Reduction by stages on W-algebras

Naoki Genra, Thibault Juillard

Representation Theory XVIII, Inter-University Centre
Dubrovnik, Croatia, Jun. 27, 2023

Motivations

History on W-algebras

Let Vir be the Virasoro algebra, which is an infinite-dimensional Lie algebra

$$\mathrm{Vir} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$$

with the defining relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}C, \quad [L_n, C] = 0.$$

The Virasoro algebra plays an important role in the 2d CFTs and also has a rich mathematical structure in the rep theory.

In the classification of 2d CFTs, Zamolodchikov found a generalization of Vir, called the W_3-algebra, which is generated by L_n, S_n, and L_n satisfies the Virasoro relations. However $[S_m, S_n]$ contains infinite sum of quadratic forms of L_n and thus W_3 is not just a Lie algebra – W_3 forms a vertex algebra.

Fateev and Lukyanov also found a family of generalizations of W_3-algebra: W_n-algebras (=WA_{n-1}), WB_n, WC_n, ... etc.

Feigin and Frenkel gave mathematical definitions of these algebras: let $\mathfrak{g}$ be a simple Lie algebra and $V^k(\mathfrak{g})$ the affine vertex algebra of $\mathfrak{g}$ at level k. Then the (principal) W-algebra of $\mathfrak{g}$ at level k is defined by the Drinfeld-Sokolov reduction

$$W^k(\mathfrak{g}) = H^0_{DS}(V^k(\mathfrak{g})).$$

By construction, the W-algebras are vertex algebras. For $\mathfrak{g} = \mathfrak{sl}_n$, $W^k(\mathfrak{sl}_n)$ is isomorphic to the W_n-algebra. In particular, $W^k(\mathfrak{sl}_2)$ is isomorphic to the Virasoro (vertex) algebra of the central charge $c(k) = 1 - 6(k+1)^2/(k+2)$.

Feigin and Semikhatov also found a family of genralizations of $V^k(\mathfrak{sl}_2)$, called the $W_n^{(2)}$-algebras. For $n = 2$, $W_2^{(2)} = V^k(\mathfrak{sl}_2)$ and $W_3^{(2)}$ is isomorphic to the Bershadosky-Polyakov algebra, which is geberated by e_n, h_n, f_n, L_n, and $[e_m, f_n]$ contains a linear term of L_n and infinite sum of quadratic forms of h_n, and thus not a Lie algebra. $W_n^{(2)}$ don't appear as examples of $W^k(\mathfrak{g})$.

Kac, Roan and Wakimoto found generalizations of $W^k(\mathfrak{g})$ by generalizing the DS-reductions: let $\mathfrak{g}$ be a simple Lie (super)algebra and f an (even) nilpotent element in $\mathfrak{g}$. Then

$$W^k(\mathfrak{g}, f) := H^0_{DS,f}(V^k(\mathfrak{g})),$$

the W-algebra associated to $\mathfrak{g}, f$ at level k.

- $W^k(\mathfrak{g}, 0) = V^k(\mathfrak{g})$.
- $W^k(\mathfrak{g}, f_{\mathrm{prin}}) = W^k(\mathfrak{g})$, where f_{prin} is a principal nilp ele.
- $W^k(\mathfrak{sl}_n, f_{\mathrm{sub}}) = W_n^{(2)}$, where f_{sub} is a subregular nilp ele.

Madsen-Ragoucy observations

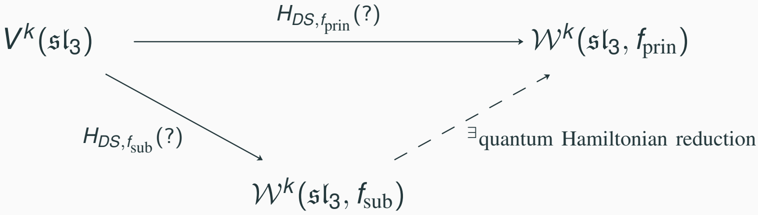
I'd like to introduce Madsen-Ragoucy observations. Let f be a nilpotent element in $\mathfrak{sl}_3$. Then the Jordan form has only 0 in the diagonal entries and thus is one of the followings:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which corresponds to the partitions (3), $(2,1)$, (1^3) of 3, called principal, subregular, zero, respectively. Thus we obtain two families of W-algebras from $V^k(\mathfrak{sl}_3)$:

- $W^k(\mathfrak{sl}_3, f_{\mathfrak{prin}})$ = the Zamolodchikov W_3-algebra.
- $W^k(\mathfrak{sl}_3, f_{\mathfrak{sub}})$ = the Bershadosky-Polyakov (BP) algebra.

Madsen and Ragoucy suggested the W_3-algebra is obtained from the BP-algebra by a quantum Hamiltonian reduction commuting the following diagram:



Now we have questions: 1) Is this true? 2) Generalizations?

Answer for 1) is (of course) true.

Our goal is to prove the Madsen-Ragoucy observations and to find generalizations by using reduction by stages.

Slodowy slices

Vertex algebras have canonical filtrations (Li filtrations) and the associated graded spaces become commutative vertex algebras equipped with additional Poisson (λ-)brackets (= Poisson vertex algebras). In case of W-algebras,

$$\mathrm{gr}\, W^k(\mathfrak{g}, f) \simeq \mathbb{C}[J_\infty \mathcal{S}_f],$$

where $\mathcal{S}_f$ is the Slodowy slice at f in $\mathfrak{g}$

$$\mathcal{S}_f := f + \mathfrak{g}^e \subset \mathfrak{g} \simeq \mathfrak{g}^*.$$

The Jacobson-Morozov theorem implies the existence of an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ in $\mathfrak{g}$ containing our choice of f. Then e is chosen from the $\mathfrak{sl}_2$-triple.

Thus $\mathcal{S}_f$ is a classical analog of the W-algebra $W^k(\mathfrak{g}, f)$.

Recall that $\mathfrak{g}^*$ is a Poisson variety equipped with the Kirillov-Kostant Poisson structure. The Slodowy slice is also a Poisson variety and transversal to the coadjoint orbits in $\mathfrak{g}^*$.

The Poisson structure of $\mathcal{S}_f$ is obtained by a moment map. I'll sketch a construction of $\mathcal{S}_f$ by the moment map: there exists a nilpotent subalgebra $\mathfrak{m} \subset \mathfrak{g}$ and let $M = \exp(\mathfrak{m})$ a unipotent Lie subgroup of an algebraic group G such that $\mathrm{Lie}\, G = \mathfrak{g}$. Define a Hamiltonian M-action on $\mathfrak{g}^*$ by the moment map

$$\mu \colon \mathfrak{g}^* \simeq \mathfrak{g} \to \mathfrak{m}^*, \quad u \mapsto (a \mapsto (a|u - f)).$$

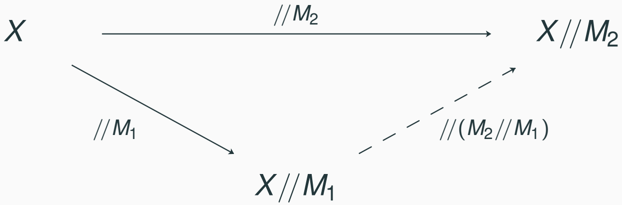
Then

$$\mathcal{S}_f \simeq \mu^{-1}(0)/M =: \mathfrak{g}^*//M.$$

The RHS is called the Hailtonian reduction of $\mathfrak{g}^*$ and has a canonical Poisson structure induced from $\mathfrak{g}^*$.

Reduction by stages

Let X be a Poisson variety with a Hamiltonian M_2-action and M_1 a normal Lie subgroup of M_2. Then we obtain two Poisson varieties $X//M_1$, $X//M_2$ from X by using the Hamiltonian reductions. But, under suitable assumptions, we may define a Hamiltonian $M_2//M_1$-action on $X//M_1$ such that the following diagram commutes:



This procedure is called the reduction by stages since we obtain $X//M_2$ by stages. We will apply for $X = \mathfrak{g}^*$.

Morgan conjectures

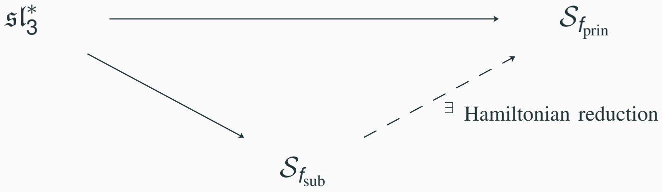
Morgan applied the reduction by stages for the Slodowy slices.

Conjecture (Morgan, PhD thesis)

Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathcal{O}_1$, $\mathcal{O}_2$ nilpotent orbits in $\mathfrak{sl}_n$ at f_1, f_2 such that $\mathcal{O}_1 < \mathcal{O}_2$ (i.e. $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$). Then $\mathcal{S}_{f_2}$ is obtained as a Hamiltonian reduction of $\mathcal{S}_{f_1}$.

Theorem (Morgan)

This is true for $n = 3$.



This is a classical analog of the Madsen-Ragoucy observations.

Reduction by stages for Slodowy slices

Gan-Ginzburg theory

Let $\mathfrak{g}$ be a simple Lie algebra, f a nilpotent element and x a semisimple element such that
(1) $\operatorname{ad} x$ defines a $\mathbb{Z}$-grading on $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$
(2) $f \in \mathfrak{g}_{-2}$
(3) $\operatorname{ad} f\colon \mathfrak{g}_j \to \mathfrak{g}_{j-2}$ is injective for $j \geq 1$ and surjective for $j \leq 1$.
Then (f, x) is called a good pair. For example, we may choose $x = h$ in the $\mathfrak{sl}_2$-triple $\{e, h, f\}$. Thanks to the good condition,

$$\langle a, b\rangle := (f|[a, b]) = ([f, a]|b), \quad a, b \in \mathfrak{g}_1$$

defines a symplectic structure on $\mathfrak{g}_1$. Let $\mathfrak{l}$ be a Lagrangian in $\mathfrak{g}_1$ (= a maximal isotropic subspace in $\mathfrak{g}_1$) and $\mathfrak{m}$ a nilpotent subalgebra

$$\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{g}_{\geq 2}.$$

Let $M = \exp(\mathfrak{m})$ be a unipotent Lie subgroup of an algebraic group G with $\mathrm{Lie}\, G = \mathfrak{g}$. Define a Hamiltonian M-action on $\mathfrak{g}^*$ by

$$\mu\colon \mathfrak{g}^* \simeq \mathfrak{g} \to \mathfrak{m}^*, \quad u \mapsto (a \mapsto (a|u - f)).$$

Define an affine morphism

$$\alpha\colon M \times \mathcal{S}_f \to \mu^{-1}(0), \quad (g, u) \mapsto \mathrm{Ad}_g(u).$$

Theorem (Gan-Ginzburg)

α *is an isomorphism.*

Therefore $M \times \mathcal{S}_f \simeq \mu^{-1}(0)$ so that

$$\mathcal{S}_f \simeq \mu^{-1}(0)/M = \mathfrak{g}^*//M.$$

Reduction by stages for Slodowy slices

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$,
f_1, f_2 nilpotent elements of $\mathfrak{g}$ s.t. $f_0 := f_2 - f_1$ is nilpotent,
x_1, x_2 semisimple elements in $\mathfrak{h}$ s.t. (f_1, x_1), (f_2, x_2) are good pairs. Set $x_0 := x_2 - x_1$. We have

$\mathfrak{l}_1$, $\mathfrak{l}_2$: Lagrangian subspaces,
$\mathfrak{m}_1$, $\mathfrak{m}_2$: nilpotent subalgebras as in Gan-Ginzburg theory.

Then $\mathcal{S}_{f_1} \simeq \mu_1^{-1}(0)/M_1$ and $\mathcal{S}_{f_2} \simeq \mu_2^{-1}(0)/M_2$.

Suppose the following conditions $(\star)$:

1) $\mathfrak{m}_1 \subset \mathfrak{m}_2$ and there exists a Lie subalgebra $\mathfrak{m}_0$ of $\mathfrak{m}_2$ s.t. $\mathfrak{m}_2 = \mathfrak{m}_1 \oplus \mathfrak{m}_0$ and $\mathfrak{m}_1$ is an ideal of $\mathfrak{m}_2$.
2) $\mathfrak{m}_0$ is $\mathfrak{h}$-invariant.
3) $[f_1, x_0] = [f_1, \mathfrak{m}_0] = 0$.
4) $[x_1, f_0] = [x_1, \mathfrak{m}_0] = 0$.

Recall

$$\alpha_1\colon M_1 \times \mathcal{S}_{f_1} \xrightarrow{\sim} \mu_1^{-1}(0), \quad \alpha_2\colon M_2 \times \mathcal{S}_{f_2} \xrightarrow{\sim} \mu_2^{-1}(0).$$

Define a Hamiltonian M_0-action on $\mathcal{S}_{f_1}$ by the moment map

$$\mu_0\colon \mathcal{S}_{f_1} \to \mathfrak{m}_0^*, \quad u = f_1 + u' \mapsto (a \mapsto (a|u - f_2) = (a|u' - f_0)).$$

Define an affine morphism

$$\alpha_0\colon M_1 \times \mu_0^{-1}(0) \to \mu_2^{-1}(0), \quad (g, u) \mapsto \mathrm{Ad}_g(u).$$

Claim
α_0 is an isomorphism.

Therefore $M_2 \times \mathcal{S}_{f_2} \overset{\text{GG}}{\simeq} \mu_2^{-1}(0) \overset{\text{Claim}}{\simeq} M_1 \times \mu_0^{-1}(0)$

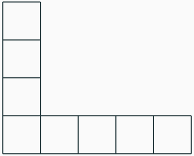
$\overset{/M_1}{\rightsquigarrow} M_0 \times \mathcal{S}_{f_2} \simeq \mu_0^{-1}(0).$

Hence $\mathcal{S}_{f_2} \simeq \mu_0^{-1}(0)/M_0 = \mathcal{S}_{f_1}/\!/M_0$.

Theorem (G.-Juillard)
Under the assumption $(\star)$*, we have* $\mathcal{S}_{f_2} \simeq \mathcal{S}_{f_1} /\!/ M_0$.

Examples:

- Let $\mathfrak{g} = \mathfrak{sl}_n$, $a_1, a_2 \in \mathbb{N}$ such that $1 \leq a_1 < a_2 \leq n$ and $f_1 = (a_1, 1^{n-a_1})$, $f_2 = (a_2, 1^{n-a_2})$. These are called hook-type nilpotent elements



  Now $\mathcal{O}_{f_1} < \mathcal{O}_{f_2}$ and f_1, f_2 satisfies the condition $(\star)$. Thus $\mathcal{S}_{f_2} \simeq \mathcal{S}_{f_1} /\!/ M_0$.
- Let $\mathfrak{g} = \mathfrak{sl}_4$, $f_1 = (2, 1^2)$ and $f_2 = (2^2)$. Then f_1, f_2 satisfies the condition $(\star)$.

- Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$, f_1 is subregular and f_2 is principal. Then f_1, f_2 satisfies the condition $(\star)$.
- Let $\mathfrak{g} = \mathfrak{sp}_{2n}$, $f_1 = (2^2, 1^{2n-4})$ (short nilpotent) and f_2 is principal. Then f_1, f_2 satisfies the condition $(\star)$.
- Let $\mathfrak{g} = G_2$, f_1 is $\widetilde{A}_1$ and f_2 is subregular. Then f_1, f_2 satisfies the condition $(\star)$.
- (Maybe) more...

Reduction by stages for finite W-algebras

Reduction by stages for finite W-algebras

Let $\chi(u) = (f|u)$, $I_\chi = (u - \chi(u) \mid u \in \mathfrak{m})$ be a two-sided ideal in $U(\mathfrak{g})$ and $Q_\chi = U(\mathfrak{g})/I_\chi$. Then

$$U(\mathfrak{g}, f) := Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$$

has an associative algebra structure induced from $U(\mathfrak{g})$ and is called the finite W-algebra associated to $\mathfrak{g}, f$. There exists the Kazhdan filtration on $U(\mathfrak{g}, f)$ such that

$$\mathrm{gr} U(\mathfrak{g}, f) \simeq \mathbb{C}[\mathcal{S}_f],$$

so $U(\mathfrak{g}, f)$ is a quantization of $\mathcal{S}_f$.

Theorem (G.-J.)
Under the assumption ($\star$), $U(\mathfrak{g}, f_2)$ is obtained by a quantum Hamiltonian reduction of $U(\mathfrak{g}, f_1)$.

Skryabin equivalence by stages

A $\mathfrak{g}$-module E is called Whittaker for χ if $u - \chi(u)$ acts on E locally nilpotently for all $u \in \mathfrak{m}$.

Let $\mathfrak{g}-\mathrm{mod}_\chi$ be the category of finitely generated Whittaker $\mathfrak{g}$-modules for χ and $U(\mathfrak{g}, f)-\mathrm{mod}$ the category of finitely generated $U(\mathfrak{g}, f)$-modules. For $E \in \mathfrak{g}-\mathrm{mod}_\chi$,

$$\mathrm{Wh}(E) := \{m \in E \mid (u - \chi(u))m = 0, u \in \mathrm{m}\}$$

becomes a $U(\mathfrak{g}, f)$-module, while for $V \in U(\mathfrak{g}, f)-\mathrm{mod}$,

$$\mathrm{Ind}(V) := Q_\chi \underset{U(\mathfrak{g},f)}{\otimes} V$$

becomes a Whittaker $\mathfrak{g}$-module.

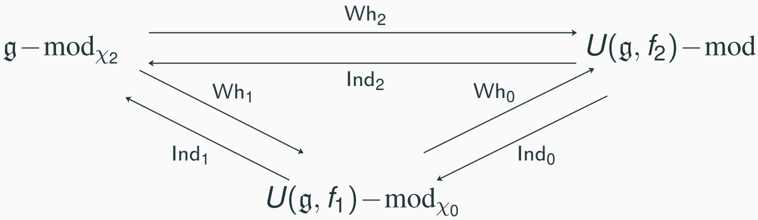
Moreover,

$$\mathfrak{g}-\mathrm{mod}_\chi \underset{\mathsf{Ind}}{\overset{\mathsf{Wh}}{\rightleftarrows}} U(\mathfrak{g}, f)-\mathrm{mod}$$

gives a quasi-inverse equivalence of categories and is called the Skryabin equivalence.

Theorem (G.-J.)

Under the assumption ($\star$), the following diagram commutes



and each $\rightleftarrows$ are quasi-inverse equivalences.

Reduction by stages for affine W-algebras

Theorem (Madsen-Ragoucy, G.-J.)
Under the assumption $(\star)$ with $\mathfrak{l}_1 = \mathfrak{l}_2 = 0$,

$$W^k(\mathfrak{g}, f_2) \simeq H^0_{DS,f_0}(W^k(\mathfrak{g}, f_1)).$$

Conjecture (G.-J.)
Theorem is also true for $\mathfrak{l}_1, \mathfrak{l}_2 \neq 0$.

Why $\mathfrak{l}_1 = \mathfrak{l}_2 = 0$?

The main difficulty is the convergence of spectral sequences of a double complex: let

$$W^k(\mathfrak{g}, f_1) = H^0(V^k(\mathfrak{g}) \otimes F(\mathfrak{m}_1), d_1),$$
$$W^k(\mathfrak{g}, f_2) = H^0(V^k(\mathfrak{g}) \otimes F(\mathfrak{m}_2), d_2),$$

where $F(\mathfrak{m})$ is the $(\dim \mathfrak{m})$-tensor products of *bc*-system vertex superalgebras. Under the assumption $(\star)$, we have

$$V^k(\mathfrak{g}) \otimes F(\mathfrak{m}_2) \simeq V^k(\mathfrak{g}) \otimes F(\mathfrak{m}_1) \otimes F(\mathfrak{m}_0),$$
$$d_2 = d_1 + d_0, \quad d_2^2 = d_1^2 = d_0^2 = 0.$$

Then Theorem is equivalent to the isomorphism of the total complex cohomology with the double complex cohomology.

Lemma (Kac-Wakimoto)

If $\mathfrak{l} = 0$, there exists a complex decomposition

$$V^k(\mathfrak{g}) \otimes F(\mathfrak{m}) = C_+ \otimes C_-$$

such that $C_\pm = \bigoplus_{n=0}^{\infty} C_\pm^{\pm n}$ and $H^\bullet(C_-) = \mathbb{C}$. Thus

$$H^0(C_+ \otimes C_-) \simeq H^0(C_+) = Ker_{C_+^0} d \subset C_+^0.$$

Therefore $W^k(\mathfrak{g}, f)$ is a vertex subalgebra of C_+^0. Under the assumption $(\star)$, we have

$$W^k(\mathfrak{g}, f_1) \subset C_{+,1}^0, \quad W^k(\mathfrak{g}, f_2) \subset C_{+,2}^0.$$

Moreover we have a natural morphism

$$H^0(W^k(\mathfrak{g}, f_1) \otimes F(\mathfrak{m}_0), d_0) \to W^k(\mathfrak{g}, f_2).$$

Claim

$C^0_{+,1} \otimes F(\mathfrak{m}_0)$ is closed under the action of d_2 and there exists a complex decomposition

$$C^0_{+,1} \otimes F(\mathfrak{m}_0) = \widetilde{C}_1^- \otimes C^0_{+,2}$$

such that $H^\bullet(\widetilde{C}_1^-) = \mathbb{C}$. Thus

$$W^k(\mathfrak{g}, f_2) = H^0(C^0_{+,1} \otimes F(\mathfrak{m}_0), d_2).$$

Now we define a filtration on $C^0_{+,1} \otimes F(\mathfrak{m}_0)$ such that the associated spectral sequences imply the isomorphism of the natural morphism.