Unitary representations of minimal W-algebras

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§1. Two Equivalent Definitions of a Vertex Algebra

A *vertex algebra* is a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with a *vacuum vector* $\mathbb{1} \in V_{\bar{0}}$ and a *translation operator* $T \in (\mathrm{End}\ V)_{\bar{0}}$ endowed with a product with values in $V((z))$, written as $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, $a_{(n)} \in \mathrm{End}\ V$, satisfying the following axioms $(a, b \in V)$:

- *vacuum*: $Y(\mathbb{1}, z) = I_V$, $Y(a,z)\mathbb{1} = a + zV[[z]]$;
- *translation invariance*: $[T, Y(a,z)] = \frac{d}{dz}Y(a,z)$, $T \cdot \mathbb{1} = 0$;
- *locality*: $(z-w)^N [Y(a,z), Y(b,w)] = 0$ for some $N \in \mathbb{Z}_+$.

Note that $a_{(n)}b$ is a product $V \otimes V \to V$ for each $n \in \mathbb{Z}$, such that $a_{(n)}b = 0$ for $n >> 0$. The product $a_{(-1)}b$ is denoted by $: ab :$ and is called the *normally ordered product*.

One has

$$a_{(-n-1)}b = \frac{1}{n!} : (T^n(a)b) : \text{ for } n \geq 0.$$

Define the *λ-bracket* $V \otimes V \to V[\lambda]$ by

$$[a_\lambda b] = \sum_{n \in \mathbb{Z}_+} \frac{\lambda^n}{n!}(a_{(n)}b).$$

It satisfies the following axioms of a *Lie conformal superalgebra* $(a, b, c \in V)$

- *sesquilinearity*: $[Ta_\lambda b] = -\lambda[a_\lambda b], \; [a_\lambda Tb] = (T + \lambda)[a_\lambda b]$;
- *skew skymmetry*: $[b_\lambda a] = -(-1)^{p(a)p(b)}[a_{-\lambda-T}b]$;
- *Jacobi identity*: $[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu}c] + (-1)^{p(a)p(b)}[b_\mu[a_\lambda c]]$.

An equivalent definition of a vertex algebra [BK03]

A *vertex algebra* is a quintuple $(V, \mathbb{1}, T, [._\lambda.], ::)$, where

(1) $(V, T, [._\lambda.])$ is a Lie conformal superalgebra

(2) $(V, \mathbb{1}, T, ::)$ is a unital differential superalgebra (with derivation T) satisfying

- *quasicommutativity*: $:ab: -(-1)^{p(a)p(b)} :ba: = \int_{-T}^{0} [a_\lambda b] d\lambda$,

- *quasiassociativity*:

$$
\begin{aligned}
&::ab:c: - :a:bc:: \\
&= :\left(\int_0^T d\lambda |a\right) [b_\lambda c]: + (-1)^{p(a)p(b)} :\left(\int_0^T d\lambda b\right) [a_\lambda c]:;
\end{aligned}
$$

(3) the λ-bracket $[._\lambda.]$ and the product $::$ are related by the *noncommutative Wick formula*

$$[a_\lambda :bc:] =: [a_\lambda b]c: + (-1)^{p(a)p(b)} :b[a_\lambda c]: + \int_0^\lambda [[a_\lambda b]_\mu c] d\mu.$$

If the "quantum corrections" (red terms) are removed, we get axioms of a *Poisson vertex algebra*.

Given a Lie conformal superalgebra R, its *universal enveloping vertex algebra* $V(R)$ is defined in the same way as for Lie superalgebras:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & V(R) \arrow[d, dashed] \\ & V \end{tikzcd}$$

PBW Theorem. Let $a_1, a_2, \dots$ be a basis of R over $\mathbb{C}$. Then the ordered monomials $: a_{i_1} a_{i_2} a_{i_3} \dots a_{i_s} :$ form a basis of $V(R)$. Here the normally ordered product is from right to left and $i_r < i_{r+1}$ if both a_{i_r} and $a_{i_{r+1}}$ are odd and $\leq$ otherwise.

Remark. Let R be a Lie conformal superalgebra. Then $S(R)$ is a Poisson vertex algebra. Its *quantization* is the universal enveloping vertex algebra $V(R)$ of R.

Basic examples.

(a) Let $\mathfrak{g}$ be a Lie superalgebra with a non-degenerate invariant supersymmetric bilinear form $(.|.)$. The *current* Lie conformal superalgebra of level $k \in \mathbb{C}$ is:

$$\mathrm{Cur}_k\mathfrak{g} = \mathbb{C}[T] \otimes \mathfrak{g}$$

with the (non-linear) λ-bracket $(a, b \in \mathfrak{g})$:

$$[a_\lambda b] = [a, b] + \lambda k(a|b)\mathbb{1}.$$

Its universal enveloping vertex algebra $V^k(\mathfrak{g})$ is called the *universal affine vertex algebra* of *level* k. Its quasiclassical limit is the PVA $\mathcal{V}^k(\mathfrak{g})$

If $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra and $k \neq -h^\vee$ ($\frac{1}{2}$ eigenvalue of the Casimir operator on $\mathfrak{g}$), then $V^k(\mathfrak{g})$ has a unique simple quotient $V_k(\mathfrak{g})$.

(b) Let A be a vector superspace with a superskewsymmetric bilinear form $\langle .,. \rangle$. The fermionic Lie conformal superalgebra

$$\mathbb{C}[T] \otimes A$$

with the λ-bracket ($\varphi, \psi \in A$)

$$[\varphi_\lambda \psi] = \langle \varphi, \psi \rangle \mathbb{1}.$$

The universal enveloping vertex algebra $F(A)$ is called the *fermionic vertex algera* based on A. It is simple.

The corresponding to $\mathrm{Cur}_k \mathfrak{g}$ Poisson vertex algebra is denoted by $\mathcal{V}^k(\mathfrak{g})$.

Note. Lie conformal superalgebras encode an important class of infinite-dimensional Lie superalgebras. Namely, given a Lie conformal superalgebra R, the corresponding Lie superalgebra is

$$\mathrm{Lie}_R = \mathrm{span}\{a_{(n)} | a \in R, n \in \mathbb{Z}\} / \langle (Ta)_{(n)} = -na_{(n-1)} \rangle$$

with bracket

$$[a_{(m)}, b_{(n)}] = \sum_{j \in \mathbb{Z}^+} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}.$$

Example 1. $\mathrm{Cur}_k \mathfrak{g}$ corresponds to the affine Lie superalgebra $\hat{\mathfrak{g}}$ with bracket $(a, b \in \mathfrak{g},\ m, n \in \mathbb{Z})$

$$[a_m, b_n] = [a, b]_{m+n} + km\delta_{m,-n}(a|b), \text{ where } a_m = a_{(m)}.$$

Example 2. Virasoro Lie conformal algebra $\mathrm{Vir} = \mathbb{C}[T]L$ with λ-bracket $[L_\lambda L] = (T + 2\lambda)L + \frac{\lambda^3}{12}c$ corresponds to Virasoro algebra $(m, n \in \mathbb{Z})$

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n}\frac{m^3 - m}{12}c, \text{ where } L_m = L_{(m+1)}.$$

§2. Classical and quantum affine Hamiltoninan reduction

Classical affine Hamiltonian reduction of a PVA $\mathcal{V}$ is associated to the data:

$$\varphi : \mathcal{V}_0 \to \mathcal{V} \text{ a PVA homomorphism}, I_0 \subset \mathcal{V}_0 \text{ a PVA ideal.}$$

Construction:

$$\mathcal{W}(\mathcal{V}, \varphi, I_0) = (\mathcal{V}/\mathcal{V}\varphi(I_0))^{\mathrm{ad}_\lambda \varphi(\mathcal{V}_0)}. \tag{1}$$

The *classical affine W-algebra* $\mathcal{W}^k(\mathfrak{g}, \mathfrak{s})$ is obtained by CHR from the affine PVA $\mathcal{V}^k(\mathfrak{g})$. From now on $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a non-degenerate supersymmetric invariant bilinear form and reductive $\mathfrak{g}_{\bar{0}}$, and $\mathfrak{s} = \langle e, 2x, f \rangle \subset \mathfrak{g}_{\bar{0}}$ is an sl_2 triple. Let

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j \tag{2}$$

be the ad x-eigenspace decomposition.

Let $\mathcal{V}_0 = \mathcal{V}(\mathfrak{g}_{>0}) \overset{\varphi}{\to} \mathcal{V}^k(\mathfrak{g})$ and I_0 be the differential ideal of $\mathcal{V}_0$, generated by $\{m - (f|m) | m \in \mathfrak{g}_{\geq 1}\}$.

Then $\mathcal{W}^k(\mathfrak{g}, \mathfrak{s})$, as defined by (1), is

$$(\mathcal{V}^k(\mathfrak{g})/\mathcal{V}^k(\mathfrak{g})I_0)^{\mathrm{ad}_\lambda \mathcal{V}_0}. \tag{3}$$

The *quantum affine W-algebra* $W^k(\mathfrak{g}, \mathfrak{s})$ is a quantization of the PVA $\mathcal{W}^k(\mathfrak{g}, \mathfrak{s})$.

In the case when the PVA $\mathcal{V}$ is a Poisson algebra, i.e. the λ-bracket is a Lie bracket and $T = 0$, CHR is given again by (1), and we obtain the *Slodowy slice* $e + \mathfrak{g}^f$ (Gan-Ginzburg):

$$\mathcal{W}_{\mathrm{fin}}(\mathfrak{g}, \mathfrak{s}) = (S(\mathfrak{g})/S(\mathfrak{g})\{m - (f|m) | m \in \mathfrak{g}_{\geq 1}\})^{\mathrm{ad}\ \mathfrak{g}_{>0}}.$$

This Poisson algebra is easy to quantize, just replace $S(\mathfrak{g})$ by $U(g)$:

$$W_{\mathrm{fin}}(\mathfrak{g}, \mathfrak{s}) = (U(\mathfrak{g})/U(\mathfrak{g})\{m - (f|m) | m \in \mathfrak{g}_{\geq 1}\})^{\mathrm{ad}\ \mathfrak{g}_{>0}}.$$

Unfortunately this approach doesn't work for PVA, and one should turn to BRST (Feigh-Frenkel 90, principal $\mathfrak{s}$; Kac-Roan-Wakimoto 03, arbitrary $\mathfrak{s}$).

Consider the following vertex algebra

$$C^k(\mathfrak{g},\mathfrak{s}) = V^k(\mathfrak{g}) \otimes F^{\mathrm{ch}} \otimes F^{\mathrm{ne}}, \tag{4}$$

where $F^{\mathrm{ch}} = F(\Pi(\mathfrak{g}_{<0} + \mathfrak{g}_{>0}))$, $F^{\mathrm{ne}} = F(\mathfrak{g}_{1/2})$, the corresponding superskewsymmetric bilinear forms being pairing between $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{>0}$ using $(.|.)$ and $\langle a,b\rangle^{\mathrm{ne}} = (f|[a,b])$ on $\mathfrak{g}_{1/2}$.

Define $\mathbb{Z}$-grading $C^k(\mathfrak{g},\mathfrak{s}) = \bigoplus_{j\in\mathbb{Z}} C^k_j$ by

$$\deg V^k(\mathfrak{g}) = 0 = \deg F^{\mathrm{ne}},\ \deg \mathfrak{g}_{>0} = -\deg \mathfrak{g}_{<0} = 1,$$

and let

$$\begin{aligned} d = \sum_{j\in S_{>0}} ((-1)^{p(u_j)} :\varphi^j u_j: + (f|u_j)\varphi^j) + \sum_{j\in S_{1/2}} :\varphi^j \Phi_j: \\ + \frac{1}{2}\sum_{i,j\in S_{>0}} (-1)^{p(u_j)} :\varphi^i\varphi^j\varphi_{[u_j,u_i]}:. \end{aligned} \tag{5}$$

Here $\{u_j\}_{j\in S_{>0}}$ is a basis of $\mathfrak{g}_{>0}$, compatible with the grading (2), $\{u^j\}$ the dual basis of $\mathfrak{g}_{<0}$, $\{\varphi_j\}$ and $\{\varphi^j\}$ are the corresponding bases of $\Pi\mathfrak{g}_{>0}$ and $\Pi\mathfrak{g}_{<0}$, and $\{\Phi_j\}_{j\in S_{1/2}}$ the corresponding basis of $\mathfrak{g}_{1/2}$.

The element d is an odd element of the vertex algebra $C^k(\mathfrak{g}, \mathfrak{s})$ of degree -1, and a direct calculation gives

$$[d_\lambda d] = 0. \tag{6}$$

In particular, $d_{(0)}d = 0$. Hence

$$d_{(0)}^2 = 0.$$

Moreover, for any a in a VA, $a_{(0)}$ is a derviation. Hence $(C^k(\mathfrak{g}, \mathfrak{s}), d_{(0)})$ is a homology complex.

Definition. $W^k(\mathfrak{g}, \mathfrak{s})$ is the 0^{th} homology of this complex, called the *quantum affine W-algebra*, associated with $(\mathfrak{g}, \mathfrak{s}, k)$.

This definition is motivated by

Theorem. [DSK06] Replacing in the above construction the vertex algebra by the corresponding Poisson vertex algebra (taking quasi-classical limit) produces the classical affine W-algebra $\mathcal{W}^k(\mathfrak{g}, \mathfrak{s})$.

Theorem. [KW04] Assume that $k \neq -h^\vee$.

(a) The vertex algebra $C^k(\mathfrak{g}, \mathfrak{s})$ is conformal with the Virasoro element

$$L = L_{Sug} + Tx + L^{\mathrm{ch}} + L^{\mathrm{ne}},$$

which is $d_{(0)}$-closed.

(b) For $a \in \mathfrak{g}$ let

$$J^{(a)} = a + \sum_{j \in S_{>0}} :\varphi^j \varphi_{[u_j, a]}: \in C^k(\mathfrak{g}, \mathfrak{s}).$$

Then for each $a \in \mathfrak{g}^f_{-j}$ $(j \geq 0)$ there exists a $d_{(0)}$-closed element $J^{\{a\}} \in C^k(\mathfrak{g}, \mathfrak{s})$ of conformal weight $1 + j$, such that $J^{\{a\}} - J^{(a)}$ is a linear combination of normally ordered products of elements $J^{(b)},\ b \in \mathfrak{g}_{-s},\ 0 \leq s < j$, the elements $\Phi_j, j \in S_{1/2}$, and of their derivatives (images under powers of T).

(c) The homology classes of the elements $J^{\{a_i\}}$, where $\{a_i\}$ is a basis of $\mathfrak{g}^f$, compatible with the gradation (2), strongly generate $W^k(\mathfrak{g},\mathfrak{s})$ and obey the PBW theorem.

(d) $H_j(C^k(\mathfrak{g},\mathfrak{s}),d_0)=0$ if $j\neq 0$.

(e) The central charge of L is

$$c_k(\mathfrak{g},x)=\text{sdim }\mathfrak{g}_0-\frac{1}{2}\text{sdim }\mathfrak{g}_{\frac{1}{2}}-\frac{12}{k+h^\vee}|\rho-(k+h^\vee)x|^2.$$

(f) The vertex algebra $W^k(\mathfrak{g},\mathfrak{s})$ is realized as a subalgebra of $V^{\alpha_k}(\mathfrak{g}_{\leq 0})\otimes F^{\mathrm{ne}}$, where α_k is a 2-cocycle on $\mathfrak{g}_{\leq 0}$ given by

$$\alpha_k(u,v)=(k+h^\vee)(u|v)-\frac{1}{2}\kappa_0(u,v),\ \kappa_0 \text{ is the Killing form.} \quad (7)$$

§3. Minimal quantum affine W-algebra

The $\frac{1}{2}\mathbb{Z}$-gradation (2) of $\mathfrak{g}$, defined by ad x, is called *minimal* if it is of the form

$$\mathfrak{g} = \mathbb{C}f \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathbb{C}e. \tag{8}$$

They correspond to the nilpotent orbit of f in a simple component of $\mathfrak{g}_{\bar{0}}$ of minimal dimension (with the exception of $\mathfrak{g} = \mathrm{osp}(3|n)$, $n \geq 2$, and component $= \mathrm{so}_3$). The central charge is (we assume $k \neq -h^\vee$):

$$c_k(\mathfrak{g}, x) = \frac{k \text{ sdim } \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4, \tag{9}$$

if the root θ of e is normalized by $(\theta|\theta) = 2$, which we shall assume. Then $x = \frac{1}{2}\theta$, and we have an orthogonal decomposition:

$$\mathfrak{g}_0 = \mathbb{C}x \oplus \mathfrak{g}^\natural,$$

and the centralizer of f is

$$\mathfrak{g}^f = \mathbb{C}f \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}^\natural.$$

Hence a minimal W-algebra is generated by the Virasoro element L, the elements $J^{\{u\}}$, $u \in \mathfrak{g}^\natural$, of conformal weight 1, and $G^{\{v\}}$, $v \in \mathfrak{g}_{-\frac{1}{2}}$, of conformal weight $\frac{3}{2}$. The λ-brackets between them are

$$
\begin{aligned}
[L_\lambda L] &= (T+2\lambda)L + \frac{c_k(\mathfrak{g},x)}{2}\lambda^3;\\
[L_\lambda J^{\{u\}}] &= (T+\lambda)J^{\{u\}};\\
[L_\lambda G^{\{v\}}] &= (T+\frac{3}{2}\lambda)G^{\{v\}};\\
[J^{\{u\}}_\lambda G^{\{v\}}] &= G^{\{u,v\}};\\
[J^{\{u\}}_\lambda J^{\{v\}}] &= J^{\{[u,v]\}} + \lambda\beta_k(u,v), \text{ where}\\
\beta_k(u,v) &= (k+\frac{1}{2}h^\vee)(u|v) - \frac{1}{2}\kappa_0(u,v);
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
[G^{\{u\}}{}_\lambda G^{\{v\}}] = -2(k+h^\vee)\langle u,v\rangle^{\mathrm{ne}} L + \langle u,v\rangle^{\mathrm{ne}} \sum_{i=1}^{d=\dim\mathfrak{g}^\natural} :J^{\{u^i\}}J^{\{u_i\}}:\\
+2\sum_{i,j=1}^{d}\langle [u_i,u],[v,u^j]\rangle^{\mathrm{ne}} :J^{\{u^i\}}J^{\{u_j\}}: +2(k+1)(T+2\lambda)J^{\{[[e,u],v]\}^\natural}\\
+2\lambda\sum_{i,j=1}^{d}\langle [u_i,u],[v,u^j]\rangle^{\mathrm{ne}} J^{\{u^i,u_j\}} + 2p(k)\lambda^2\langle u,v\rangle^{\mathrm{ne}},
\end{aligned}
$$

where $\{u_i\}$ and $\{u^i\}$ are dual bases of $\mathfrak{g}^\natural$ and $p(k)$ is a mysterious *monic quadratic polynomial in k*.

By abuse of notation we denote the W-algebra $W^k(\mathfrak{g}, \mathfrak{s})$, where $\mathfrak{s} = \{e, 2x, f\}$ with ad x defining a minimal gradation (8), by $W^k_{min}(\mathfrak{g})$, and its simple quotient by $W_k^{min}(\mathfrak{g})$.

There are one, two, or three minimal gradations of $\mathfrak{g}$, but at most one corresponds to unitary W-algebras.

Remarks.

(a) $W_k^{\mathrm{min}}(\mathfrak{g})$ collapses to its affine part iff $p(k) = 0$ [A K MF P P].

(b) Negatives of the roots of $p(k)$ are singularities of the R-matrix of $Y(\mathfrak{g})$ in (the adjoint representation) $\oplus\mathbb{C}$ [K MF P].

§4. Unitary representation of vertex algebras

Joint work with P. Moseneder Frajria and P. Papi

We shall assume that the vertex algebra V in question is *conformal*, i.e. there exists a Virasoro vector $L \in V$, i.e. $Y(L,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, where L_n satisfy relations of the Virasoro algebra, $L_{-1} = T$, and L_0 is diagonalizable with the eigenspace decomposition

$$V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+} V_n,\ \dim V_n < \infty,\ V_0 = \mathbb{C}\mathbb{1}. \tag{11}$$

Let ϕ be a conjugate linear involution of V, and let

$$g(a) = e^{-\pi\sqrt{-1}(\frac{1}{2}p(a)+\Delta_a)}\phi(a),\ a \in V_{\Delta_a}, \tag{12}$$

where $p(a) = 0,\ 1 \in \mathbb{Z}$ is the parity of a.

A Hermitian form $H(.,.)$ on V is called *ϕ-invariant* if for all $a \in V$ one has

$$H(u, Y(a,z)v) = H(Y(A(z)a, z^{-1})u, v),$$
$$\text{where } A(z) = e^{zL_1}z^{-2L_0}g,\ u,v \in V. \tag{13}$$

The definition for u, v in a V-module M is similar.

Comments.

- Formula (13) with $g(a) = (-1)^{\Delta_a}$ first appeared in [Borcherds 86] for the construction of the coadjoint module for V purely even and $\mathbb{Z}$-grading.
- If (11) is compatible with parity, then $g(a) = (-1)^{\Delta_a + 2\Delta_a^2}\phi(a)$ (Dong-Lin)
- If $a \in V$ is *quasiprimary*, i.e. $L_1(a) = 0$, then (13) means that

$$a_n^* = (ga)_{-n},$$

where $Y(a,z) = \sum_{n \in \mathbb{Z} - \Delta_a} a_n z^{-n-\Delta_a}$. For example,

$$L_n^* = L_{-n}.$$

- Formula $\langle Y^{M'}(a,z)m', m\rangle = \langle m', Y^M(A(z)a, z^{-1})m\rangle$ defines a V-module structure on M'.

Theorem A.

(a) Let ϕ be a conjugate linear involution of the Lie superalgebra $\mathfrak{g}$, such that
$$\phi(e) = e,\ \phi(f) = f,\ \phi(x) = x. \tag{14}$$
Then ϕ descends to a conjugate linear involution of the W-algebra $W^k(\mathfrak{g}, \mathfrak{s}),\ k \in \mathbb{R}$.

(b) There exists a unique ϕ-invariant Hermitiain form $H(.,.)$ on $W^k(\mathfrak{g}, \mathfrak{s})$, such that $H(1, 1) = 1$.

We denote by $W_k(\mathfrak{g}, \mathfrak{s})$ the quotient of $W^k(\mathfrak{g}, \mathfrak{s})$ by the kernel of the form $H(.,.)$. It is a simple vertex algebra. We shall assume that $k \neq -h^\vee$. The vertex algebra $W_k(\mathfrak{g}, \mathfrak{s})$ is called *unitary* if H is positive definite.

§5. Theorems on unitarity of minimal W-algebras

Theorem B. If $W_k^{\min}(\mathfrak{g})$ is a unitary simple minimal quantum affine W-algebra with *non-collapsing level* (i.e. $W_k^{\min}(\mathfrak{g})$ is not an affine vertex algebra), then

(a) the parity of $\mathfrak{g}$ is compatible with its ad x-gradation,

(b) the conjugate linear involution ϕ is *almost compact*, i.e. (14) holds and $\phi|_{\mathfrak{g}^\natural}$ is a compact conjugate linear involution. Such a conjugate linear involution exists and is essentially unique.

From part (a) of Theorem B it follows that a minimal simple W-algebra $W_k^{\min}(\mathfrak{g})$ with non-collapsing k can be unitary only for the following $\mathfrak{g}$:

$$\mathrm{psl}(2|2),\ \mathrm{spo}(2|m) \text{ for } m \geq 0,\ D(2,1;a) \text{ for } a \in \mathbb{R},\ F(4),\ G(3) \tag{15a}$$

$$\mathrm{sl}(2|m) \text{ for } m \geq 3,\ \mathrm{osp}(4|m) \text{ for } m > 2 \text{ even.} \tag{15b}$$

It is not hard to show that (15b) doesn't give unitary non-collapsing W-algebras (in the first case only $k = -1$ is unitary level, but it is collapsing; in the second case no k is unitary).

The remaining cases (15a) are very interesting, and include all vertex algebras corresponding to superconformal algebras ($\Longleftrightarrow$ Lie conformal superalgebras with linear λ-bracket):

(a) $W_k^{\min}(\mathrm{spo}(2|N))$ corresponds to Virasoro algebra for $N = 0$, Neveu-Schwarz algebra for $N = 1$, $N = 2$ superconformal algebra for $N = 2$, $N = 3$ superconformal algebra for $N = 3$ after tensoring with one fermion;

(b) $W_k^{\min}(\mathrm{psl}(2|2))$ corresponds to $N = 4$ superconformal algebra;

(c) $W_k^{\min}(D(2,1;a))$ corresponds to the big $N = 4$ superconformal algebra after tensoring with four fermions and one boson.

(N is the number of linearly independent elements of the W-algebra of conformal weight $\frac{3}{2}$.)

The cases where $N = 0, 1, 2$ (in the language of Lie superalgebras) were extensively studied in the mid 80's. Putting $k = \frac{1}{p} - 1$ in all three cases, we obtain from (9) the following formulas for central charges of $W^k_{\min}(\mathfrak{g})$, respectively:

$$c_k = 1 - \frac{6}{p(p+1)} \text{ for Virasoro } (N = 0) \tag{16}$$

$$c_k = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right) \text{ for Neveu-Schwarz } (N = 1) \tag{17}$$

$$c_k = 3\left(1 - \frac{2}{p}\right) \text{ for } N = 2. \tag{18}$$

Theorem. (Many authors, mid 80's) The complete list of unitary $N = 0$, 1, and 2 minimal W-algebras is given by (16), (17), (18) respectively for integers $p \geq 2$, or for $c_k \geq 1$, $\frac{3}{2}$ or 3 respectively. (These are all $W_k^{\min}(\mathfrak{g})$ with $\mathfrak{g}^\natural$ abelian.)

We shall assume that $\mathfrak{g}^\natural$ is not abelian. If $\mathfrak{g}$ is from the list (15a), then $\mathfrak{g}^\natural$ is semisimple: $\mathfrak{g}^\natural = \bigoplus_{i=1}^{s} \mathfrak{g}_i^\natural$, where $\mathfrak{g}_i^\natural$ are simple and $s = 1$ or 2. Let $M_i(k)$ be the level of the affine subalgebra of $W_k^{\min}(\mathfrak{g})$, generated by the $J^{\{u\}}$ with $u \in \mathfrak{g}_i^\natural$.

Table 1.

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$h^\vee$	$M_i(k)$	χ_i
$\mathrm{psl}(2\|2)$	sl_2	0	$-k-1$	-1
$\mathrm{spo}(2\|3)$	sl_2	$\frac{1}{2}$	$-4k-2$	-2
$\mathrm{spo}(2\|m)$, $m > 4$	so_m	$2-\frac{m}{2}$	$-2k-1$	-1
$D(2,1;a)$	$\mathrm{sl}_2 \oplus \mathrm{sl}_2$	0	$-(1+a)k-1,\ -\frac{1+a}{a}k-1$	-1, -1
$F(4)$	so_7	-2	$-\frac{3}{2}k-1$	-1
$G(3)$	G_2	$-\frac{3}{2}$	$-\frac{4}{3}k-1$	-1

Theorem C. The complete list of nontrivial unitary vertex algebras $W_{-k}^{\min}(\mathfrak{g})$ with $\mathfrak{g}_0$ non-abelian is as follows:

- $\mathfrak{g} = \mathrm{psl}(2|2): \; k \in \mathbb{N}+1$
- $\mathfrak{g} = \mathrm{spo}(2|3): \; k \in \frac{1}{4}(\mathbb{N}+2)$
- $\mathfrak{g} = \mathrm{spo}(2|m), \; m > 4: \; k \in \frac{1}{2}(\mathbb{N}+1)$
- $\mathfrak{g} = D(2,1;\frac{m}{n}), \; m,n$ coprime $\in \mathbb{N}: \; k \in \frac{mn}{m+n}\mathbb{N}, \; k \neq \frac{1}{2}$
- $\mathfrak{g} = F(4): \; k \in \frac{2}{3}(\mathbb{N}+1)$
- $\mathfrak{g} = G(3): \; k \in \frac{3}{4}(\mathbb{N}+1)$

That these conditions on k are necessary for unitarity follows from $M_i(k) \in \mathbb{Z}_+$ for all i and looking at Table 1.

Definition. The set of k above is called the *unitary range* for $W_k^{\min}(\mathfrak{g})$.

In order to prove that these conditions are sufficient, we use the KW realization of $W^k_{\min}(\mathfrak{g})$ in $V^{\alpha_k}(\mathfrak{g}_{\leq 0}) \otimes F^{\mathrm{ne}}$, where α_k is the cocycle given by (7). Using the obvious homomorphism $\mathfrak{g}_{\leq 0} \to \mathfrak{g}_0$, we obtain a vertex algebra homomorphism

$$\Psi : W^k_{\min}(\mathfrak{g}) \to V_{\alpha_k}(\mathfrak{g}_0) \otimes F^{\mathrm{ne}}. \tag{19}$$

Explicitly, it is given on the generators of $W^k_{\min}(\mathfrak{g})$ by

$$\begin{aligned}
\Psi(J^{\{a\}}) &= a + \frac{1}{2} \sum_{j \in S_{\frac{1}{2}}} :\Phi^j \Phi_{[u_j,a]},\ \ a \in \mathfrak{g}^\natural \qquad (20)\\
\Psi(G^{\{v\}}) &= \sum_{j \in S_{\frac{1}{2}}} :[v,u_j]\Phi^j: - (k+1) \sum_{j \in S_{\frac{1}{2}}} (v|u_j) T\Phi^j \\
&\quad + \frac{1}{3} \sum_{i,j \in S_{\frac{1}{2}}} :\Phi^i \Phi^j \Phi_{[u_j,[u_i,v]]}:,\ \ v \in \mathfrak{g}_{-\frac{1}{2}}, \\
\Psi(L) &= \frac{1}{2(k+h^\vee)} \sum_{j \in S_0} :u_j u^j: + \frac{k+1}{k+h^\vee} Tx \\
&\quad + \frac{1}{2} \sum_{j \in S_{\frac{1}{2}}} :(T\Phi^j)\Phi_j:,\ \ \text{where } \langle \Phi_i, \Phi^j \rangle^{\mathrm{ne}} = \delta_{ij}.
\end{aligned}$$

Note that the level of the affine vertex superalgebra of $V^{\alpha_k}(\mathfrak{g}_0)$, generated by $\mathfrak{g}_i^\natural$ is equal to

$$M_i(k) + \chi_i, \tag{21}$$

where the $M_i(k)$ and χ_i are integers given in Table 1. But for the numbers $M_i(k) \in \mathbb{Z}_+$, one of the numbers (21) is negative if and only if k is a collapsing level, when we have unitary. Hence we may assume that the numbers (21) are nonnegative, hence we have unitarity on the right of (19), since F^{ne} is unitary.

Unfortunately, the map Ψ is not an isometry, due to the term $\frac{k+1}{k+h^\vee}Tx$ in (20).

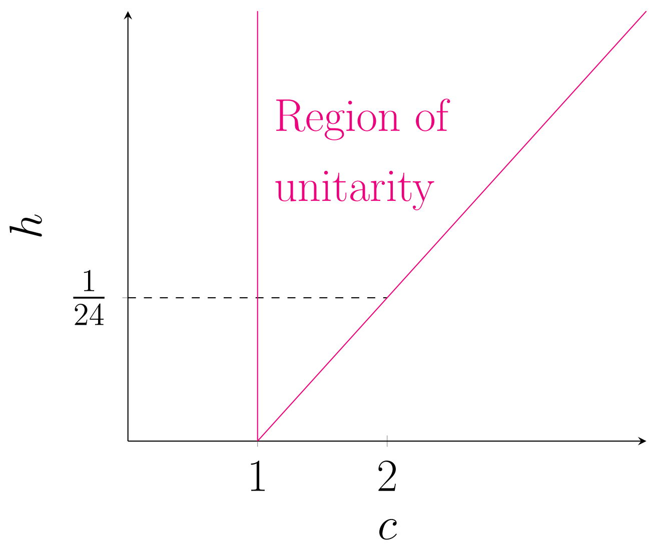
Example. $\mathfrak{g} = \mathrm{sl}_2$, then $W^k_{\min}(\mathfrak{g}) = \mathrm{Vir}$ and $\Psi(L) = \frac{1}{2} : a^2 : + \gamma T a$ (*quantum Miura map*) where $[a_\lambda a] = \lambda \cdot \mathbb{1}$ is the free boson LCA and $\gamma \in \mathbb{C}$. Hence

$$\Psi(L_n) = \frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j} a_{j+n} : - \gamma (n+1) a_n, \ \gamma \in \mathbb{C}.$$

But $a_n^* = -a_n$, hence $\Psi(L_n)^* \neq \Psi(L_{-n})$. However the *Fairlie modification* restores isometry:

$$\tilde{L}_n = \frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j} a_{j+n} : + i \lambda n a_n \text{ for } n \neq 0,$$
$$\tilde{L}_0 = \frac{1}{2}(\lambda^2 + \mu^2) + \sum_{j \geq 1} a_{-j} a_j,$$

which produces a unitary representation of the Virasoro Lie algebra, provided that $\lambda, \mu \in \mathbb{R}$, with central charge $1 + 12\lambda^2$:



Extend, using positivity of the determinant of the Hermitian form, to get unitarity in the region $h \geq 0, c \geq 1$.

In order to generalize Fairlie's modification, note that the FFR (20) is actually

$$\Psi : W^k_{\min}(\mathfrak{g}) \to B \otimes V_{\alpha_k}(\mathfrak{g}^\natural) \otimes F^{\mathrm{ne}}, \tag{22}$$

where B is a free boson associated to the LCA $\mathbb{C}[T]x$ with $[x_\lambda x] = \frac{1}{2}(k+h^\vee)\lambda$. We modify the RHS of (22), replacing B by its irreducible highest weight module M with highest weight $\in i\mathbb{R}$, and modifying Ψ as follows:

$$\begin{aligned}
&\Psi_{\mathrm{mod}} : J^{\{u\}}_n \mapsto \Psi(J^{\{u\}})_n \text{ (unmodified)}, \\
&\Psi_{\mathrm{mod}} : G^{\{v\}}_n \mapsto \Psi(G^{\{v\}})_n - (k+1)(\Phi_{[e,v]})_n, \\
&\Psi_{\mathrm{mod}} : L_n \mapsto \Psi(L)_n + \frac{k+1}{k+h^\vee} x^M_n - \frac{(k+1)^2}{4(k+h^\vee)} \mathbb{1}_n.
\end{aligned} \tag{23}$$

These formulas define an irreducible highest weight module over $W^k_{\min}(\mathfrak{g})$, which is unitary if $V_{\alpha_k}(\mathfrak{g}^\natural)$ is unitary.

Irreducible highest weight modules over $W^k_{\min}(\mathfrak{g})$ are parametrized by pairs ν, l_0, where $\nu \in (\mathfrak{h}^\natural)^*$ and $l_0 \in \mathbb{C}$ is the lowest eigenvalue of L_0, and denoted by $L^W(\nu, l_0)$. *Unitarity* of this module implies the following conditions:

(a) k must be in the unitary range, hence $M_i(k) \in \mathbb{Z}_+$,

(b) $\nu \in \hat{P}^+_k = \{\nu$ dominant integral for $\mathfrak{g}^\natural$ and $\nu(\theta_i^\vee) \leq M_i(k)\}$,

(c) l_0 is a nonnegative real number,

(d) $l_0 \geq A(\nu) := \frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)} + \frac{(\xi|\nu)}{k+h^\vee}((\xi|\nu) - k - 1)$, where $\xi \in (\mathfrak{h}^\natural)^*$ is the highest weight of the $\mathfrak{g}^\natural$-module $\mathfrak{g}_{-\frac{1}{2}}$.

Note that $W_k^{\min}(\mathfrak{g}) = L^W(0,0)$.

Definition. $\nu \in \hat{P}_k^+$ is called an *extremal weight* if $\nu + \xi \notin \hat{P}_k^+$. Equivalently, if $(\nu + \xi)(\theta_i^\vee) = M_i(k) + \chi_i < 0$ for some i.

Proposition. If the module $L^W(\nu, l_0)$ is unitary, then conditions (a), (b), (c), (d) hold and also

(e) $l_0 = A(\nu)$ if ν is extremal.

Theorem D. If conditions (a), (b), (c), (d) hold and ν is not extremal, then the module $L^W(\nu, l_0)$ is unitary. Consequently the vertex algebra $W_k^{\min}(\mathfrak{g})$ is nontrivial unitary iff k is in the unitary range for $W_k^{\min}(\mathfrak{g})$.

Conclusion. The unitary $W^k_{\min}(\mathfrak{g})$-modules $L^W(\nu, l_0)$ are contained in the following list:

- k is in the unitary range,
- $\nu \in \hat{P}^+_k$,
- $l_0 \geq A(\nu)$ if ν is not extremal,
- $l_0 = A(\nu)$ if ν is extremal.

It is still an open problem whether the "extremal" modules $L^W(\nu, A(\nu))$ are unitary.

Conjecture 1. They are unitary.

Remark. The answer coincides with Eguchi-Taormina for $N = 4$ and with Miki for $N = 3$, but they don't provide a proof.

Theorem E. [joint with D. Adamovic] All these $W^k_{\min}(\mathfrak{g})$-modules descend to $W_k^{\min}(\mathfrak{g})$.

Conjecture 2. A QFT type vertex algebra is unitary iff it has a unitary module.

Conjecture 3. If V is a unitary vertex algebra of CFT type, then every of its unitary modules descends to its irreducible quotient.

Sketch of proof of the theorems. The necessary conditions of unitarity follow from conditions of unitarity of the affine vertex subalgebra $V_{\beta_k}(\mathfrak{g}^\natural)$ and the condition $\|G^{\{a\}}_{-\frac{1}{2}}\mathcal{V}_{\nu,l_0}\|^2 \geq 0$.

Using the generalized Fairlie modification we show that $L^W(\nu, l_0)$ is unitary, provided that ν is not extremal (recall that the levels of $V_{\alpha_k}(\mathfrak{g}^\natural)$ are, by (21), $M_i(k) + \chi_i$), and $l_0 \geq B(\nu)$ for some $B(\nu) > A(\nu)$.

To prove unitarity for $l_0 \geq A(\nu)$ it suffices to show that the determinant of the Hermitian form is positive for $l_0 > A(\nu)$. It is a polynomial $P(l_0)$, whose factors are explicitly known [KW04], and miraculously, $P(l_0) > 0$ for $l_0 > A(\nu)$.

Hence we can go down from $B(\nu)$ all the way to $A(\nu)$.

Finally, $\nu = 0$ is an extremal weight iff $M_i(k) + \chi_i < 0$ for some i. This happens iff k is a collapsing level, in which case we do have unitarity.

§6. Characters of unitary $W^k_{\min}(\mathfrak{g})$ modules

The characters of the unitary irreducible highest weight $W^k_{\min}(\mathfrak{g})$-modules (=$W_k^{\min}(\mathfrak{g})$-modules) are computed by making use of the quantum Hamiltonian reduction (QHR). Given a $\hat{\mathfrak{g}}$-module M, its QHR is the $W^k(\mathfrak{g}, \mathfrak{s})$-module $H_0(M)$, the 0^{th} homology of the complex

$$(M \otimes F^{\text{ch}} \otimes F^{\text{ne}}, d_{(0)}).$$

Since M is a $V^k(\mathfrak{g})$-module, $M \otimes F^{\text{ch}} \otimes F^{\text{ne}}$ is a $C^k(\mathfrak{g}, s)$-module.

The functor H_0 maps Verma modules to Verma modules [KW04], and irreducible $\hat{\mathfrak{g}}$-modules to irreducible $W^k_{\min}(\mathfrak{g})$-modules or 0 [Arakawa 05].

The highest weights of irreducible highest weight $V^k(\mathfrak{g})$-modules $L(\Lambda)$ are

$$\Lambda = \hat{\nu}_h = k\Lambda_0 + \nu + h\theta,\ \nu \in \hat{P}^k_+,\ h \in \mathbb{R}_{\geq 0}.$$

Then $H_0(L(\Lambda)) = L^W(\nu, l_0)$, where

$$l_0 = l(h) := \frac{(\hat{\nu}_h|\hat{\nu}_h + 2\hat{\rho})}{2(k+h^\vee)} - h.$$

Thus, if

$$\text{ch } L(\hat{\nu}_h) = \sum_\lambda c_\lambda \text{ch } M(\lambda),$$

then

$$\text{ch } L^W(\nu, l_0) = \sum_\lambda c_\lambda \text{ch } M^W(\lambda),$$

where $M(\lambda)$ is a $V^k(\mathfrak{g})$-Verma module, and $M^W(\lambda)$ is a $W^k_{\min}(\mathfrak{g})$-Verma module.

Thus, the problem reduces to the computation of characters of the $\hat{\mathfrak{g}}$-modules $L(\hat{\nu}_h)$. There are two cases to consider:

(1) typical (massive), which happens if $l_0 > A(k, \nu)$;

(2) maximally atypical (massless), which happens if $l_0 = A(k, \nu)$.

Let $\hat{W}^\natural$ be the affine Weyl group for the Lie algebra $\mathfrak{g}^\natural$, and $\Pi_{\bar{1}}$ be the set of simple isotropic roots of $\mathfrak{g}$. Then

Case 1: $\mathrm{ch}\, L(\hat{\nu}_h) = \sum_{w \in \hat{W}^\natural} (\det w) \mathrm{ch}\, M(w.\hat{\nu}_h)$.

Case 2: $\mathrm{ch}\, L(\hat{\nu}_h) = \sum_{w \in \hat{W}^\natural} \sum_{\gamma \in \mathbb{Z}_+ \Pi_{\bar{1}}} (-1)^\gamma (\det w) \mathrm{ch}\, M(w.(\hat{\nu}_h - \gamma))$, where $(-1)^\gamma := (-1)^{n_1+n_2+\dots}$ for $\gamma = \sum_i n_i \gamma_i,\ \Pi_{\bar{1}} = \{\gamma_i\}_i$.

Proof of Case 1 is easy. Case 2 is a special case of the KW84-conjecture, proved in all the above cases in [GK2015], except for $\mathfrak{g} = D(2,1;a),\ \nu \neq 0$.