Varieties of affine vertex operator algebras of type A_2 at non-admissible Levels

Cuipo Jiang

Shanghai Jiao Tong University

j/w Jingtian Song

Represention Theory XVIII (Vertex Algebras)

Dubrovnik

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1. Varieties of VAS.

Let V be a vertex algebra, and $S \subset V$. Recall that V is called strongly generated by S, if V is linearly spanned by elements of the form:

$$\{ x^1_{-m_1} \cdots x^s_{-m_s} \mathbb{1} \mid x^i \in S, \ m_i \in \mathbb{Z}_{\geq 1} \}.$$

If further, $|S| < \infty$, V is called finitely strongly generated.

Recall [Zhu]

$$C_2(V) = \{ u_{-2}v \mid u, v \in V \}$$

and

$$R_V = V / C_2(V).$$

Then R_V is a Poisson algebra with relations

$$\bar{u} \cdot \bar{v} = \overline{u_{-1}v}, \quad \{ \bar{u}, \bar{v} \} = \overline{u_0 v}.$$

R_V is usually called the C_2-Zhu alg of V.

It is easy to see that V is finitely strongly generated if and if R_V is finitely generated.

Recall that V is called C_2-cofinite if R_V is finite-dimensional [Zhu, 96]

Let

$$X_V = \mathrm{Specm}(R_V).$$

X_V is called the assoc. variety of V. [Arakawa, 10].

V is called lisse if $X_V = \{0\}$ [Arakawa, 10]

Theorem [Arakawa, 10] Let V be a finitely strongly generated VA. Then V is C_2-cofinite iff V is Lisse.

Since R_V is a Poisson alg, we have [Brown - Gordon, 03]

$$X_V = \bigsqcup_{i=1}^{r} X_i ,$$

X_i are smooth analytic Poisson varieties. So X_V is a union of symplectic leaves.

V is called quasi-lisse if X_V has only finitely many symplectic leaves [Arakawa-Kawasetsu, 18]

Theorem [Arakawa-Kawasetsu, 18]

If V is quasi-lisse, then V has finitely many irreducible ordinary modules.

Given a VA V, $X_V = ?$ (fairly open).

2. Varieties of affine VAs and affine W-algebras

Let $\mathfrak{g}$ be a f.d. simple Lie alg $/\mathbb{C}$ with $(\cdot|\cdot) = \frac{1}{2h^\vee}$ Killing form.

For $k \in \mathbb{C}$, Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra assoc. to k and $\mathfrak{g}$. It is easy to see that

$$R_{V^k(\mathfrak{g})} \cong \mathbb{C}[\mathfrak{g}^*].$$

So

$$X_{V^k(\mathfrak{g})} = \mathfrak{g}^* \cong \mathfrak{g}.$$

Let V be a quotient of $V^k(\mathfrak{g})$.

Then X_V is a closed G-invariant subvariety of $\mathfrak{g}^*$.

Q: $X_V = ?$

In particular, $X_{L_k(\mathfrak{g})} = ?$

In general, we have

Theorem [Arakawa – J – Moreau, 20]

For any k and f.d. simple Lie algebra $\mathfrak{g}$, we have

$$X_{V^k(\mathfrak{g})} = X_{L^k(\mathfrak{g})}$$

if and only if $V^k(\mathfrak{g}) = L^k(\mathfrak{g})$.

Recall

Theorem [Gorelik – Kac, 07]

For $k \in \mathbb{C}$, $V^k(\mathfrak{g}) \neq L_k(\mathfrak{g})$ if and only if

$$r(k+h^\vee) \in \mathbb{Q}_{\geq 0} \setminus \{ \tfrac{1}{m} \mid m \in \mathbb{Z}_{\geq 1} \}$$

where r is the lacing number of $\mathfrak{g}$.

Theorem [I. Frenkel – Zhu, 92], [Li, 96], [Dong – Mason 04], [Arakawa, 10]

$L_k(\mathfrak{g})$ is C_2-cofinite $\iff k \in \mathbb{Z}_{\geq 0}$

$\iff X_{L_k(\mathfrak{g})} = \{0\}$.

Theorem [E. Frenkel – Gaitsgory 07], [Arakawa, 11]

If $k + h^\vee = 0$, then

$X_{L_k(\mathfrak{g})} = \mathcal{N}$ (the nilpotent cone).

Recall that $k \in \mathbb{Q}$ is called admissible [Kac–Wakimoto 03, ...], if

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{Z}_{\geq 0}, \quad (p, q) = 1$$

and

$$p \geq \begin{cases} h^\vee & (r^\vee, q) = 1 \\ h & (r^\vee, q) = r^\vee \end{cases}$$

If further, $q \geq h$ for $(q, r^\vee) = 1$

or

$$q \geq r^\vee h^\vee_{L_g} \text{ for } (q, r^\vee) = r^\vee,$$

then k is called a non-degenerate admissible number.

Theorem [Arakawa. 11] Let k be an admissible number, then

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}}_q \subset \mathcal{N}.$$

Furthermore, $X_{L_k(\mathfrak{g})} = \mathcal{N}$ if and only if k is a non-degenerate admissible number.

Q: What about $X_{L_k(\mathfrak{g})}$ when k is not admissible ?

Q: Is $X_{L_k(\mathfrak{g})}$ always irr ?

Up to now, complete results only for $L_k(\mathfrak{sl}_2)$.

Following [Collingwood – McGovern, 93]

Let $\mathfrak{g}/\mathbb{C}$ be a f.d. simple Lie alg.

Let $\mathfrak{l}$ be a Levi subalg of $\mathfrak{g}$, and

$\mathcal{O}_{\mathfrak{l}}$ a nilpotent orbit in $\mathfrak{l}$.

Then there is a unique nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ in $\mathfrak{g}$ meeting $\mathcal{O}_{\mathfrak{l}} \oplus \mathfrak{n}$ on an open dense set. $\mathcal{O}_{\mathfrak{g}}$ is called the induced orbit from $\mathcal{O}_{\mathfrak{l}}$.

- If $O_{\mathfrak{l}} = \{0\}$, then $O_{\mathfrak{g}}$ is called a Richardson orbit.

- A nilpotent orbit in $\mathfrak{g}$ which is not reduced from any proper subalgebra is called rigid.

- It is known that if $\mathfrak{g} = \mathfrak{sl}_n$, then every nilpotent orbit is a Richardson orbit. So the only rigid orbit in $\mathfrak{sl}_n$ is $\{0\}$.

Let $x \in \mathfrak{g}$, and $x = x_s + x_n$ be the Jordan-Chevalley decomposition of x.

The Jordan class of x is defined as

$$J_G(x) = G.\left(z(\mathfrak{g}^{x_s})^{reg} + x_n \right),$$

where for $X \subset \mathfrak{g}$,

$$X^{reg} = \{ y \in X \mid \dim O_y \text{ is maximal} \}.$$

Remark.

1. $Z(\mathfrak{g}^{x_s})^{reg}$ is a dense open subset of $z(\mathfrak{g}^{x_s})$.

2. $J_G(x)$ is a G-invariant, irreducible and locally closed subset of $\mathfrak{g}$.

Let $\mathfrak{l} = \mathfrak{g}^{x_s}$, then $\mathfrak{l}$ is a Levi subalg of $\mathfrak{g}$. Let $\mathcal{O}_{\mathfrak{l}}$ be the nilpotent orbit in $\mathfrak{l}$ of x_n.

- Jordan classes $\xleftrightarrow{\text{one to one}}$ $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$.

For $m \in \mathbb{Z}_{\geq 0}$, denote

$$\mathfrak{g}^{(m)} = \{ x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = m \}.$$

An irreducible component of $\mathfrak{g}^{(m)}$ is called a sheet of $\mathfrak{g}$.

- A sheet is a finite disjoint union of Jordan classes. So a sheet S contains a unique dense open Jordan class J, and

$$\bar{S} = \bar{J}, \quad S = (\bar{J})^{reg}.$$

In particular, $\bar{S}$ is irreducible.

Fact: A Jordan class with datum $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ is open dense in a sheet if and only if $\mathcal{O}_{\mathfrak{l}}$ is rigid.

- A sheet with datum $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ is called Dixmier if $\mathcal{O}_{\mathfrak{l}} = \{0\}$.

Theorem [Arakawa - Moreau, 16]

- For $n \geqslant 4$, $X_{L_{-1}(\mathfrak{sl}_n)} \cong \overline{\mathbb{S}}_{\mathbb{1}_1}$
- For $m \geqslant 2$, $X_{L_{-m}(\mathfrak{sl}_{2m})} \cong \overline{\mathbb{S}}_{\mathbb{1}_m}$
- $X_{L_{2-r}(\mathfrak{so}_{2r})} = \overline{\mathbb{S}}_{\mathbb{1}_1} \cup \overline{\mathbb{S}}_{\mathbb{1}_2}$ (reducible).

We will consider $X_{L_k(\mathfrak{sl}_3)}$, for non-admissible k, that is, $k+3 = \frac{2}{2m+1}$, $m \geqslant 0$.

Note that if $m=0$, then $k = -1$.

Representations of $V^k(\mathfrak{sl}_3)$, $L_k(\mathfrak{sl}_3)$

and assoc. W algebras.

[Adamovic - Kawasetsu - Ridout, 23], [Adamovic - Creutzig - Genra, 21]. [Fehily - Kawasetsu - Ridout, 21], [Kawasetsu - Ridout - Wood, 21], [Adamovic - Kontrec, 20, 19], [Creutzig, 18], [Arakawa, 16], [Adamovic, 16], [Wang, 97],

..., ...

Theorem [J-Song, 23]

(1) $X_{L_{-4}(sl_3)}$ is the closure of a Dixmier sheet:

$$X_{L_{-4}(sl_3)} = \overline{S_1} = \overline{G\cdot\mathbb{C}^*(\alpha_1-\alpha_2)} = \overline{J}$$
$$= G\cdot\mathbb{C}^*(\alpha_1-\alpha_2) \cup \overline{\mathbb{O}_{min}},$$

where $J = G\cdot\mathbb{C}^*(\alpha_1-\alpha_2)$,

$S_1 = G\cdot\mathbb{C}^*(\alpha_1-\alpha_2) \cup \mathbb{O}_{min}$, $\mathbb{O}_{min} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)$,

$\mathfrak{l} = \mathfrak{h} \oplus \mathbb{C}e_{\alpha_1+\alpha_2} \oplus \mathbb{C}f_{\alpha_1-\alpha_2}$.

In particular, $\dim X_{L_{-4}(sl_3)} = 5$.

(2) $L_{-4}(sl_3)$ is not quasi-lisse.

It is known that for $\mathfrak{g} = \mathfrak{sl}_3$,

$\mathfrak{g}^*$ contains G-invariant subvarities

of dimension 7. For example,

- Let h be a regular semisimple element of $\mathfrak{sl}_3$. Then

$$\mathcal{X}_h = \overline{G \cdot \mathbb{C}^* h} \supset \mathcal{N}$$

and $$\dim \mathcal{X}_h = 7.$$

- Let $\lambda = \alpha_2 - \alpha_1$, then

$$\mathcal{X}_{\lambda + f_\theta} = \overline{G \cdot (\mathbb{C}^* \lambda + f_\theta)} \supset \mathcal{N}$$

and $$\dim \mathcal{X}_{\lambda + f_\theta} = 7.$$

Q: Is there k such that $\dim X_{L_k(\mathfrak{sl}_3)} = 7$?

In particular, is there k such that

$$X_{L_k(\mathfrak{sl}_3)} = \overline{G \cdot \mathbb{C}^* h},$$

where h is regular.

- Is there k such that

$$X_{L_k(\mathfrak{sl}_3)} = \overline{G \cdot (\mathbb{C}^*(\alpha_1 - \alpha_2) + f_\theta)}$$

Theorem [J-Song]

Let $k+3=\frac{2}{2m+1}$, $m\geq 1$. Then

1) $\mathcal{N}\subset X_{L_k(\mathfrak{sl}_3)}\subset\overline{G\cdot(\mathbb{C}^*\lambda+f_\theta)}$,

where $\lambda=\alpha_2-\alpha_1$.

2) If $m=1$, then $X_{L_k(\mathfrak{sl}_3)}=\overline{G\cdot(\mathbb{C}^*(\alpha_1-\alpha_2)+f_\theta)}$.

3) There is no k such that

$$X_{L_k(\mathfrak{sl}_3)}=\mathcal{X}_h,$$

where $h=\alpha_1+\alpha_2$ is principal.

Corollary:

For $k \in \mathbb{C}$, $X_{L_k(\mathfrak{sl}_3)}$ is one of the following:

$$\mathfrak{g}^*, \quad \{0\}, \quad \mathcal{N}, \quad \overline{\mathbb{O}}_{min}, \quad \overline{G\cdot(\mathbb{C}^*(\alpha_1-\alpha_2))},$$

$$\overline{G\cdot(\mathbb{C}^*(\alpha_1-\alpha_2)+f_\theta)},$$

where $\mathfrak{g}=\mathfrak{sl}_3$.

Conjecture: For all $k=-3+\frac{2}{2m+1}$, $m\geq 1$,

$$X_{L_k(\mathfrak{sl}_3)} = \overline{G\cdot(\mathbb{C}^*(\alpha_1-\alpha_2)+f_\theta)}.$$

3. Varieties of assoc. affine W-algebras

Let $\mathfrak{g}/\mathbb{C}$ be a simple Lie algebra and f a nilpotent element of $\mathfrak{g}$. By the Jacobson-Morozov theorem, there exists an $\mathfrak{sl}_2$-triple $\{e, f, h\}$. Let

$$\mathcal{S}_f = f + \mathfrak{g}^e$$

be the Slodowy slice associated with $\{e, f, h\}$.

Recall from [Kac-Roan-Wakimoto, 03], [Kac-Wakimoto, 04], [E. Frenkel-Ben-Zvi, 04], ...

Let M be a $V^k(\mathfrak{g})$-module, denote by $H_f^{\frac{\infty}{2}+0}(M)$ the BRST cohomology with coefficients in M.

If M is a quotient of $V^k(\mathfrak{g})$, then $H_f^{\frac{\infty}{2}+0}(M)$ is a VA. In particular, $H_f^{\frac{\infty}{2}+0}(V^k(\mathfrak{g}))$, denoted by $W^k(\mathfrak{g}, f)$ is called the universal affine W alg.

Denote by $W_k(\mathfrak{g}, f)$ the simple quotient of $W^k(\mathfrak{g}, f)$.

Theorem [De Sole – Kac, 06. Arakawa, 15]

1) $X_{W^k(\mathfrak{g}, f)} \cong \mathcal{S}_f = f \oplus \mathfrak{g}^e$.

2) $X_{H_f^{\frac{\infty}{2}+0}(M)} = X_M \cap \mathcal{S}_f$.

3) $H_f^{\frac{\infty}{2}+0}(M) \neq 0$ iff $\overline{G \cdot f} \subset X_M$.

In particular, for a quotient V of $V^k(\mathfrak{g})$,

$$X_{H_f^{\frac{\infty}{2}+0}(V)} = X_V \cap \mathcal{S}_f,$$

and

$$H_f^{\frac{\infty}{2}+0}(V) \neq 0 \text{ iff } \overline{G \cdot f} \subset X_V.$$

Ex. For degenerate admissible level k and regular nilpotent element f,

$$\overline{G\cdot f} = \mathcal{N}$$

$$X_{L_k(\mathfrak{g})} \subsetneq \mathcal{N} \quad \text{[Arakawa, 12]}$$

So

$$H_f^{\frac{\infty}{2}+0}(L_k(\mathfrak{g})) = 0.$$

Theorem [Arakawa – J – Moreau, 20]

Let f be a nilpotent element of $\mathfrak{g}$

The following are equivalent

(1) $V^k(\mathfrak{g})$ is simple ;

(2) $W^k(\mathfrak{g}, f) = H_f^{\frac{\infty}{2}+0}(L_k(\mathfrak{g}))$;

(3) $X_{H_f^{\frac{\infty}{2}+0}(L_k(\mathfrak{g}))} = \mathcal{S}_f$.

Theorem [J - Song, 23]

Let $f \in \mathbb{O}_{min}$ be a nilpotent element of $\mathfrak{sl}_3$. Then

(1) $\dim X_{W_k(\mathfrak{sl}_3, f)} \geq 2$, for $k = -3 + \frac{2}{2m+1}$, $m \geq 1$.

(2) If $m = 1$, $\dim X_{W_k(\mathfrak{sl}_3, f)} = 3$.

Conjecture: for all $k = -3 + \frac{2}{2m+1}$, $m \geq 1$,

$$\dim X_{W_k(\mathfrak{sl}_3, f)} = 3.$$

Let f be a regular nilpotent element of $\mathfrak{g}$.

Denote by $W_k(\mathfrak{g})$ the simple quotient of $W^k(\mathfrak{g})$.

By the Feigin-Frenkel Langlands duality,

$$W^k(\mathfrak{g}) \cong W^{k'}({}^L\mathfrak{g}),$$

where ${}^L\mathfrak{g}$ is the Langlands dual

Lie algebra of $\mathfrak{g}$ and

$$r^\vee(k+h^\vee)(k'+{}^L h^\vee)=1.$$

In particular, for $\mathfrak{g}=\mathfrak{sl}_n$, if

$k+n=\frac{q}{p}$, then $k'+n=\frac{p}{q}$.

If $X_{L_k(\mathfrak{sl}_3)}=\overline{G\cdot\mathbb{C}^*(\alpha_2-\alpha_1+f_\theta)}$ for $k+3=\frac{2}{2m+1}$,

then $$X_{L_k(\mathfrak{sl}_3)}\cap \mathcal{S}_f \neq \{f\}.$$

and

$$\dim H_f^{\frac{\infty}{2}+0}(L_k(\mathfrak{sl}_3))=1.$$

Therefore $W_k(\mathfrak{sl}_3)$ is not Lisse. If $k=-\frac{7}{3}$, it is true [Wang, 97]

Conjecture:

For all non-admissible numbers k or dedengerate admissible k, the simple principal W-algebras $W_k(\mathfrak{sl}_n)$ are not Lisse.

Thanks!