Rationality and orthosymplectic Feigin–Semikhatov duality

Dubrovnik - Representation Theory XVII

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The subregular $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$ ($n \geqslant 2$)

Fix $f^+ = f_{\mathrm{subreg}}$ that we embed in an $\mathfrak{sl}_2$-triple $\{e^+, 2x^+, f^+\}$ where x^+ defines a good grading on $\mathfrak{so}_{2n+1}$:

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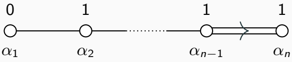

$$\mathcal{W}^k(\mathfrak{so}_{2n+1}, f_{\mathrm{subreg}}) \simeq \mathcal{W}(1, 2, 4, \ldots, 2n-2, (n)^2),$$

is strongly generated by

- the Virasoro field L of conformal weight 2,
- J^+ of conformal weight 1, which generates a Heisenberg vertex subalgebra $\pi^{J^+} \subset \mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{subreg}})$ for $k \neq -(2n-1), -(2n-2)$, and satisfies the OPE
$$J^+(z)J^+(w) \sim \frac{(k+h_+^\vee)-1}{(z-w)^2}, \quad h_+^\vee = 2n-1,$$
- $W_i(z)$ ($i = 4, \ldots, 2n-2$), which commute with $J^+(z)$,
- and $G^\pm(z)$, of conformal weight n, satisfy
$$J^+(z)G^\pm(w) \sim \frac{\pm G^\pm(w)}{(z-w)}.$$

Spectral flow twist action

Introduce the Li's Δ-operators with respect to the semisimple action of J_0^+

$$\Delta_{\theta J^+}(z) = z^{\theta J_0^+} \exp\left(-\theta \sum_{m>0} \frac{J_m^+}{m}(-z)^{-m}\right), \quad (\theta \in \mathbb{Z}).$$

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For a $\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{subreg}})$-module M,

$$\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{subreg}}) \times M \to M(\!(z)\!), \quad (a, m) \mapsto (S_{\theta J^+} a)(z) m := (\Delta_{\theta J^+}(z) a)(z) m$$

defines another $\mathcal{W}^k(\mathfrak{so}_{2n+1}, f_{\mathrm{subreg}})$-module structure on M denoted by $S_{\theta J^+}(M)$.

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In particular, $S_{\theta J^+}$ twists the actions of J^+ and L as

$$J_m^+ \mapsto J_m^+ + \delta_{m,0}(k + h_+^\vee - 1)\theta, \quad L_m \mapsto L_m + \theta J_n^+ + \tfrac{1}{2}(k + h_+^\vee - 1)\theta^2 \delta_{m,0}.$$

Accordingly, the characters $\mathrm{ch}_M(z,q) := \mathrm{tr}_M z^{J_0^+} q^{L_0}$ and the set of J_0^+-eigenvalues $\mathrm{wt}(M)$ transform as

$$\mathrm{ch}_{S_\theta M}(z,q) = q^{\frac{1}{2}\theta^2(k+h_+^\vee-1)} z^{(k+h_+^\vee-1)\theta} \mathrm{ch}_M(zq^\theta, q),$$
$$\mathrm{wt}_{J_0^+}(S_\theta M) = (k + h_+^\vee - 1)\theta + \mathrm{wt}(M).$$

Simple modules for $\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$

At admissible levels

$$k = -h_+^\vee + \frac{p}{q}, \text{ where } h_+^\vee = 2n-1 \text{ and } q = 2n-1, 2n,$$

$\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$ is rational [McRae].

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Theorem (F.-Nakatsuka)

The set of simple $\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$-modules is given by

$$\mathrm{Irr}\,\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) = \{S_{\theta J^+}\mathbf{L}_k(\lambda),\ \lambda \in \mathrm{Pr}_{\mathbb{Z}}^k,\ 0 \leqslant \theta < \bar{q}\},$$

where $\bar{q} = 2n-1$ when $q = 2n-1$ and $\bar{q} = n$ when $q = 2n$.

$$\text{where } \mathrm{Pr}_{\mathbb{Z}}^k = \begin{cases} \{\lambda \in \mathrm{P}_+ \mid (\lambda, \theta) \leqslant p - h^\vee\} & (q = 2n-1), \\ \{\lambda \in \mathrm{P}_+ \mid (\lambda, \theta_s^\vee) \leqslant p - h\} & (q = 2n), \end{cases}$$

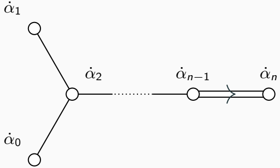
The classification provides an alternative proof for the rationality of exceptional $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$.

Compatibility of spectral flows

Using that spectral flows preserve simple currents, we show that

$$S_{\bar{q}J^+}\mathbf{L}_k(0) \simeq \mathbf{L}_k(\sigma \circ 0)$$

where σ is the symmetry of the Dynkin diagram



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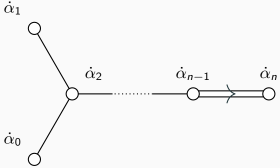

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Conjecture

For all $\lambda \in \mathrm{Pr}^k_{\mathbb{Z}}$, $S_{\bar{q}J^+}\mathbf{L}_k(\lambda) \simeq \mathbf{L}_k(\sigma \circ \lambda)$ and more generally, for $0 \leqslant \theta < \bar{q}$, there exists $\mu \in \mathfrak{h}^*$ such that $S_{\theta J^+}\mathbf{L}_k(\lambda) \simeq \mathbf{L}_k(\mu)$.

Fusion rules

Under the previous conjecture, the fusion ring of $\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) - \mathrm{Mod}$ is given by

$$\mathcal{K}\left(\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})\right) \simeq \mathcal{K}^0\left(\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})\right) \underset{\mathbb{Z}[\mathbb{Z}_2]}{\otimes} \mathbb{Z}[\mathbb{Z}_{2\bar{q}}]$$

where the tensor product identifies $\mathbf{L}_k(\sigma \circ \lambda) \otimes [\theta] \simeq \mathbf{L}_k(\lambda) \otimes [\theta + \bar{q}]$.

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Moreover, when $q = 2n - 1$,

$$\mathcal{K}^0\left(\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})\right) \simeq \mathcal{K}(L_{p-h_+^\vee}(\mathfrak{so}_{2n+1})).$$

Example: $k = -h_+^\vee + \frac{p}{q}$ with $(p, q) = (2n, 2n-1)$

$$\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) \simeq L(\tfrac{1}{2}, 0) \otimes V_{2\sqrt{q}\mathbb{Z}} \oplus L(\tfrac{1}{2}, \tfrac{1}{2}) \otimes V_{\sqrt{q}+2\sqrt{q}\mathbb{Z}},$$

where $L(\frac{1}{2}, 0)$ is the simple Virasoro module of central charge $\frac{1}{2}$.

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$L(\frac{1}{2},0)$: $\mathcal{K}(L(\frac{1}{2},0)) \simeq \mathcal{K}(L_2(\mathfrak{sl}_2))$

$$L_2(a\varpi) \mapsto H_f^0(L_2(a\varpi))$$
$$0 \mapsto L(\tfrac{1}{2},0)$$
$$1 \mapsto L(\tfrac{1}{2},\tfrac{1}{16})$$
$$2 \mapsto L(\tfrac{1}{2},\tfrac{1}{2})$$

	0	1	2
0	0	1	2
1	1	$0 \oplus 2$	1
2	2	1	0

$V_{2\sqrt{q}\mathbb{Z}}$-modules: $\{V_{\frac{a}{2\sqrt{q}}+2\sqrt{q}\mathbb{Z}},\ 0 \leqslant a < 4q\}$

$$V_{\frac{a}{2\sqrt{q}}+2\sqrt{q}\mathbb{Z}} \boxtimes V_{\frac{b}{2\sqrt{q}}+2\sqrt{q}\mathbb{Z}} \simeq V_{\frac{a+b}{2\sqrt{q}}+2\sqrt{q}\mathbb{Z}}$$

$\mathrm{Irr}\left(L(\frac{1}{2},0) \otimes V_{2\sqrt{q}\mathbb{Z}}\right) = \{M(a,b),\ a = 0, \frac{1}{16}, \frac{1}{2},\ 0 \leqslant b < 4q\}$

$$M(a_1,b_1) \boxtimes M(a_2,b_2) \simeq M(a_3, b_1+b_2).$$

$$\operatorname{Irr}\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) \simeq \operatorname{Irr}\left(M(0,0)\right)^{loc} = \left\{ \begin{array}{lll} M(0,b), & (b \equiv 0) & \\ M(\frac{1}{2},b), & (b \equiv 0), & 0 \leqslant b < 4q \mod 2 \\ M(\frac{1}{16},b), & (b \equiv 1) & \end{array} \right\}$$

where $M(\frac{1}{2}, 2q) \boxtimes M(a,b) \sim M(a,b)$.

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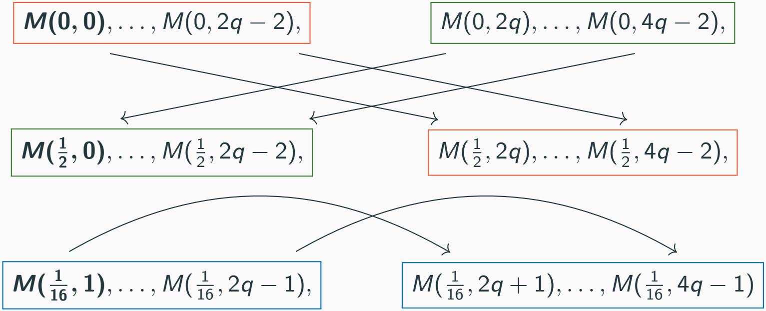

$$\{M(0,0), M(\tfrac{1}{2},0), M(\tfrac{1}{16},q)\}$$

$$\bigoplus_{b\in\mathbb{Z}_{2q}} \mathbb{Z}\, M(0,2b)$$

$$\mathcal{K}\left(\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})\right) \simeq \mathcal{K}(L_1(\mathfrak{so}_{2n+1})) \underset{\mathbb{Z}[\mathbb{Z}_2]}{\otimes} \mathbb{Z}[\mathbb{Z}_{2q}]$$

$$[q] \leftrightarrow M(0,2q) \sim M(\tfrac{1}{2},0)$$

$\mathcal{W}^{\ell}(\mathfrak{osp}_{2|2n})$ and Fegin–Semikhatov duality

$$\mathcal{W}^{\ell}(\mathfrak{osp}_{2|2n}) \simeq \mathcal{W}(1, 2, 4, \cdots, 2n, (n+\tfrac{1}{2})^2).$$

Its structure is very similar to the one of $\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$.

J^- of weight 1 generates a Heisenberg vertex subalgebra $\pi^{J^-} \subset \mathcal{W}_{\ell}(\mathfrak{osp}_{2|2n})$ for $\ell \neq -n, -n+\frac{1}{2}$ with OPE

$$J^-(z)J^-(w) \sim \frac{-2(\ell + h_-^{\vee}) + 1}{(z-w)^2}, \qquad h_-^{\vee} = n.$$

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$$J^-(z)J^-(w) \sim \frac{-2(\ell + h^\vee_-) + 1}{(z-w)^2}, \qquad h^\vee_- = n.$$

We have the following isomorphism of vertex superalgebras [CGN,CL]:

$$\mathrm{Com}\left(\pi^{J^+}, \mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})\right) \simeq \mathrm{Com}\left(\pi^{J^-}, \mathcal{W}_\ell(\mathfrak{osp}_{2|2n})\right)$$

$$\text{with } 2(k + h^\vee_+)(\ell + h^\vee_-) = 1, \quad (k, \ell) \neq (-2n + 2, -n + \tfrac{1}{2}).$$

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and we recover the Kazama–Suzuki type coset constructions [CGN]:

$$\mathcal{W}^k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) \simeq \mathrm{Com}\left(\pi^{J_\Delta^-}, \mathcal{W}^\ell(\mathfrak{osp}_{2|2n}) \otimes V_{\sqrt{-1}\,\mathbb{Z}}\right),$$

$$\mathcal{W}^\ell(\mathfrak{osp}_{2|2n}) \simeq \mathrm{Com}\left(\pi^{J_\Delta^+}, \mathcal{W}^k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\right).$$

where $\pi^{J_\Delta^\pm}$ denote the Heisenberg vertex algebras generated by the fields

$$J_\Delta^+(z) = J^+(z) - x(z), \quad J_\Delta^-(z) = J^-(z) + y(z).$$

IHR and Feigin–Frenkel duality

We have the following free field realizations

$$\mathcal{W}^k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) \hookrightarrow \pi_{\mathfrak{h}_+}^{k+h_+^\vee} \otimes \Pi \quad \text{and} \quad \mathcal{W}^k(\mathfrak{so}_{2n+1}) \hookrightarrow \pi_{\mathfrak{h}_+}^{k+h_+^\vee},$$

where $\Pi = \bigoplus_{m\in\mathbb{Z}} \pi^{x,y}_{m(x+y)}$ These embeddings are characterized by the "same" screening operators.

Theorem (BM, FN)

For $k \neq -h_+^\vee$, there exists an embedding of vertex algebras

$$\mu_{\mathrm{iHR}} \colon \mathcal{W}^k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) \hookrightarrow \mathcal{W}^k(\mathfrak{so}_{2n+1}) \otimes \Pi.$$

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Similarly, on the super side, the Miura maps give

$$\mathcal{W}^\ell(\mathfrak{osp}_{2|2n}) \hookrightarrow \mathcal{W}^\ell(\mathfrak{sp}_{2n}) \otimes V_{\mathbb{Z}} \otimes \pi.$$

The previous realizations relate the Feigin–Semikhatov and Feigin–Frenkel dualities:

For k, ℓ satisfying $2(k + h_+^\vee)(\ell + h_-^\vee) = 1$,

$$
\begin{array}{ccc}
\mathcal{W}^k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) & \xhookrightarrow{\mu_{\mathrm{iHR}}} & \mathcal{W}^k(\mathfrak{so}_{2n+1}) \otimes \Pi \\
\Big\updownarrow \text{F.-S.} & & \Big\updownarrow \text{F.-F.} \qquad \Big\updownarrow \mathrm{H}^{\frac{\infty}{2}}_{\mathrm{rel}} \\
\mathcal{W}^\ell(\mathfrak{osp}_{2|2n}) & \xhookrightarrow{\qquad} & \mathcal{W}^\ell(\mathfrak{sp}_{2n}) \otimes V_{\mathbb{Z}} \otimes \pi.
\end{array}
$$

Simple modules of $\mathcal{W}_\ell(\mathfrak{osp}_{2|2n})$

Let $2(k+h_+^\vee)(\ell+h_-^\vee)=1$ with $k=-h_+^\vee+\frac{p}{q}$ admissible s.t. $q=2n-1,\ 2n$.
For such levels, the $\mathcal{W}$-superalgebra $\mathcal{W}_\ell(\mathfrak{osp}_{2|2n})$ is rational [CGN,CL].

Theorem (F.–N.)

Moreover, $\operatorname{Irr}\mathcal{W}_\ell(\mathfrak{osp}_{2|2n})=\{S_{\theta J^-}\,\mathbf{L}_k^-(\lambda),\ \lambda\in\mathrm{Pr}_{\mathbb{Z}}^k,\ 0\leqslant\theta<p\}$, where

$$\mathbf{L}_k^-(\lambda)\overset{\text{def}}{=}\operatorname{Hom}_{V_{2\sqrt{\overline{pq}}\,\mathbb{Z}}}\big(V_{\frac{1}{2}\sqrt{\frac{q}{p}}a+2\sqrt{\overline{pq}}\,\mathbb{Z}},\mathbf{L}_k(\lambda)\otimes V_{\mathbb{Z}}\big),$$

where $a=0$ if $\operatorname{Supp}\mathbf{L}_k(\lambda)=\mathbb{Z}$ and $a=1$ if $\operatorname{Supp}\mathbf{L}_k(\lambda)=\frac{1}{2}+\mathbb{Z}$ and

$$2\overline{p}=\begin{cases}2p & (q=2n-1)\\ p & (q=2n).\end{cases}$$

Relative semi-infinite cohomology

The Kazama–Suzuki type coset constructions can be rephrased using the *relative semi-infinite cohomology* [Fei, FGZ]

$$\mathrm{H}_{\mathrm{rel}}^{\frac{\infty}{2}+m}(\widehat{\mathfrak{gl}}_1, \mathfrak{gl}_1; \pi_p^{A_*} \otimes \pi_q^{A}) \simeq \delta_{m,0}\delta_{p+q,0}\mathbb{C}[\mathbf{e}^p \otimes \mathbf{e}^q].$$

We get

$$\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) \simeq \mathrm{H}^0_{\mathrm{rel}}(\mathfrak{gl}_1, \mathcal{W}_\ell(\mathfrak{osp}_{2|2n}) \otimes K_-),$$
$$\mathcal{W}_\ell(\mathfrak{osp}_{2|2n}) \simeq \mathrm{H}^0_{\mathrm{rel}}(\mathfrak{gl}_1, \mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) \otimes K_+).$$

where

$$K_+ := V_{\mathbb{Z}} \otimes \pi^y, \quad K_- := V_{\sqrt{-1}\,\mathbb{Z}} \otimes \pi^x.$$

$\rightsquigarrow$ relate modules of $\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$ and $\mathcal{W}_\ell(\mathfrak{osp}_{2|2n})$.

Kazhdan-Lusztig categories

Consider the full subcategories

$$\mathsf{KL}_+^k, \quad \mathsf{KL}_-^\ell$$

of the category of modules over $\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$ (resp. $\mathcal{W}_\ell(\mathfrak{osp}_{2|2n})$) which decompose into direct sums of Fock modules.

Since $\mathrm{wt}_{J_0^+}(\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})) = \mathrm{wt}_{J_0^-}(\mathcal{W}_\ell(\mathfrak{osp}_{2|2n})) = \mathbb{Z}$ we have the decompositions

$$\mathsf{KL}_+^k = \bigoplus_{[\lambda]\in\mathbb{C}/\mathbb{Z}} \mathsf{KL}_+^{k,[\lambda]}, \quad \mathsf{KL}_-^\ell = \bigoplus_{[\lambda]\in\mathbb{C}/\mathbb{Z}} \mathsf{KL}_-^{\ell,[\lambda]}.$$

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For $\lambda \in \mathbb{C}$, define the functors

$$\mathrm{H}_\lambda^{\pm}\colon\ \mathsf{KL}_\pm^\star \ni M \mapsto \mathrm{H}^0_{\mathrm{rel}}(\mathfrak{gl}_1, M \otimes K_\mp^\lambda),$$

where

$$K_+^\lambda := V_{\mathbb{Z}} \otimes \pi^y_{\lambda y}, \quad K_-^\lambda := V_{\sqrt{\text{-}1}\,\mathbb{Z}} \otimes \pi^x_{\lambda x},$$

are the modules over K_+ and K_- respectively.

Correspondence of categories

Set $\varepsilon := \sqrt{k + h_+^\vee} = 1/\sqrt{2(\ell + h_-^\vee)}$ and $\lambda_\theta^+ = \lambda - \theta\varepsilon^{-1}$ ($\theta \in \mathbb{Z}$).

Theorem (F.–N.)

The functors $\mathrm{H}_\lambda^\pm$ can be restricted block-wise into

$$\mathrm{H}^+_{\lambda_\theta^+} \colon \mathsf{KL}_+^{k,[\lambda\varepsilon]} \rightleftarrows \mathsf{KL}_-^{\ell,[\lambda_\theta^+/\varepsilon]} \colon \mathrm{H}^-_{\lambda_\theta^+}, \quad (\theta \in \mathbb{Z}),$$

which are quasi-inverse to each other and give an equivalence of categories.

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The functors $\mathrm{H}^+_{\lambda_\theta^+}$ are related to each other through spectral flow twits: for $\theta_i \in \mathbb{Z}$ $(1 \leqslant i \leqslant 4)$ and $\lambda, \lambda' \in \mathbb{C}$ with $\theta_4 = \theta_2 + \theta_3$ and $\lambda' = \lambda + \varepsilon\theta_1$

$$\begin{array}{ccc}
\mathsf{KL}_+^{k,[\lambda\varepsilon]} & \xrightarrow{\mathrm{H}^+_{\lambda^+_{\theta_3}}} & \mathsf{KL}_-^{\ell,[\lambda^+_{\theta_3}/\varepsilon]} \\
\Big\downarrow {\scriptstyle S_{\theta_1 J^+}} & & \Big\downarrow {\scriptstyle S_{\theta_2 J^-}} \\
\mathsf{KL}_+^{k,[\lambda'\varepsilon]} & \xrightarrow{\mathrm{H}^+_{\lambda'^+_{\theta_4}}} & \mathsf{KL}_-^{\ell,[\lambda'^+_{\theta_4}/\varepsilon]}
\end{array}$$

commutes up to a natural isomorphism and all the arrows give equivalences of categories.

Using the correspondence of category, we have

$$\mathbf{L}_k^-(\lambda) = H_\mu^+(\mathbf{L}_k(\lambda))$$

with $\mu = 0$ if $\operatorname{Supp}\mathbf{L}_k(\lambda) = \mathbb{Z}$ and $\mu = \frac{1}{2}\sqrt{\frac{q}{p}}$ if $\operatorname{Supp}\mathbf{L}_k(\lambda) = \frac{1}{2} + \mathbb{Z}$.

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with $\mu = 0$ if $\operatorname{Supp} \mathbf{L}_k(\lambda) = \mathbb{Z}$ and $\mu = \frac{1}{2}\sqrt{\frac{q}{p}}$ if $\operatorname{Supp} \mathbf{L}_k(\lambda) = \frac{1}{2} + \mathbb{Z}$.

The conjecture on the periodicity of the spectral flow twist for $\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$ implies that $S_{2\overline{p}J^-}\, \mathbf{L}_k^-(\lambda) = \mathbf{L}_k^-(\lambda)$, for $\lambda \in \mathrm{Pr}_{\mathbb{Z}}^k$, and

$$\mathcal{K}(\mathcal{W}_\ell(\mathfrak{osp}_{2|2n})) \simeq \left(\mathcal{K}(\mathcal{W}_k(\mathfrak{so}_{2n+1}, f_{\mathrm{subreg}})) \underset{\mathbb{Z}[L^*/L]^G}{\otimes} \mathbb{Z}[L^*/L]\right)^{G'},$$

where $L^*/L \simeq \mathbb{Z}_{4pq}$ for $q = 2n-1$ and $\mathbb{Z}_{pq}$ for $q = 2n$.

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Thank you for your attention!