PBW Bases of Ising Modules

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Virasoro Lie algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3 - m}{12} \delta_{m,-n} C$$

$M(h,c)$ Verma module   cyclic vector $|h,c\rangle$

$$L_0 |h,c\rangle = h |h,c\rangle \qquad\qquad L_n |h,c\rangle = 0 \quad n > 0$$

$$C |h,c\rangle = c |h,c\rangle \qquad\qquad h, c \in \mathbb{C}$$

basis $L_{-\lambda_1} L_{-\lambda_2} \cdots L_{-\lambda_u} |h,c\rangle =: L_\lambda |h,c\rangle$

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_u \geqslant 1$$

Unique irreducible quotient

$$M(h,c) \twoheadrightarrow L(h,c)$$

Question: What is the induced
basis of $L(h,c)$?

- "Generic" h, c ✓ $M(h,c) = L(h,c)$

$$C = 1 - \frac{6(P-P')^2}{PP'}$$

$$h = \frac{(PS-P'r)^2 - (P-P')^2}{4PP'}$$

$2 \leq P < P'$ $(P,P') = 1$

$1 \leq S < P'$

$1 \leq r < P$

$\left.\begin{array}{c}\end{array}\right\}$ $M(h,c) \neq L(h,c)$

Feigin - Frenkel ('93)
Bruschek - Moustada - Schepers ('13) $P=2$
Van Eneren . H ('18) $P' = 2U + 1$
                                         $r = S = 1 \Rightarrow h = 0$

Theorem: a basis of $L(h,c)$ is given by
$L_\lambda|h,c\rangle$ for partitions $\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_n \geqslant 2$
   such that $\lambda_i - \lambda_{i+U-1} \geqslant 2$

Proof $(U=2, \; P'=5)$ $\lambda_n \geqslant 2$ $\lambda_{n-1} \geqslant 4$ $\lambda_{n-2} \geqslant 6 \cdots$
                          $\lambda_n - 2 \leqslant \lambda_{n-1} - 4 \leqslant \cdots \leqslant \lambda_1 - 2k$

Count partitions of $n - k(n+1)$

$$\sum_{n \geqslant 1} \frac{q^{n(n+1)}}{(q)_n}   \qquad (q)_n = \prod_{i=1}^{n} (1 - q^i)$$

The character of $L(h,c)$:

$$L_0\left(L_\lambda|h,c\rangle\right) = (\lambda_1 + \cdots + \lambda_u + h)L_\lambda|h,c\rangle$$

$$\chi(q) = Tr_{L(h,c)}\, q^{L_0} =$$

$$\chi^{p,p'}_{r,s} = \frac{q^{h^{p,p'}_{r,s}}}{(q)_\infty} \sum_{\lambda \in \mathbb{Z}}{}' q^{\lambda^2 pp' + \lambda(p'r - ps)} - q^{(\lambda p + r)(\lambda p' + s)}$$

In particular we have

$$\chi^{2,5}_{1,1} = \frac{1}{(q)_\infty} \sum_{\lambda \in \mathbb{Z}}{}' q^{10\lambda^2 + 3\lambda} - q^{10\lambda^2 + 7\lambda + 1} =$$

$$\prod_{u \neq 0, \pm 1 (5)} \frac{1}{(q)_u} = \sum_{u \geq 1} \frac{q^{u(u+1)}}{(q)_u}$$

Rogers-Ramanujan

- Andrews, Van-Ewern - H. Case $c = \frac{1}{2}$ $(P, P') = (3, 4)$

  $h = 0$ $(r, s) = (1, 1)$

. Selezer   Case $c = \frac{1}{2}$ $h = \frac{1}{2}$ or $h = \frac{1}{16}$

$M(h, c)$ grading (conformal weight)

hiltration (Li decreasing, increasing, Selezer)

$$\langle_\lambda | h, c \rangle = L_{-\lambda_1} L_{-\lambda_2} \cdots L_{-\lambda_u} \underbrace{L_{-1} L_{-1} \cdots L_{-1}}_{u \text{ Times}} | h, c \rangle \qquad (*)$$

$$\deg \langle_\lambda | h, c \rangle = (h + \lambda_1 + \cdots + \lambda_u + u) \longleftarrow \begin{array}{l} \text{We can forget } h \\ \text{here} \end{array}$$

$$G^P M(h, c) = \left\{ \langle_\lambda | h, c \rangle \ \middle| \ 2\lambda_1 + \cdots + 2\lambda_u + u \geqslant P \right\}$$

The Kernel $\quad K \hookrightarrow M(h,c) \longrightarrow L(h,c)$

is generated by $0, 1$ or 2 singular vectors

Example $h = 0$ $\quad L_{-1}|h,c\rangle$ is singular

Another singular vector of degree $(P-1)(P'-1)$

$$\omega_{p,p'} := \underbrace{L_{-2} L_{-2} \cdots L_{-2}}|h,c\rangle + \cdots \quad \left(\begin{array}{c}\text{lower Terms in the}\\ \text{filtration}\end{array}\right)$$

$m := \dfrac{(P-1)(P'-1)}{2}$ - Times

$$L_{-2}^m \in \mathrm{gr}_{2m}^G K \implies L_{-2}^m \text{ is zero in } \mathrm{gr}^G L(h,c)$$

The symbol (wrt filtration) of $L^u_{-1}(W_{P,P'})$

equals the symbol of $L^u_{-1}(L^m_{-2})$

Example $(P,P') = (2,5)$ $h=0$

$L_{-2}L_{-2}L_{-2}, \quad L_{-3}L_{-2}L_{-2}, \quad L_{-4}L_{-2}L_{-2} + \boxed{L_{-3}L_{-3}L_{-2}},$

$L_{-5}L_{-2}L_{-2}^+ \quad L_{-4}L_{-3}L_{-2} + \boxed{L_{-3}L_{-3}L_{-3}}, \quad \cdots$

Choose a global monomial ordering
(for example reverse lexicografic).

$[P,P,P], \; [P+1, P, P], \; [P+1, P+1, P], \; [P+1, P+1, P+1], \cdots$

$\Rightarrow \quad \lambda_u - \lambda_{u+2} \geqslant 2$

For any P, P' and $h = 0$ we have

$$m = \frac{(P-1)(P'-1)}{2} = \frac{(2-1) \cdot \left[(2m+1) - 1 \right]}{2}$$

$L(0, C_{P,P'})$ is "smaller" than $L(0, C_{2, 2m+1})$!

$$K \lhook\joinrel\longrightarrow M(0, C_{P,P'}) \longtwoheadrightarrow L(0, C_{P,P'})$$

$$\text{12 - Vect spaces} \qquad \neq$$

$$K' \lhook\joinrel\longrightarrow M(0, C_{2, 2m+1}) \longtwoheadrightarrow L(0, C_{2, 2m+1})$$

$$L_{-2}^{m} \in K \not\subseteq K' \qquad \text{if } P > 2$$

Example $(P, P') = (3, 4)$ $h = 0$

$L(0, \frac{1}{2})$ is smaller than $L(0, C_{2,7})$

$m = \frac{(3-1)(4-1)}{2} = 3$

$L_{-1}|0,c\rangle$ and $L_{-2} L_{-2} L_{-2} |0,c\rangle$ are singular

$L_{-2} L_{-2} L_{-2}, \quad L_{-3} L_{-2} L_{-2}, \quad L_{-4} L_{-2} L_{-2} + \boxed{L_{-3} L_{-3} L_{-2}},$

$\frac{1}{6} L_{-5} L_{-2} L_{-2} + L_{-4} L_{-3} L_{-2} + \boxed{L_{-3} L_{-3} L_{-3}}, \quad \cdots$

$\in \mathrm{gr}^G_r K$

$L_{-4} L_{-3} L_{-2}$ (more generally $[P+2, P+1, 7]$ $P \geq 2$)

is a LM of $\mathrm{gr}^G_r K$ for $L(0, \frac{1}{2})$

Two variable characters

$$\chi(t,q) = \sum_{n,P \geq 0} \dim \operatorname{gr}_P^G L(h,c)_{h+n} \, q^{h+n} \, t^P$$

$$\chi(q) = \chi(t=1, q)$$

Theorem (Andrews, Van-Ekeren, H. '20 – Jolozar '23) $c = \frac{1}{2}$

$$\chi_h(t,q) = \sum_{\substack{k_1, k_2 \geq 0 \\ h=0}} t^{4u_1 + 2k_2} \, \frac{q^{4u_1^2 + 3k_1 k_2 + k_2^2}}{(q)_{u_1} (q)_{k_2}} \left(1 - q^{k_1} + q^{u_1 + k_2}\right)$$

$$\chi_{h=\frac{1}{2}}(t,q) = q^{\frac{1}{2}} \sum_{k_1, u_2 \geq 0} \quad \overset{''}{} \quad \left(q^{3u_1 + 2u_2} + t q^{5k_1 + 2k_2 + 1} + t^2 q^{6k_1 + 3k_2 + 2}\right)$$

$$\chi_{h=\frac{1}{16}}(t,q) = q^{\frac{1}{16}} \sum \quad \overset{''}{} \quad \left(q^{u_1 + u_2} + t q^{4k_1 + 2k_2 + 1} + t^3 q^{7k_1 + 3k_2 + 3}\right)$$

Theorem (Andrews, Von-Eweren, H. '20)

The module $L(0, \frac{1}{2})$ admits as basis

$$L_{-\lambda_1} L_{-\lambda_2} \cdots L_{-\lambda_u} |0, \frac{1}{2}\rangle \quad \text{where}$$

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_u \geqslant 2 \qquad \text{are partitions that}$$

do not contain:

$$[P,P,P], [P+1,P,P], [P+1,P+1,P], [P+2,P+1,P], [P+2,P+2,P] \qquad P \geqslant 2$$

$$P \geqslant 3$$

$$[P+2,P,P]$$

$$[P+3,P+3,P,P], [P+4,P+3,P,P], [P+4,P+3,P+1,P], [P+4,P+4,P+1,P] \quad P \geqslant 2$$

$$P \geqslant 2$$

$$[P+6,P+5,P+3,P+1,P]$$

$$[5,4,2,2], [7,6,4,2,2], [7,7,4,2,2], [9,8,6,4,2,2]$$

Sketch of the proof.

1) Use a recursion to count the partitions of the theorem. [Construct partitions of n from partitions of $n-k$, for $k \geq 1$].

Difficulty: Need to decompose this set into S different subsets.

Outcome: The function

$$X(t,q) = \sum t^{4u_1 + 2u_2} q^{4u_1 + 3u_1 u_2 + u_2^2} \frac{}{(q)_{u_1} (q)_{u_2}} \left(1 - q^{u_1} + q^{u_1 + u_2} \right)$$

Counts these partitions.

2) Prove that this counts the right graded dimension of $L(0, \frac{1}{2})$.

$$\chi(t=1, q) = Tr_{L(0,\frac{1}{2})} q^{l_0} = \sum_{k \geq 0} \frac{q^{2k^2}}{(q)_{2k}}$$

Technique: Find polynomials in q

$$L_n(q) \quad , \quad R_n(q)$$

such that

a) $\lim_{n \to \infty} L_n(q) = LHS \qquad \lim_{n \to \infty} R_n(q) = RHS$

b) $L_n(q) = R_n(q) \; \forall n \quad \left(\begin{array}{l} \text{both satisfy some} \\ \text{recurrence} \end{array} \right)$

3) Prove that all partitions enumerated in the theorem are leading monomials of elements in the Kernel

$$K \hookrightarrow M(0, \tfrac{1}{2}) \twoheadrightarrow L(0, \tfrac{1}{2})$$

Compute explicitly $\forall k \geq 0$

$$L_{-1}^{k} \left(L_{-2}^{3} \right) \qquad L_{-1}^{k} \left(\tfrac{1}{6} L_{-5} L_{-2}^{2} + L_{-4} L_{-3} L_{-2} \right)$$

Do this by induction - keeping only the first few-terms.

□

Theorem (Andrews - Van Ekeren, H '20)

step 2 of the proof can be done
for the modules $L(\frac{1}{2}, \frac{1}{2})$ and $L(\frac{1}{16}, \frac{1}{2})$

$$\chi_{h=\frac{1}{2}}(t=1, q) = q^{\frac{1}{2}} \sum \frac{q^{4k_1^2 + 3k_1 k_2 + k_2^2}}{(q)_{k_1} (q)_{k_2}} \left(1 - q^{8k_1 + 4k_2 + 6}\right) = q^{\frac{1}{2}} \sum \frac{q^{2k^2 - 2k}}{(q)_{2k-1}}$$

$$\chi_{h=\frac{1}{16}}(t=1, q) = q^{\frac{1}{16}} \sum \quad " \quad \left(q^{k_1 + k_2} + q^{4k_1 + 2k_2 + 1}\right) = q^{\frac{1}{16}} \sum \frac{q^{\frac{k(k+1)}{2}}}{(q)_k}$$

Proof of Solezer's theorem:

1) Count partitions by a recurrence, nothing new here.

Excepcional cases

$h=\frac{1}{2}$ $[2]$, $[3,3]$, $[1,1,1]$, $[3,1,1]$, $[4,4,1]$, $[5,4,1,1]$, $[6,5,3,1]$

$h=\frac{1}{16}$ $[2]$, $[3,3,1]$, ..., $[6,5,3,1,1]$, $[5,5,1,1,1]$, $[6,6,3,1,1]$
  $[7,6,4,1,1,1]$, $[8,7,5,3,1,1]$.

Singular vectors

$$(L_{-1}L_{-1} - \tfrac{4}{3}L_{-2})|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$(L_{-1}L_{-1}L_{-1} - 3L_{-2}L_{-1} - \tfrac{3}{4}L_{-3})|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

Outcome of Step 1:

$\chi_h(t,q)$ for $h = \frac{1}{2}$ or $h = \frac{1}{16}$ such that

$\chi_h(t=1,q)$ equals Andrews-V. Eueren, H. 's.

step 2). Was already done ✓.

Step 3: Why the only difference with $h=0$ lies on the exceptional case?

A tiny bit of vertex algebras.

set $V = Vir^{1/2}$ the universal Virasoro
vertex algebra of $c = 1/2$.

$$K \hookrightarrow gr^G V \simeq \mathbb{C}[L_{-2}, L_{-3}, \ldots] \twoheadrightarrow gr^G L(0, 1/2)$$

$$gr^G M(0, 1/2) \simeq \bigoplus_{u \geq 0} V \cdot L_{-1}^u |h, 1/2\rangle$$

$$12$$

$$K' \hookrightarrow gr^G M(h, 1/2) \longrightarrow\!\!\!\!\rightarrow gr^G L(h, 1/2)$$

It follows that all the of the ideal
described in AvĒ-H for $h=0$ is in the
Kernel $\qquad M(h, \frac{1}{2}) \longrightarrow L(h, \frac{1}{2})$ for $h=\frac{1}{2}, \frac{1}{16}$

Only exceptional cases need to be found. $\qquad\square$

Remarks / Open questions:

- V-module M, decreasing filtration F and increasing filtration G on both V and M.

$\mathrm{gr}\,M$ is a PVA-module over $\mathrm{gr}\,V$.

$$G^P M = \left\{ a^{i_1}_{(n_1,i)} \cdots a^{i_u}_{(-n_u,i)} m \;\middle|\; \sum_{j=1}^{u} \Delta_{i_j} + \Delta_{m-h} \geqslant P \right\}$$

$$\mathrm{gr}^F M \simeq \mathrm{gr}^G M \qquad\qquad \mathrm{gr}^F_r V \simeq \mathrm{gr}^G_r V$$

- For $V = Vir_c$ or Vir^c the "Poisson" structure is zero, but the Poisson module structure of $\mathrm{gr}\,L(h,c)$ is not trivial for $h \neq 0$