,
where $\ell = \mathrm{rank}(L)$ and φ is the Euler totient function.

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Suppose $V_L(1) \circ \sigma \cong V_{\lambda+L}(r)$ for some $\sigma \in V_L^{\hat{g}}$. Then g stabilizes $\lambda + L$.
Since the characteristic polynomial of g on L is $\Phi_n(x)^{\ell/\varphi(n)}$,

$$\dim V_L(j)_1 = \begin{cases} \frac{\ell}{\varphi(n)}, & \text{if } (j,n) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\dim V_{\lambda+L}(r)_1 = \frac{\ell}{\varphi(n)}$, also.

Moreover, $\dim V_{\lambda+L}(r)_1 = |(\lambda + L)(2)|/n$ for any $0 \le r \le n-1$. Therefore,

$$|(\lambda + L)(2)| = \frac{n}{\varphi(n)} \cdot \ell. \qquad (3)$$

Since $\Phi_n(g)\lambda = 0$ and g stabilizes $\lambda + L$, we have $\Phi_n(1)\lambda \in L$. Recall that

$$\Phi_n(1) = \begin{cases} 1 & \text{if } n \text{ is not a prime power} \\ p & \text{if } n = p^t. \end{cases}$$

Now set $N = \mathrm{Span}_{\mathbb{Z}}\{L, \lambda\}$. Then we have $|N/L| = 1$ or $|N/L| = p$.

By our assumption, $|N/L| > 1$; hence $n = p^t$ and $|N/L| = p$.

Since g stabilizes $\lambda + L$, g also acts on N. Let $\hat{g}$ be a lift of g on V_N.

Now assume that $n = p^t$ and $m = n/p = p^{t-1}$. Let $h = g^m$.
Then h is fixed point free of order p on L. Moreover, we have

- $|N(2)| = \sum_{i=1}^{p-1} |(i\lambda + L)(2)| = (p-1)\frac{p^t \ell}{p^{t-1}(p-1)} = p\ell$.
- $h(\lambda + L) = \lambda + L$.

Lemma

The sublattice of N spanned by $N(2)$ is isometric to the orthogonal sum of k copies of A_{p-1}, where $k = \ell/(p-1)$. Therefore, N can be obtained by construction A from a certain code C over $\mathbb{Z}_p$ and L can be obtained by construction B from the same code C.

There is a standard lift $\hat{h}$ of h and an automorphism $\sigma \in V_L^{\hat{h}}$ such that $V_L \circ \sigma \cong V_N^{\hat{h}}$ as $V_L^{\hat{h}}$-modules.

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By adjusting the lift $\hat{g}$ of g, we may also assume $\hat{h} = \hat{g}^m$, where $m = n/p$.

In this case, we have

$$V_L(1; \hat{h}) \circ \sigma \cong V_{\lambda+L}^{\hat{h}};$$

$V_L(j; \hat{h}) = \{v \in V_L \mid \hat{h}v = e^{2\pi\sqrt{-1}\frac{j}{p}} v\}$ and $V_{\lambda+L}^{\hat{h}} = \bigoplus_{i=1}^{m-1} V_{\lambda+L}(ip; \hat{g})$.

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Since $V_L(1) \circ \sigma \cong V_{\lambda+L}(r)$ and n is the smallest integer such that $V_L(1)^{\boxtimes n} \cong V_L(0)$, we have

$$V_{\lambda+L}(r)^{\boxtimes s} \cong V_{s\lambda+L}(sr) \not\cong V_L(0) \text{ if } s < n.$$

Therefore, $V_{\lambda+L}(r)^{\boxtimes s} \cong V_{s\lambda+L}(sr) \cong V_L(0)$
if and only if $p|s$ and $sr \equiv 0 \mod n$.
Thus, $(m, r) = 1$.

On the other hand,

$$V_L(1;\hat{h})\circ\sigma=\bigoplus_{i=1}^{m-1}V_L(1+ip;\hat{g})\circ\sigma=\bigoplus_{i=1}^{m-1}V_{\lambda+L}(r+irp;\hat{g}).$$

Therefore, we have $r\equiv 0 \mod p$ and thus $(p,m)=1$;
nevertheless, $n=p^t$ is a prime power and thus $m=n/p=1$ and $n=p$ is a prime number.

General cases: g is fixed point free of order n and $L_2 = \emptyset$

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Case 1: $V_L(1) \circ \tau \cong V_{\lambda+L}(r)$ for some $\lambda + L \in \mathcal{D}(L) \setminus \{L\}$ and $\tau \in \mathrm{Aut}\,(V_L^{\hat{g}})$.

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Set $N = \mathrm{Span}\{L, \lambda\}$. Then N is also an even lattice since $V_L(1)$ has integral weights.
Moreover, $(1-g)\lambda \in L$; therefore, g stabilizes each coset $i\lambda + L$ for $i \in \mathbb{Z}$.
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Let $\hat{g}$ be a lift of g on V_N. Then $\hat{g}$ also acts on $V_{\lambda+L}$ and we use $V_{\lambda+L}(j)$ to denote the eigenspace
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Then we have

$$V_L(i) \circ \tau \cong V_{\lambda+L}(r)^{\boxtimes i} = V_{i\lambda+L}(ri). \tag{4}$$

Now suppose $[N : L] = m \gneq 1$. Since $1 + g + \cdots + g^{n-1} = 0$ and $(1 - g)\lambda \in L$, $n\lambda \in L$ and thus m divides n.
Set $k = n/m$ and let $h = g^k$. We also denote $\hat{h} = \hat{g}^k$.

Lemma

We have $(r, k) = 1$.

Proof.

Since $V_L(1) \circ \tau \cong V_{\lambda+L}(r)$, $V_{\lambda+L}(r)$ is also a simple current modules and has order n with respect to the fusion product. By (4),

$$V_{\lambda+L}(r)^{\boxtimes i} = V_{i\lambda+L}(ri)$$

Suppose $V_{\lambda+L}(r)^{\boxtimes j} \cong V_L(0)$. Then

$$j\lambda \in L, i.e,\ m \text{ divides } j, \quad rj = 0 \mod n.$$

That $V_{\lambda+L}(r)$ has order n implies $(r, k) = 1$. □

Lemma

The automorphism $\tau \in \mathrm{Aut}\,(V_L^{\hat{g}})$ stabilizes the orbifold subVOA $V_L^{\hat{h}}$. In particular, τ can be lift to an automorphism of $V_L^{\hat{h}}$.

Proof.

Since $\hat{h} = \hat{g}^k$ on V_L, $\hat{h}$ has order m on V_L and $V_L^{\hat{h}} = \oplus_{i=0}^{k-1} V_L(mi)$; note that $e^{2\pi\sqrt{-1}mi/n}$ are k-th roots of unity for $0 \le i \le k-1$. By (4), we have

$$V_L(mi) \circ \tau \cong (V_L(1) \circ \tau)^{\boxtimes mi} \cong V_{mi\lambda+L}(mij) = V_L(mij) \subset V_L^{\hat{h}}.$$

Therefore, $V_L^{\hat{h}} \circ \tau \cong V_L^{\hat{h}}$ as desired. □

Lemma

There exists a lift $\tilde{h} \in \mathrm{Aut}\,(V_N)$ *of* h *such that* $\tilde{h}|_{V_L} = \hat{h}|_{V_L}$ *and* $V_N^{\tilde{h}} \cong V_L \circ \tau$.

Proof.

Since $[N : L] = m$, there is $\mu \in L^*$ such that $\langle \mu, \lambda \rangle \equiv 1/m \mod \mathbb{Z}$.
Then $\tilde{h} = \hat{g}^k \cdot \sigma_{r\mu}$ will be the desired automorphism, where
$\sigma_{r\mu} = \exp(-2\pi\sqrt{-1} r \mu_{(0)})$.

□

Let $R = \mathrm{Span}_{\mathbb{Z}}\{N_2\}$. Then R is a root lattice associated with a simple laced root system. Moreover, g acts on R since g must preserve N_2.
Let $R = R_1 \oplus \cdots \oplus R_t$ be the sum of simple root lattices.
Then $(V_N)_1 = (V_R)_1 \oplus \mathbb{C}R^\perp$ and $\dim(V_N^{\tilde{h}})_1 = \dim(V_R^{\tilde{h}})_1 + \dim(\mathbb{C}R^\perp)^h$.
Since $\tilde{h}$ is regular on $(V_R)_1$, $\dim(V_R^{\tilde{h}})_1 \le \dim \mathbb{C}R$. Moreover, we have $\dim(V_N^{\tilde{h}})_1 = \dim(V_L)_1 = \mathrm{rank}(L) = \dim \mathbb{C}R + \dim \mathbb{C}R^\perp$.
Therefore, we have $\dim(V_R^{\tilde{h}})_1 = \dim \mathbb{C}R$ and $\dim(\mathbb{C}R^\perp)^h = \dim(\mathbb{C}R^\perp)$.

Proposition

The isometry h preserves all irreducible components of R and h acts trivially on $R^\perp$. Moreover, the order of $\tilde{h}|_{(V_{R_i})_1}$ is the Coxeter number of R_i.

Remark: $\tilde{h}|_{(V_{R_i})_1}$ is conjugate to a lift of a Coxeter element of R_i.

Lemma

All irreducible components of R are of type A.

Proof.

Let $\alpha \in R_i$ be a root. Since $L_2 = \emptyset$, $\alpha \notin L$.
Consider the set $\{\alpha, h\alpha, \dots, h^{s-1}\alpha\}$. Then we have $\langle \alpha, \sum_{i=1}^{s-1} h^i\alpha \rangle = -2$.
Moreover, $\langle \alpha, h^i\alpha \rangle \in \{0, -1, -2\}$ for all $1 \le i \le s-1$.
Suppose $\langle \alpha, h^i\alpha \rangle = -2$ for some i. Then $\langle \alpha, h^{s-i}\alpha \rangle = -2$ and $\langle \alpha, h^j\alpha \rangle = 0$ for any $j \neq i$ and $i = s-i$,
that implies s is even and $i = s/2$. In particular, $\{\alpha, h\alpha, \dots, h^{s-1}\alpha\}$ spans a lattice of type $A_1^{s/2}$ in R_i and h induces a cyclic permutation on $A_1^{s/2}$.
It is not possible except for the case that $R_i = A_1$.

Assume that $\mathrm{rank}(R_i) \gneq 1$. Then $\langle \alpha, h^i\alpha \rangle \in \{0, -1\}$. Then $\langle \alpha, h^i\alpha \rangle = \langle \alpha, h^{s-i}\alpha \rangle = -1$ and $\langle \alpha, h^j\alpha \rangle = 0$ for any $j \neq i, s-1 \mod s$.

In this case, R_i is an orthogonal sum of simple root lattice of type A. □

Lemma

We have $|g|_{\tilde{R}}| = |h|_{\tilde{R}}|$. Moreover, g preserves every irreducible component of R and $|g|_{R_i}| = |h|_{R_i}|$ for each irreducible component R_i of R.

Proof.

Suppose $|g|_{\tilde{R}}| \gneq |h|_{\tilde{R}}|$.
Then there exists a root $\alpha \in R$ such that the set $\{\alpha, h\alpha, \dots, h^{s-1}\alpha\}$ is a proper subset of $\{\alpha, g\alpha, \dots, g^{t-1}\alpha\}$, where s and t are the smallest positive integers such that $h^s\alpha = \alpha$ and $g^t\alpha = \alpha$.

Since $\sum_{i=0}^{t-1} g^i\alpha = 0$ and $\{\alpha, h\alpha, \dots, h^{s-1}\alpha\}$ spans a lattice of type A_{s-1}, the sublattice spanned by $\{\alpha, g\alpha, \dots, g^{t-1}\alpha\}$ is isometric to A_{s-1}^a,

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Such a case is not possible.

□

Lemma

We have $GCD(m,k) = 1$ *and* $g|_{N^h}$ *has order* k.

Proof.

Suppose $g|_{N^h}$ has order q. Then q divides k. Moreover,

$$mk = |g| = LCM(|g|_{\tilde{R}}|, |g|_{N^h}|) = \frac{mq}{(m,q)}.$$

Since $q|k$, we have $mk/q = \frac{m}{(m,q)}$. Then $(m,q) = 1$ and $k = q$ as desired. □

Lemma

We have $L = \tilde{R}' \perp Ann_L(R')$.

Thank You

T. Arakawa, C.H. Lam and H. Yamada, Zhu's algebras, C_2-algebras and C_2-cofiniteness of parafermion vertex operator algebras, *Adv. Math.*, **264** (2014), 261-295

R.E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Nat'l. Acad. Sci. U.S.A.* **83** (1986), 3068–3071.

A. M. Cohen and R. L. Griess, Jr. , On finite simple subgroups of the complex Lie group of type E_8, Proc. Symp. Pure Math., 47, 1987, 367-405.

H.Y. Chen and C.H. Lam, Quantum dimensions and fusion rules of the VOA $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$, *J. Algebra* **459** (2016), 309–349.

H.Y. Chen, C.H. Lam and H. Shimakura, On $\mathbb{Z}_3$-orbifold construction of the Moonshine vertex operator algebra, Math. Z. **288** (2018), no. 1-2, 75-100; arXiv:1606.05961

S. Carnahan and M. Miyamoto, Regularity of fixed-point vertex operator subalgebras; arXiv:1603.05645.

C. Dong, Vertex algebras associated with even lattices, *J. Algebra* **161** (1993), 245–265.

C. Dong and J. Lepowsky, The algebraic structure of relative twisted vertex operators, *J. Pure Appl. Algebra* **110** (1996), 259–295.

C. Dong, H. Li, and G. Mason, Simple Currents and Extensions of Vertex Operator Algebras, *Comm. Math. Phys.* **180** (1996), 671-707.

C. Dong, H. Li, and G. Mason, Modular-invariance of trace functions in orbifold theory and generalized Moonshine, *Comm. Math. Phys.* **214** (2000), 1–56.

C. Dong and G. Mason, Holomorphic vertex operator algebras of small central charge, *Pacific J. Math.* **213** (2004), 253–266.

C. Dong and G. Mason, Rational vertex operator algebras and the effective central charge, *Int. Math. Res. Not.* (2004), 2989–3008.

C. Dong and K. Nagatomo, Automorphism groups and twisted modules for lattice vertex operator algebras, *in* Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), 117–133, *Contemp. Math.*, **248**, Amer. Math. Soc., Providence, RI, 1999.

C. Dong and L. Ren, Representations of the parafermion vertex operator algebras; arXiv:1411.6085.

C. Dong and Q. Wang, The structure of parafermion vertex operator algebras: general case, *Comm. Math. Phys.* 299 (2010), no. 3, 783–792.

J. van Ekeren, S. Möller and N. Scheithauer, Construction and Classification of Holomorphic Vertex Operator Algebras, *J. Reine Angew. Math.*, Published Online.

J. van Ekeren, S. Möller and N. Scheithauer, Dimension Formulae in Genus Zero and Uniqueness of Vertex Operator Algebras, to appear in *Int. Math. Res. Not.* ; arXiv:1704.00478.

I.B. Frenkel, Y. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Mem. Amer. Math. Soc.* **104** (1993), viii+64 pp.

- I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator algebras and the Monster, Pure and Appl. Math., Vol.134, Academic Press, Boston, 1988.
- I. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123–168.
- R. L. Griess, Jr., A vertex operator algebra related to E_8 with automorphism group $\mathrm{O}^{+}(10,2)$. *The Monster and Lie algebras* (Columbus, OH, 1996), 43–58, Ohio State Univ. Math. Res. Inst. Publ., 7, de Gruyter, Berlin, 1998.
- R. L. Griess, Jr. and C. H. Lam, A moonshine path for $5A$ node and associated lattices of ranks 8 and 16, *J. Algebra*, 331(2011), 338-361.
- S. Helgason, Differential geometry, Lie groups, and symmetric spaces. Pure and Applied Mathematics. **80**. Academic Press, New York-London (1978)
- G. Höhn, On the Genus of the Moonshine Module, preprint.

G. Höhn and N.R. Scheithauer, A generalized Kac-Moody algebra of rank 14, *J. Algebra*, **404**, (2014), 222–239.

Y.Z. Huang and J. Lepowsky, A theory of tensor product for module category of a vertex operator algebra, III, *J. Pure Appl. Algebra*, **100** (1995), 141–171.

V.G. Kac, Infinite-dimensional Lie algebras, Third edition, Cambridge University Press, Cambridge, 1990.

K. Kawasetsu, C.H. Lam and X. Lin, $\mathbb{Z}_2$-orbifold construction associated with (-1)-isometry and uniqueness of holomorphic vertex operator algebras of central charge 24, *Proc. Amer. Math. Soc.* **146**, No. 5 (2018), 1937–1950.

M. Krauel and M. Miyamoto, A modular invariance property of multivariable trace functions for regular vertex operator algebras, *J. Algebra* **444** (2015), 124–142.

C.H. Lam, Orbifold vertex operator algebras associated with coinvariant lattices of Leech lattice, preprint.

C.H. Lam, Automorphism group of an orbifold vertex operator algebra associated with the Leech lattice, to appear in the Proceedings of the Conference on Vertex Operator Algebras, Number Theory and Related Topics, Contemporary Mathematics.

C.H. Lam, Some observations about the automorphism groups of certain orbifold vertex operator algebras, to appear in RIMS Kôkyûroku Bessatsu.

C.H. Lam and M. Miyamoto, Niemeier Lattices, Coxeter elements and McKay's E_8 observation on the Monster simple group, *Intern. Math. Res. Notices*, Vol 2006 (2006).

C.H. Lam and H. Shimakura, Construction of holomorphic vertex operator algebras of central charge 24 using the Leech lattice and level p lattices, *Bull. Inst. Math. Acad. Sin. (N.S.)*, Vol. 12 No. 1 (2017), 39 –70.

C.H. Lam and H. Shimakura, Reverse orbifold construction and uniqueness of holomorphic vertex operator algebras; arXiv:1606.08979.

C.H. Lam and H. Shimakura, On orbifold constructions associated with the Leech lattice vertex operator algebra, to appear in Mathematical Proceedings of the Cambridge Philosophical Society.

C.H. Lam and H. Shimakura, Inertia subgroups and uniqueness of holomorphic vertex operator algebras; arXiv:1804.02521.

, C.H. Lam and H. Yamauchi, On 3-transposition groups generated by σ-involutions associated to $c = 4/5$ Virasoro vectors, *J. Algebra*, 416 (2014), 84-121.

J. Lepowsky, Calculus of twisted vertex operators, *Proc. Natl. Acad. Sci. USA* **82** (1985), 8295–8299.

H. Li, Symmetric invariant bilinear forms on vertex operator algebras, *J. Pure Appl. Algebra*, **96** (1994), 279–297.

H. Li, Extension of vertex operator algebras by a self-dual simple module, *J. Algebra* **187** (1997), 236–267.

X. J. Lin, Mirror extensions of rational vertex operator algebras, *Trans. Amer. Math. Soc.* 369 (2017), no. 6, 3821–3840.

- M. Miyamoto, C_2-cofiniteness of cyclic-orbifold models, *Comm. Math. Phys.* **335** (2015), 1279–1286.
- M. Miyamoto and K. Tanabe, Uniform product of $A_{g,n}(V)$ for an orbifold model V and G-twisted Zhu algebra, *J. Algebra* **274** (2004), 80–96.
- R. Scharlau and B. Blaschke, Reflective integral lattices, *J. Algebra* **181** (1996), 934-–961.
- R. Scharlau and B.B. Venkov, Classifying lattices using modular forms- a preliminary report; in: M. Ozeki, E. Bannai, M. Harada (eds.): Codes, Lattices, Modular Forms and Vertex Operator Algebras, Conference Yamagata University, October 2 - 4, 2000 (Proceedings 2001).
- S. Sakuma and H. Yamauchi, Vertex operator algebra with two Miyamoto involutions generating S_3, *J. Algebra* **267** (2003), no. 1, 272–297.

- A.N. Schellekens, Meromorphic $c = 24$ conformal field theories, *Comm. Math. Phys.* **153** (1993), 159–185.
- H. Shimakura, The automorphism group of the vertex operator algebra V_L^+ for an even lattice L without roots, *J. Algebra* **280** (2004), 29–57.
- H. Shimakura, Lifts of automorphisms of vertex operator algebras in simple current extensions, *Math. Z.* **256** (2007), no. 3, 491–508.
- H. Shimakura, Automorphism groups of the holomorphic vertex operator algebras associated with Niemeier lattices and the -1-isometries ; arXiv:1811.05119.
- R.A. Wilson, The finite simple groups, Graduate Texts in Mathematics, **251**, Springer-Verlag London, Ltd., London, 2009.