The Path of Hooks

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Representation Theory XVIII

W-Algebras

Given a simple finite-dimensional Lie algebra $\mathfrak{g}$, a nilpotent element $f \in \mathfrak{g}$ and $\mathsf{k} \in \mathbb{C}$, the W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f)$ is the homology of certain complex involving the universal affine vertex algebra $\mathsf{V}^{\mathsf{k}}(\mathfrak{g})$ [Kac, Roan, Wakimoto, '03].

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Nonrational W-algebras

- Have very few well-understood examples, still appear in applications.
- Not even clear which module category is the 'right' one (Khazdan-Lusztig? Weight modules? Fin.dim. weight-spaces?).

Goal

Better understand the structure and representation theory of W-algebras.

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$$\lambda \leq \lambda' \quad \leftrightarrow \quad \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \lambda'_i \qquad \forall k \geq 1.$$

We say that $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f) \geq \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f')$ if $\lambda(f) \leq \lambda(f')$.

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Partial Reduction

Is there a way to 'reduce' from $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f)$ to $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f')$ like quantum hamiltonian reduction? Strong signs pointing to yes, e.g. [Genra, Juillard, '23]

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Can we reconstruct $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f')$ from $\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f)$ along with some other easy to understand pieces?

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Concretely, we are looking for embeddings

$$\mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f) \hookrightarrow \mathsf{W}^{\mathsf{k}}(\mathfrak{g}, f') \otimes \mathsf{V}$$

where V is some manageable VOA. This idea goes back to work by [Semikhatov, '94] and [Adamović, '17] who both considered the following example:

Inverse Reduction for $\mathfrak{sl}_2$

There is one non-affine $\mathfrak{sl}_2$ W-algebra: $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_2, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$ is the Virasoro vertex algebra $\mathsf{Vir}^{\mathsf{k}}$ which is generated by its conformal field $L(z)$.

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Inverse Reduction [Adamović, '17]

For V, choose the half lattice vertex algebra Π (generators denoted $c(z), d(z)$ and $\mathrm{e}^{mc}(z)$ for $m \in \mathbb{Z}$). Then, $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_2) \hookrightarrow \mathsf{Vir}^{\mathsf{k}} \otimes \Pi$ given by

$$h(z) \mapsto 2a^{+}(z) \qquad e(z) \mapsto \mathrm{e}^{c}(z)$$

$$f(z) \mapsto :\Big((\mathsf{k}+2)L(z) - (\mathsf{k}+1)\partial a^{-}(z) - a^{-}(z)a^{-}(z)\Big)\mathrm{e}^{-c}(z):$$

where $a^{\pm}(z) = \pm\frac{\mathsf{k}}{4}c(z) + \frac{1}{2}d(z)$. This descends to an embedding of simple quotients if and only if $\mathsf{k}+1 \notin \mathbb{Z}_{\geq 1}$.

More Inverse Reductions

Can brute force all inverse reductions for $\mathfrak{sl}_3$
[Adamović, Kawasetsu, Ridout, '20 / Adamović, Creutzig, Genra , '21]:

$$\mathsf{W}^{\mathsf{k}}\left(\mathfrak{sl}_3, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right) \hookrightarrow \mathsf{W}^{\mathsf{k}}\left(\mathfrak{sl}_3, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right) \otimes \Pi,$$

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_3) \hookrightarrow \mathsf{W}^{\mathsf{k}}\left(\mathfrak{sl}_3, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right) \otimes \Pi \otimes \mathsf{B}.$$

Also descend to simple quotients for certain known k.

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These known inverse reductions proven to be very useful in analysing the representation theory and important-to-physics data for the W-algebras/affine VOAs involved.

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Questions

Where does these come from? Why Π and B?

Even More Inverse Reductions? $\mathfrak{sl}_4$

$$j(z)j(w) \sim \frac{(3k+8)\mathbb{1}(w)}{4(z-w)^2}, \qquad j(z)g^{\pm}(w) \sim \frac{\pm G^{\pm}(w)}{(z-w)},$$

$$g^+(z)g^-(w) \sim \frac{(k+2)(2k+5)(3k+8)\mathbb{1}(w)}{(z-w)^4} + \frac{4(k+2)(2k+5)j(w)}{(z-w)^3} - \frac{(k+2)\big((k+4)\tilde{T}(w) - 6{:}j(w)j(w){:} - 2(2k+5)\partial j(w)\big)}{(z-w)^2}$$
$$+(k+2)\Bigg(W(w) + \frac{8(11k+32)}{3(3k+8)^2}{:}j(w)^3{:} - \frac{4(k+4)}{3k+8}{:}\tilde{T}(w)j(w){:} + 6{:}j(w)\partial j(w){:}$$
$$-\frac{1}{2}(k+4)\partial\tilde{T}(w) + \frac{4(3k^2+17k+26)}{3(3k+8)}\partial^2 j(w)\Bigg)(z-w)^{-1},$$

$$W(z)g^{\pm}(w) \sim \pm\frac{2(k+4)(3k+7)(5k+16)g^{\pm}(w)}{(3k+8)^2(z-w)^3} + \left(\pm\frac{3(k+4)(5k+16)}{2(3k+8)}\partial g^{\pm}(w) - \frac{6(k+4)(5k+16)}{(3k+8)^2}{:}j(w)g^{\pm}(w){:}\right)(z-w)^{-2}$$
$$+\Bigg(-\frac{8(k+3)(k+4)}{(k+2)(3k+8)}{:}j(w)\partial g^{\pm}(w){:} - \frac{4(k+4)(3k^2+15k+16)}{(k+2)(3k+8)^2}{:}\partial j(w)g^{\pm}(w){:} \pm \frac{(k+3)(k+4)}{k+2}\partial^2 g^{\pm}(w)$$
$$\mp\frac{2(k+4)^2}{(k+2)(3k+8)}{:}\tilde{T}(w)g^{\pm}(w){:} \pm \frac{4(k+4)(5k+16)}{(k+2)(3k+8)^2}{:}j(w)^2g^{\pm}(w){:}\Bigg)(z-w)^{-1},$$

$$W(z)W(w) \sim \frac{2(k+4)[\text{illegible}]}{(3k+8)(z-w)^6} + \frac{3(k+4)[\text{illegible}]}{(3k+8)(z-w)^4} + \frac{3(k+4)^2(5k+16)\partial\tilde{T}_\perp(w)}{2(3k+8)(z-w)^3}$$
$$+\Bigg(-\frac{3(k+4)^2(5k+16)(12k^2+59k+74)}{4(3k+8)[\text{illegible}]}\partial^2\tilde{T}_\perp(w)$$
$$+\frac{8(k+4)^3(5k+16)}{(3k+8)(20k^2+93k+102)}{:}\tilde{T}_\perp(w)\tilde{T}_\perp(w){:} + 4(k+4)\Lambda(w)\Bigg)(z-w)^{-2}$$
$$+\Bigg(-\frac{(k+4)^2(5k+16)(12k^2+59k+74)}{6(3k+8)(20k^2+93k+102)}\partial^3\tilde{T}_\perp(w)$$
$$+\frac{8(k+4)^3(5k+16)}{(3k+8)(20k^2+93k+102)}{:}\partial\tilde{T}_\perp(w)\tilde{T}_\perp(w){:} + 2(k+4)\partial\Lambda(w)\Bigg)(z-w)^{-1},$$

where $\tilde{T}(z) = T_{(3,1)}(z) - \partial j(z)$, $\tilde{T}_\perp(z) = \tilde{T}(z) - \frac{2}{3k+8}{:}j(z)j(z){:}$ and

$$(k+2)^2\Lambda(z) = {:}g^+(z)g^-(z){:} - \frac{k+2}{2}\partial W(z) - \frac{4(k+2)}{3k+8}{:}W(z)j(z){:} + \frac{3(k+2)^2(k+4)(6k^2+33k+46)}{2(3k+8)(20k^2+93k+102)}\partial^2\tilde{T}_\perp(z)$$
$$-\frac{(k+2)(k+4)^2(11k+26)}{2(3k+8)(20k^2+93k+102)}{:}\tilde{T}_\perp(z)\tilde{T}_\perp(z){:} + \frac{2(k+2)(k+4)}{3k+8}\partial{:}\tilde{T}_\perp(z)j(z){:} + \frac{8(k+2)(k+4)}{(3k+8)^2}{:}\tilde{T}_\perp(z)j(z)j(z){:}$$
$$-\frac{(k+2)(2k+5)}{3k+8}\left(\frac{8}{3}{:}\partial^2 j(z)j(z){:} + 2{:}\partial j(z)\partial j(z){:} + \frac{16}{3k+8}{:}\partial j(z)j(z)j(z){:} + \frac{32}{3(3k+8)^2}{:}j(z)^4{:} + \frac{3k+8}{6}\partial^3 j(z)\right).$$

$f = M_{3,2} + M_{4,3}$

5 generating fields

$$H(z)H(w) \sim \frac{2(k+1)\mathbb{1}(w)}{(z-w)^2}, \quad J(z)J(w) \sim \frac{4(k+2)\mathbb{1}(w)}{(z-w)^2},$$

$$H(z)E(w) \sim \frac{2E(w)}{z-w}, \quad H(z)F(W) \sim \frac{-2F(w)}{z-w}, \quad H(z)G^{i,\pm}(w) \sim \frac{(3-2i)G^{i,\pm}(w)}{z-w}$$

$$J(z)G^{1,\pm}(w) \sim \frac{\pm 2G^{1,\pm}(w)}{(z-w)}, \quad J(z)G^{2,\pm}(w) \sim \frac{\pm 2G^{2,\pm}(w)}{(z-w)}, \quad E(z)F(w) \sim \frac{(k+1)\mathbb{1}(w)}{(z-w)^2} + \frac{H(w)}{z-w},$$

$$E(z)G^{2,-}(w) \sim \frac{G^{1,-}(w)}{z-w}, \quad E(z)G^{2,+}(z) \sim \frac{-G^{1,+}(w)}{z-w}, \quad F(z)G^{1,-}(w) \sim \frac{G^{2,-}(w)}{z-w}, \quad F(z)G^{1,+}(w) \sim \frac{-G^{2,+}(w)}{z-w},$$

$$G^{1,-}(z)G^{1,+}(w) \sim \frac{2(k+2)E(w)}{(z-w)^2} + \frac{{:}J(w)E(w){:} - (k+2)\partial E(w)}{z-w},$$

$$G^{2,-}(z)G^{2,+}(w) \sim \frac{-2(k+2)F(w)}{(z-w)^2} + \frac{{:}J(w)F(w){:} - (k+2)\partial F(w)}{z-w},$$

$$G^{1,-}(z)G^{2,+}(w) \sim \frac{-2(k+1)(k+2)[\text{illegible}]}{(z-w)^3} + \frac{[\text{illegible}]H(w)}{(z-w)^2}$$
$$+\frac{(k+4)T_{(2,1,1)}(w) - 2{:}E(w)F(w){:} - \frac{1}{2}{:}H(w)H(w){:} + \frac{1}{2}{:}H(w)J(w){:} - \frac{3}{8}{:}J(w)J(w){:} - \frac{k}{2}\partial H(w) + \frac{1}{2}(k+1)\partial J(w)}{z-w},$$

$$G^{1,+}(z)G^{2,-}(w) \sim \frac{2(k+1)(k+2)\mathbb{1}(w)}{(z-w)^3} + \frac{(k+1)J(w) + (k+2)H(w)}{(z-w)^2}$$
$$+\frac{-(k+4)T_{(2,1,1)}(w) + 2{:}E(w)F(w){:} + \frac{1}{2}{:}H(w)H(w){:} + \frac{1}{2}{:}H(w)J(w){:} + \frac{3}{8}{:}J(w)J(w){:} + \frac{k}{2}\partial H(w) + \frac{1}{2}(k+1)\partial J(w)}{z-w}.$$

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9 generating fields

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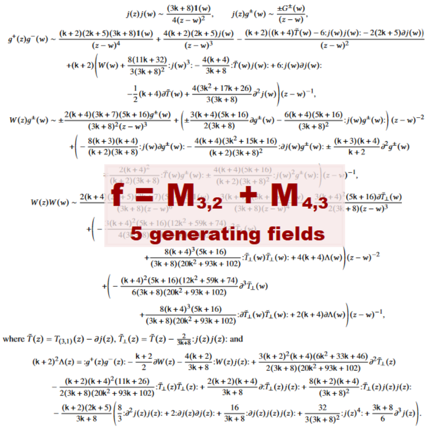

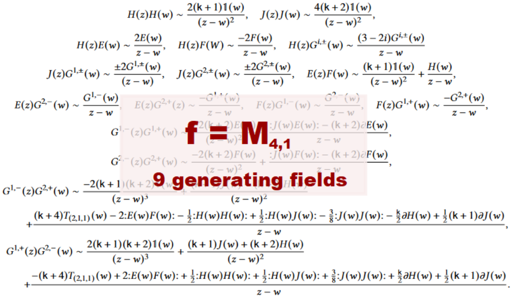

Surely there's a better way than brute force.

Wakimoto Realisation of $\mathsf{V}^k(\mathfrak{g})$

This free-field realisation requires two pieces: A Heisenberg vertex algebra $\mathsf{H}(\mathfrak{g})$ with generating fields $\{a_i(z)\}_{i=1}^r$ and a $\beta\gamma$-ghost system B_α with generating fields $\{\beta_\alpha(z), \gamma_\alpha(z)\}$ for each positive root $\alpha \in \Delta_+$.

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$$a_i(z)a_j(w) \sim \frac{2(\mathsf{k}+\mathsf{h}^\vee)A_{i,j}\mathbb{1}(w)}{(z-w)^2} \qquad \beta_\alpha(z)\gamma_{\alpha'}(w) \sim \frac{-\delta_{\alpha,\alpha'}\mathbb{1}(w)}{z-w}.$$

where $[A]$ is the Cartan matrix of $\mathfrak{g}$.

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Question

Can we describe the image of the embedding another way?

Screening Operators

Let's stick to $\mathfrak{sl}_{n+1}$. Denote the positive roots by $\{\alpha_{i,j} \mid 1 \leq i \leq j \leq n\}$.
Define fields

$$S^i(z) = :\left(\beta_{\alpha_{i,i}}(z) + \sum_{j=1}^{i-1} \gamma_{\alpha_{i-j,i-1}}(z)\beta_{\alpha_{i-j,i}}(z)\right) \mathrm{e}^{\frac{-1}{\mathsf{k}+\mathsf{h}^\vee} a_i}(z):.$$

and consider the operators $S^i_{(0)} = \int S^i(z)\,\mathrm{d}z$ on $\mathsf{H}(\mathfrak{sl}_{n+1}) \otimes \bigotimes_{\alpha \in \Delta_+} \mathsf{B}_\alpha$.

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and consider the operators $S^i_{(0)} = \int S^i(z)\, \mathrm{d}z$ on $\mathsf{H}(\mathfrak{sl}_{n+1}) \otimes \bigotimes_{\alpha \in \Delta_+} \mathsf{B}_\alpha$. It turns out that the image of the Wakimoto realisation embedding is

[Feigin, Frenkel, '90]

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Actually, this only works for generic k. But that's typically enough to do what we want to do since the set of generic levels is Zariski dense in $\mathbb{C}$.

Screening Operators for W-Algebras

There is a Wakimoto-style realisation for W-algebras too: For $\mathfrak{sl}_{n+1}$, choose f and pick a 'nice' h. Then there is a subset $\Delta_+^0 \subset \Delta_+$ such that

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Idea

Let's see if we can relate the free-field realisations, and therefore the W-algebras.

Explaining $\mathfrak{sl}_2$ and Foreshadowing

We have free-field realisations for both of these vertex algebras:

$$\mathsf{V}^k(\mathfrak{sl}_2) \hookrightarrow \mathsf{H}(\mathfrak{sl}_2) \otimes \mathsf{B} \qquad \mathsf{W}^k(\mathfrak{sl}_2, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)) \simeq \mathsf{Vir}^k \hookrightarrow \mathsf{H}(\mathfrak{sl}_2)$$

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- If done carefully, the only fields in $\widetilde{\mathsf{H}(\mathfrak{sl}_2)}$ that appear in the image of $\widetilde{\psi}$ are in the image of ϕ. This defines an inverse reduction.
- This also works for the $\mathfrak{sl}_3$ inverse reduction and defines one relating the principal and subregular $\mathfrak{sl}_{n+1}$ W-algebras [ZF, '21].

Making New Inverse Reductions

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$.

The Second Biggest W-algebra

Let $f = f_\theta = M_{n+1,1}$. The minimal W-algebra $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_\theta)$ is the 'closest' W-algebra to $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$.

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For the minimal W-algebra, $\Delta_+^0 = \{\alpha_{i,j} \mid 1 \le i \le j \le n-1\}$. Screening operators are (zero modes of):

$$Q^i(z) = \begin{cases} S^i(z), & i = 1, \dots n-1, \\ :\gamma_{\alpha_{1,n-1}}(z) \mathrm{e}^{\frac{-1}{\mathsf{k}+\mathsf{h}^\vee} a_n}(z):, & i = n. \end{cases}$$

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Not unique: choosing a different f conjugate to f_θ gives a different set of screening operators but an isomorphic W-algebra.

One Difference and Overcoming

Observation

Ignoring the differing domains, the only difference in the screening operators for $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$ and $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_\theta)$ is in the n'th ones:

$$Q^n(z) = :\gamma_{\alpha_{1,n-1}}(z)\mathrm{e}^{\frac{-1}{\mathsf{k}+\mathsf{h}^\vee}a_n}(z):$$

vs.

$$S^n(z) = :\left(\beta_{\alpha_{n,n}}(z) + \sum_{j=1}^{n-1}\gamma_{\alpha_{n-j,n-1}}(z)\beta_{\alpha_{n-j,n}}(z)\right)\mathrm{e}^{\frac{-1}{\mathsf{k}+\mathsf{h}^\vee}a_n}(z):.$$

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Question

Can we 'free' it by bosonising a ghost system?

Tildefication

Let's bosonise $\mathsf{B}_{\alpha_{1,n}}$ by embedding it into Π:

$$\beta_{\alpha_{1,n}}(z) \mapsto \mathrm{e}^{c}(z), \qquad \gamma_{\alpha_{1,n}}(z) \mapsto \frac{1}{2}{:}(c(z) + d(z))\,\mathrm{e}^{-c}(z){:}.$$

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Payoff

The screening operator $S^n(z)$ becomes:

$$S^n(z) = {:}\left(\beta_{\alpha_{n,n}}(z) + \sum_{j=1}^{n-1} \gamma_{\alpha_{n-j,n-1}}(z)\beta_{\alpha_{n-j,n}}(z)\right)\mathrm{e}^{\frac{-1}{\mathsf{k}+\mathsf{h}^\vee} a_n}(z){:}$$

$$\downarrow$$

$${:}\widetilde{\gamma_{\alpha_{1,n-1}}}(z)\mathrm{e}^{\frac{-1}{\mathsf{k}+\mathsf{h}^\vee} \widetilde{a}_n}(z){:}$$

where $\widetilde{a}_n(z) = a_n(z) - (\mathsf{k} + \mathsf{h}^\vee)c(z)$ and

$$\widetilde{\gamma_{\alpha_{1,n-1}}}(z) = \gamma_{\alpha_{1,n-1}}(z) + (\text{some other fields}).$$

Not So Fast

So by combining the Wakimoto realisation with bosonisation, the n'th screening operator for $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$ looks like that of $\mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_\theta)$ with tildes.

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- $\widetilde{\gamma_{\alpha_{1,n-1}}}(z)$ has nontrivial OPEs with fields that it shouldn't, so we need to reshuffle the rest of the fields so that the ghost fields all split into pairs.
- If we're reshuffling ghost fields, that will change the form of $S^i(z)$ (for $i < n$) since it contains ghost fields.

Splitting Ghosts

Define:

$$\widetilde{\beta}_\alpha(z) = \beta_\alpha(z) - \frac{1}{2} \sum_{\substack{\alpha',\alpha'' \in \Delta_+ \setminus \theta \\ \alpha'+\alpha''=\theta+\alpha}} :\beta_{\alpha'}(z)\beta_{\alpha''}(z)\mathrm{e}^{-c}(z):$$

$$\widetilde{\gamma}_\alpha(z) = \gamma_\alpha(z) + \sum_{\substack{\alpha' \in \Delta_+ \setminus \theta \\ \alpha'=\theta-\alpha}} :\beta_{\alpha'}(z)\mathrm{e}^{-c}(z): + \sum_{\substack{\alpha'',\alpha''' \in \Delta_+ \setminus \theta \\ -\alpha''+\alpha'''=\theta-\alpha}} :\gamma_{\alpha''}(z)\beta_{\alpha'''}(z)\mathrm{e}^{-c}(z):$$

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- $\widetilde{\beta_\alpha}(z)\widetilde{\gamma_{\alpha'}}(w) \sim -\delta_{\alpha,\alpha'}\mathbb{1}(w)(z-w)^{-1}$

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- $\widetilde{\beta_\alpha}(z)\widetilde{\gamma_{\alpha'}}(w) \sim -\delta_{\alpha,\alpha'}\mathbb{1}(w)(z-w)^{-1}$
- A Miracle?: Replacing all fields in $S^i(z)$ with their tilded versions and substituting the above gives $S^i(z)$ back again (for $i < n$).

Rearranging Screening Operators

Take the Wakimoto realisation of $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$, bosonise $\mathsf{B}_{\alpha_{1,n}}$, replace fields with their tilded versions to obtain an embedding

$$\begin{array}{c} \mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1}) \overset{\text{Wakimoto}}{\hookrightarrow} \mathsf{H}(\mathfrak{sl}_{n+1}) \otimes \bigotimes_{\alpha\in\Delta_+} \mathsf{B}_\alpha \\ \big\downarrow \text{ Bosonisation} \\ \mathsf{H}(\mathfrak{sl}_{n+1}) \otimes \bigotimes_{\alpha\in\Delta_+\setminus\theta} \mathsf{B}_\alpha \otimes \Pi \\ \wr\!\mid \text{ Tildefication} \\ \widetilde{\mathsf{H}(\mathfrak{sl}_{n+1})} \otimes \bigotimes_{\alpha\in\Delta_+\setminus\theta} \widetilde{\mathsf{B}_\alpha} \otimes \widetilde{\Pi} \end{array}$$

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$$\mathsf{H}(\mathfrak{sl}_{n+1}) \otimes \bigotimes_{\alpha\in\Delta_+\setminus\theta} \mathsf{B}_\alpha \otimes \Pi$$

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$$\widetilde{\mathsf{H}(\mathfrak{sl}_{n+1})} \otimes \bigotimes_{\alpha\in\Delta_+\setminus\theta} \widetilde{\mathsf{B}_\alpha} \otimes \widetilde{\Pi}$$

with screening operators description

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1}) \simeq \underbrace{\bigcap_{i=1}^{n} \ker \widetilde{Q}^i{}_{(0)}}_{\text{screening operators for } \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_\theta)} \cap \underbrace{\ker T_{(0)}}_{\text{screening operator for bosonisation}}$$

Inverse Reduction

Result [ZF, 'Soon]

For k generic, there exists an embedding

$$\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1}) \hookrightarrow \mathsf{W}^{\mathsf{k}}(\mathfrak{sl}_{n+1}, f_\theta) \otimes \Pi \otimes \mathsf{B}^{\otimes(n-1)}$$

with known screening operator (coming from bosonisation). Generic k can be upgraded to noncritical k with a little extra work.

Where To From Here

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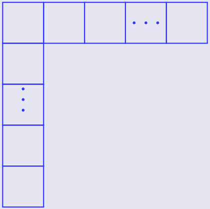
Hook-Types

Recall that $\mathfrak{sl}_{n+1}$ W-algebras are labelled by partitions of $n+1$. If that partition is of the form



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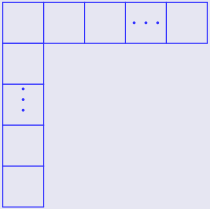

we call the W-algebra hook-type . This includes the principal/regular, subregular and minimal $\mathfrak{sl}_{n+1}$ W-algebras, as well as the affine $\mathfrak{sl}_{n+1}$ VOA. There are $n+1$ of these, choose a corresponding nilpotent element $f^{(m)}$.

Inverse Reduction for Hook-Types

Remarkably, an almost identical argument to that for the minimal-to-affine $\mathfrak{sl}_m$ inverse reduction gives:

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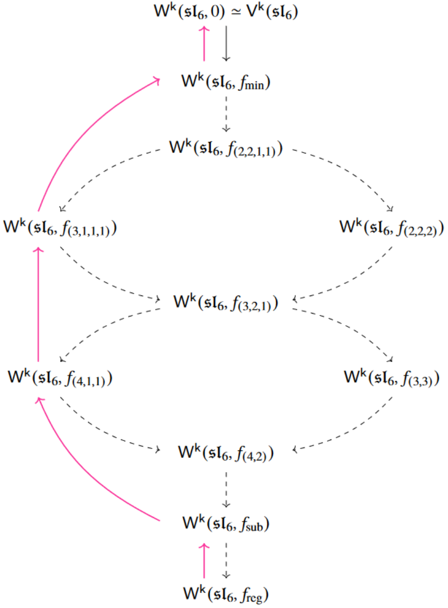
- All known $\mathfrak{sl}_{n+1}$ inverse reductions are examples of the above.
- Tildefication is also a recipe to make the inverse reduction explicit.
- Can compose these embeddings to realise $\mathsf{V}^{\mathsf{k}}(\mathfrak{sl}_{n+1})$ in terms of any hook-type $\mathfrak{sl}_{n+1}$ W-algebra.

The Path of Hooks

Now have a traversable path in the poset of W-algebras for $\mathfrak{sl}_{n+1}$ using partial and inverse reduction.

Can construct modules for any hook-type W-algebra (or affine VOA) by taking a module for a 'smaller' hook-type W-algebra and tensoring with modules for the bosonic ghost systems and half lattices.

When do these inverse reduction embeddings descend to embeddings of simple quotients? Know $\mathfrak{sl}_2, \mathfrak{sl}_3$ and $m = 2$ for general $\mathfrak{sl}_{n+1}$



Lingering Questions and Future Directions

- When else can we construct inverse reductions and why?
- What about examples outside of type A?
- Why do we need to bosonise? Some kind of localisation?
- Is there something geometric underlying all of this, since the Wakimoto realisation is very geometric?
- Representation theory (W-algebra modules by restriction, embeddings of simple quotients, highest-weight theory, …)
- Physics (modular-invariant partition functions, fusion, correlation functions, conformal blocks …)
- Mathematics (finite W-algebras/shifted Yangians, Slodowy slices and geometric representation theory, …)