Coset Constructions and Kac-Wakimoto sets

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Content

1. Introduction

What is the coset construction?

In the language of vertex operator algebra:

- V: a vertex operator algebra
- $U \subset V$: a subalgebra
- The coset vertex operator algebra U^c : the centralizer of U in V; U^c is also called the commutant of U in V.
- Coset theory studies the representions of U^c in terms of representations of U and V

1. Introduction

Rationality

- A vertex operator algebra is rational if the $\mathbb{Z}_+$-graded module category is semisimple (analogue of finite dimensional semisimple associative algebra and Lie algebra)
- (Dong-Li-Mason 1998) If V is rational then there are only finitely many inequivalent irreducible V-modules

Coset theory conjecture

If U and V are rational vertex operator algebras then U^c is rational.

Remark

This conjecture explains why coset theory is important

1. Introduction

The origin

- $L(c,h)$–the irreducible highest weight module for the Virasoro algebra with central charge c and highest weight h
- Which $L(c,h)$ is unitary?
- (Goddard-Kent-Olive 1986) Coset construction was introduced to prove the untarity of $L(c_m, h^m_{r,s})$ where

$$c_m = 1 - \frac{6}{(m+1)(m+2)}$$

for $m = 1, 2, 3, \cdots$ and

$$h = h^m_{r,s} = \frac{[(m+2)r-(m+1)s]^2 - 1}{4(m+1)(m+2)}$$

$$(r, s \in \mathbb{N}, 1 \le s \le r \le m)$$

1. Introduction

History

- There are hundreds of papers in physics on coset constructions (affine vertex operator algebras or WZW models)
- (Frenkel-Zhu 1992) Formulation of coset construction in the language of vertex operator algebras
- (Xu 1999-2000, 2007) The algebraic coset conformal field theory and mirror extensions (in the language of conformal nets)
- (MaRae 2021) If U and V are rational, U, V, U^c are C_2-cofinite then U^c is rational.
- Most study on coset theory focus on examples

1. Introduction

Representation theory of U^c :

- Rationality of U^c (Coset theory conjecture)
- Classification of irreducible U^c-modules

Our work

- Categorical coset constructions: main results are proved in categorical setting
- Apply results on categorical coset constructions to VOA coset constructions: assume $(U^c)^c = U$ and U, U^c, V are regular. We use irreducible U-modules and V-modules to determine irreducible U^c-modules

2. Basics

Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra.

- V is called rational if any $\mathbb{Z}_+$-graded module is completely reducible
- V is C_2-cofinite if $\dim V/C_2(V) < \infty$ where

  $$C_2(V) = \langle u_{-2}v | u, v \in V \rangle$$

  This condition is needed to prove the formal characters convergent to holomorphic functions in the upper half plane
- V is rational and C_2-cofinite iff V is regualr: any weak V-module is completely reducible (Li 1999, Abe-Buhl-Dong 2004, Dong-Yu 2012)

2. Basics

- $U = (U, Y, \mathbf{1}, \omega^1)$ is a vertex operator subalgebra of V. The commutant or centralizer of U in V

$$U^c = \{v \in V | [Y(u, z_1), Y(v, z_2)] = 0, u \in U\}$$

$$= \{v \in V | u_n v = 0, u \in U, n \geq 0\}$$

is a vertex operator algebra if $L(1)\omega^1 = 0$ (Frenkel-Zhu 1992). U^c is called the coset construction in physics

2. Basics

- If V is rational and C_2-cofinite, the V-module category $\mathcal{C}_V$ is a modular tensor category with tensor product $\boxtimes$ (Huang 2008)
- If V is rational then there are only finite many irreducible modules $M^1, \cdots M^p$ up to isomorphism and

$$M^i \boxtimes M^j = \sum_{k=1}^{p} N_{i,j}^k M^k$$

  where the multiplicities $N_{i,j}^k$ are called the fusion rules
- A V-module M is called a simple current if for any irreducible V-module N, $M \boxtimes N$ is irreducible

3. Fusion category

Fusion category

- A fusion category over $\mathbb{C}$ is a $\mathbb{C}$-linear semisimple rigid category with finitely many simple objects and finite dimensional morphism spaces such that the unit object $\mathbf{1}$ is simple, $\otimes$, the dual object of $X : X'$
- $K(\mathcal{C})$: Grothendieck ring of a fusion category of $\mathcal{C}$
- $\mathbf{O}(\mathcal{C})$ = set of equivalence classes of the simple objects
- The Frobenius-Perron dimension: Unique ring homo $\mathrm{FPdim} : K(\mathcal{C}) \to \mathbb{R}$, $\mathrm{FPdim}(M) \geq 1$ for any nonzero object M [Etingof-Nikshych-Ostrik2005].
- $\mathrm{FPdim}(M)$ is exactly the quantum dimension [Dong-Jiao-Xu2013, Dong-Ren-Xu2018] in the theory of vertex operator algebra
- $\mathrm{FPdim}(\mathcal{C}) = \sum_{M \in \mathbf{O}(\mathcal{C})} \mathrm{FPdim}(M)^2$.

3. Fusion category

Braided fusion category

- A braided fusion categories is a fusion category $\mathcal{C}$ endowed with a braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$
- Its reverse category $\overline{\mathcal{C}}$ is the same fusion category with a new braiding $\bar{c}_{X,Y} = c^{-1}_{Y,X}$. A braided fusion category is symmetric if $\bar{c} = c$.
- Two braided fusion categories $\mathcal{C}$ and $\mathcal{D}$ are braided equivalent if there is functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(c_{X,Y}) = c_{F(X),F(Y)}$ (preserving the braiding)

3. Fusion category

Modular tensor category

- $\mathcal{C}$ is a braided fusion category
- For any X and any $f : X \to X$, can define $\mathrm{tr}(f)$
- There is a set of isomorphisms $\theta_X : X \to X$ such that $\theta_{X\otimes Y} = c_{Y,X}c_{X,Y}\theta_X \otimes \theta_Y$, $\theta_i : X_i \to X_i$ (X_i are simple objects).
- $s = (s_{i,j})$ is nondegenerate where $s_{i,j} = \mathrm{tr}(c_{X_j,X_i}c_{X_i,X_j})$ and $c_{X_j,X_i}c_{X_i,X_j} : X_i \otimes X_j \to X_i \otimes X_j$
- $\dim(X) = \mathrm{tr}(\mathrm{id}_X)$

3. Fusion category

Regular algebra

- Let $\mathcal{C}$ be a braided fusion category, A regular (commutative and separable) algebra A is an object in $\mathcal{C}$ with a product $\mu : A \otimes A \to A$ and $\mathrm{Hom}(\mathbf{1}, A)$ is 1-dimensional
- There is a notion of A-module, local A-module in $\mathcal{C}$. $\mathcal{C}_A$ is the category of A-modules in $\mathcal{C}$, the category of local A-modules $\mathcal{C}_A^0$ in $\mathcal{C}$ is a subcategory of $\mathcal{C}_A$. $\mathcal{C}_A$ is a fusion category and $\mathcal{C}_A^0$ is a braided fusion category
- If $\mathcal{C}$ is a modular tensor category, so is $\mathcal{C}_A^0$

4. Categorical coset theory

Setting

- $\mathcal{C}_1, \mathcal{C}_2$: modular tensor categories
- $\mathbf{O}(\mathcal{C}_1) = \{W^\alpha | \alpha \in J\}$, $W^1 = 1_{\mathcal{C}_1}$
- $\mathbf{O}(\mathcal{C}_2) = \{N^\phi | \phi \in K\}$, $N^1 = 1_{\mathcal{C}_2}$
- A: regular commutative algebra in $\mathcal{C}_1 \otimes \mathcal{C}_2$
- $\mathcal{C} = (\mathcal{C}_1 \otimes \mathcal{C}_2)^0_A$ is also a modular tensor category
- $\mathbf{O}(\mathcal{C}) = \{M^i | i \in I\}$ with $1 \in I$ and $M^1 = A$.
- $M^i \cong \oplus_{\alpha \in J} W^\alpha \otimes M^{(i,\alpha)}$ as objects in $\mathcal{C}_1 \otimes \mathcal{C}_2$ for $i \in I$.

Assumptions

1. $\mathcal{C}_1, \mathcal{C}_2$ are pseudo unitary : for any X in $\mathcal{C}_1, \mathcal{C}_2$, $\mathrm{FPdim}(X) = \dim(X)$
2. $M^{(1,1)} = 1_{\mathcal{C}_2}$ and $\mathrm{Hom}_{\mathcal{C}_2}(1_{\mathcal{C}_2}, M^{(1,\alpha)}) = \delta_{1,\alpha}$

4. Categorical coset theory

Goal

1. Decompose $M^{(i,\alpha)}$ into a direct sum of simple objects in $\mathcal{C}_2$
2. Identify simple direct summands from $M^{(i,\alpha)}$ with simple direct summands from $M^{(j,\beta)}$

Notations

- $(i,\alpha) = M^{(i,\alpha)}$, $J_i = \{\alpha \mid (i,\alpha) \neq 0\}$
- For $\alpha \in J$, $\phi \in K$ set

$$a_{\alpha\otimes\phi} = A \boxtimes_{\mathcal{C}_1 \otimes \mathcal{C}_2} (\alpha \otimes \phi) \in (\mathcal{C}_1 \otimes \mathcal{C}_2)_A$$

$$a_{1\otimes(i,\alpha)} = A \boxtimes_{\mathcal{C}_1 \otimes \mathcal{C}_2} (1_{\mathcal{C}_1} \otimes (i,\alpha)) \in (\mathcal{C}_1 \otimes \mathcal{C}_2)_A$$

- $\langle X, Y \rangle = \dim \mathrm{Hom}_{\mathcal{D}}(X,Y)$ for any fusion category $\mathcal{D}$ and $X, Y \in \mathcal{D}$

4. Categorical coset theory

- $\mathcal{F}_1$: the fusion subcategory of $\mathcal{C}_1$ generated by W^α for $\alpha \in J_1$
- $\mathcal{F}_2$: the fusion subcategory of $\mathcal{C}_2$ generated by the simple objects appearing in $(1, \alpha)$ for $\alpha \in J_1$.

Recall $A = \oplus_{\alpha \in J_1} W^\alpha \otimes (1, \alpha)$

Schur-Weyl duality

Let $\alpha, \beta \in J_1$. Then

1. $(1, \alpha)$ is simple,
2. $(1, \alpha)$ and $(1.\beta)$ are isomorphic iff $\alpha = \beta$,
3. $\mathbf{O}(\mathcal{F}_1) = \{W^\alpha \mid \alpha \in J_1\}$, $\mathbf{O}(\mathcal{F}_2) = \{(i, \alpha) \mid \alpha \in J_1\}$,
4. $\mathcal{F}_1$ and $\overline{\mathcal{F}_2}$ are braided equivalent.

This result was essentially obtained by Lin in 2017.

5. Main results

Kac-Wakimoto Set

$$\mathrm{KW} = \{(i,\alpha) | \langle 1_{\mathcal{C}_2}, (i,\alpha)\rangle \geq 1, i \in I, \alpha \in J_i\}.$$

Properties

- If $(i,\alpha) \in \mathrm{KW}$ then $(i',\alpha') \in \mathrm{KW}$.
- $(i,\alpha), (i,\beta) \in \mathrm{KW}$, then $\alpha = \beta$ and $\langle 1_{\mathcal{C}_2}, (i,\alpha)\rangle = 1$.
- If $(i,\alpha) \in \mathrm{KW}$ then $M^i = a_{\alpha\otimes 1}$
- $(i,\alpha) \in \mathrm{KW}$ iff $W^\alpha \in C_{\mathcal{C}_1}(\mathcal{F}_1)$ where the Müger centralizer

$$C_{\mathcal{C}}(\mathcal{F}_1) = \{Y \in \mathcal{C}_1 | c_{Y,X} \circ c_{X,Y} = id_{X\otimes Y} \forall X \in \mathcal{F}_1\}$$

a braided fusion subcategory of $\mathcal{C}_1$

5. Main results

Kac-Wakimoto Set

- Let $\mathcal{C}^{\mathrm{KW}}$ be the fusion subcategory of $\mathcal{C}$ generated by M^i for $(i,\alpha) \in \mathrm{KW}$, $\mathcal{C}_1^{\mathrm{KW}}$ the fusion subcategory of $\mathcal{C}_1$ generated by W^α for $(i,\alpha) \in \mathrm{KW}$. Then

$$\mathbf{O}(\mathcal{C}^{\mathrm{KW}}) = \{M^i | (i,\alpha) \in \mathrm{KW}\},$$

$$\mathbf{O}(\mathcal{C}_1^{\mathrm{KW}}) = \{W^\alpha | (i,\alpha) \in \mathrm{KW}\},$$

$\mathcal{C}^{\mathrm{KW}}$ and $\mathcal{C}_1^{\mathrm{KW}}$ are braided equivalent.

Remark

The importance of KW-set was first noticed by Kac-Wakimoto in 1988 when they studied the coset vertex operator superalgebras associated to affine vertex operator superalgebras. KW-set plays an essential role for decomposing (i,α) into a direct sum of simple objects in $\mathcal{C}_2$.

5. Main results

$\mathcal{C}^{\mathrm{KW}}$-orbit

- Define an equivalence relation $\sim$ on $\mathbf{O}(\mathcal{C})$ as follows: $M \sim N$ iff there exists $(i, \alpha) \in \mathrm{KW}$ such that N is a submodule of $M \boxtimes M^i$.
- The equivalence of M is called a $\mathcal{C}^{\mathrm{KW}}$-orbit
- Orbit decomposition $\mathbf{O}(\mathcal{C}) = \cup_{\xi \in \Psi} \mathcal{O}_\xi$

Remark

If V is a vertex operator algebra, G is a finite automorphism group of V then G acts on the set of all twisted modules. If two twisted modules in the same orbit they are isomorphic as V^G-modules, the irreducible V^G-modules come from different orbits are inequivalent.

5. Main results

Example

- $A = 1_{\mathcal{C}_1} \otimes 1_{\mathcal{C}_2}$,
- $KW = \{(W \otimes 1_{\mathcal{C}_2}, W) | W \in \mathbf{O}(\mathcal{C}_1)\}$
- $\mathcal{C}^{KW} = \mathcal{C}_1 \otimes 1_{\mathcal{C}_2}$
- $\mathcal{C}^{KW}$-orbits $\mathbf{O}(\mathcal{C}_1) \otimes X$ for $X \in \mathbf{O}(\mathcal{C}_2)$

Theorem

1. If $M^s, M^t \in \mathcal{O}_\xi$ then the simple objects of $\mathcal{C}_2$ appearing in M^s and M^t are the same
2. If $M^s \in \mathcal{O}_\xi, M^t \in \mathcal{O}_\psi$ and ξ, ϕ are different, then simple objects of $\mathcal{C}_2$ appearing in M^s and M^t are inequivalent

Remark

This result is called the field identification in the literature by Fuchs-Schellekens-Schweigert

5. Main results

Kac-Wakimoto set and S-matrix

- $\dot{s}, \ddot{s}, s$ are nomalized s-matrices associated to $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}$.
- $\dot{S}, \ddot{S}, S$ are the corresponding linear on $K(\mathcal{C}_1)$, $K(\mathcal{C}_2)$, $K(\mathcal{C})$
- $\ddot{S}(i,\alpha) = \sum_{j\in I, \beta\in J_j} \overline{\dot{s}_{\alpha,\beta}} s_{i,j}(j,\beta)$ (This result was given for branching functions associated to affine Kac-Moody algebras by Kac-Perteson in 1981)
- Every simple object in $\mathcal{C}_2$ occurs in M^i for some $i \in I$ (This result was given in VOA setting by Krauel-Miyamoto in 2015)

5. Main results

For $i \in I$ and $\alpha \in J_i$, set $b(i,\alpha) = \sum_{(j,\beta)\in \mathrm{KW}} \overline{\dot{s}_{\alpha\beta}} s_{ij}$.

Theorem

for all $i \in I, \alpha \in J_i$,

$$\dim(i,\alpha) = \frac{b(i,\alpha)}{b(1,1)}$$

In particular, $b(i,\alpha) \neq 0$.

Remark

The $b(i,\alpha)$ was introduced by Kac-Peterson, Kac-Wakimoto to study the coset constructions for affine Kac-Moody algebras. Kac-Wakimoto conjectured that $b(i,\alpha) \neq 0$ for $\alpha \in J_i$. This theorem asserts that Kac-Wakimoto conjectrue in the coset setting associated with modular tensor categoies is always true. If the categories come from VOAs, $b(i,\alpha) > 0$.

5. Main results

Set $d_{(i,\alpha)} = \dim_{\mathcal{C}_2}(i,\alpha)$, $d_i = \dim_{\mathcal{C}} M^i$, $d_\alpha = \dim_{\mathcal{C}_1} W^\alpha$

Theorem

The following are equivalent:

1. $\mathbf{O}(\mathcal{C}^{\mathrm{KW}})$ forms a group
2. $d_{(i,\alpha)} = d_i d_\alpha$ for all $i \in I, \alpha \in J_i$
3. For any $\alpha \in J$, $W^{\alpha'} \boxtimes W^\alpha = \sum_{\gamma \in J_1} N_{\alpha,\alpha'}^{\gamma} W^\gamma$ where $W^{\alpha'}$ is the dual of W^α

Remark

Kac-Wakimoto also gave a Hypothesis that $s_{ij}\overline{\dot{s}_{\alpha\beta}} \geq 0$ for $(j,\beta) \in \mathrm{KW}$ and $\alpha \in J_i$ for coset constructions associated to affine Kac-Moody algebras. We prove this hypothesis in modular tensor category setting with the assumption in the previous theorem.

5. Main results

Theorem

Let $i \in I, \alpha \in J_i$.

1. $\dim \mathrm{Hom}_{\mathcal{C}_2}((i,\alpha),(i,\alpha)) \leq \sum_{(j,\beta)\in \mathrm{KW}} N^{\beta}_{\alpha',\alpha} N^{j}_{i',i}$ where $W^{\alpha'} = (W^{\alpha})'$ and $M^{j'} = (M^j)'$
2. $d_{(i,\alpha)} = d_i d_\alpha$ iff

$$\dim \mathrm{Hom}_{\mathcal{C}_2}((i,\alpha),(i,\alpha)) = \sum_{(j,\beta)\in \mathrm{KW}} N^{\beta}_{\alpha',\alpha} N^{j}_{i',i}$$

3. If $\mathrm{KW} = \{(1,1)\}$ then $O(\mathcal{C}_2) = \{(i,\alpha) | i \in I, \alpha \in \Lambda_i\}$. That is, all (i,α) are simple and inequivalent.

Remark

① gives an upper bound of $\dim \mathrm{Hom}_{\mathcal{C}_2}((i,\alpha),(i,\alpha))$. But it does not tell us the exact number of simple objects of $\mathcal{C}_2$ appearing in (i,α). An investigation is in progress

5. Main results

Assume that $G = \mathbf{O}(\mathcal{C}^{\text{KW}})$ forms a group. Set

$$H = \{W^\beta \in \mathbf{O}(\mathcal{C}_1^{\text{KW}}) | a_{\beta\otimes 1} \in \mathbf{O}(\mathcal{C}^{\text{KW}})\} \cong G$$

For $i \in I, \alpha \in J_i$ set

$$H^i = \{h \in H | a_{h\otimes 1} \boxtimes_A M^i = M^i\}$$

$$H^{(i,\alpha)} = \{h \in H^i | h \boxtimes_{\mathcal{C}_1} W^\alpha = W^\alpha, a_{h\otimes 1} \boxtimes_A M^i = M^i\}.$$

Proposition

1. If $a_{\beta\otimes 1} \in H^i$ then $\beta \in J_1$ and $a_{\beta\otimes 1} = a_{1\otimes(1,\beta')}$
2. H^i acts on the set $\{(W^\gamma | \gamma \in J_i\}$ by tensor product
3. $\{(1,\beta) | W^\beta \in H^i\}$ is a group isomorphic to H^i and acts on $\{(i,\gamma) | \gamma \in J_i\}$ by tensor product

5. Main results

Theorem

1. If M^i, M^j are in the same G-orbits then
$$\{(i,\gamma)|\gamma \in J_i\} = \{(j,\lambda)|\lambda \in J_j\}$$
2. $(i,\alpha) = \oplus_{s=1}^t nX_s$ where $X_s \in \mathbf{O}(\mathcal{C}_2)$ are inequivalent, $n^2t = o(H^{(i,\alpha)})$, and $\dim X_s = \frac{1}{nt}d_i d_\alpha$ for all s
3. If $H^{(i,\alpha)}$ is cyclic, then $(i,\alpha) = \oplus_{s=1}^t X_s$
4. $(i,\alpha) \cong (i,\gamma)$ if and only if W^γ and W^α are in the same H^i-orbit
5. $\mathrm{Hom}_{\mathcal{C}_2}((i,\alpha),(i,\beta)) = 0$ if and only if W^γ and W^α are not in the same H^i-orbit

5. Main results

Remark

- If G is a cyclic, then $H^{(i,\alpha)}$ is cyclic and ③ is true for any i, α
- If $H^{(i,\alpha)}$ is not cyclic, (i, α) is a projective representation of of $H^{(i,\alpha)}$. This is why we cannot prove the multiplicity n in ② is 1
- If G not a group, we believe that the previous Theorem holds with G replaced by the group of all simple currents in $\mathbf{O}(\mathcal{C}^{KW})$, but we do not know how to prove this in general

5. Main results

Remark

- Kac-Wakimoto (1988) studied the characters of (i, α) in coset constructions associated to affine Kac-Moody algebras
- Schellekens-Yankielowicz (1990) studied the characters of the diagonal coset constructions: field identification, fixed point resolution - how to write the characters of (i, α) as a sum of of characters of irreducible modules
- Fuchs-Schellekens-Schweigert (1995) used simple currents and out automorphisms of affine Kac-Moody algebras to study (i, α)
- Our work solved the fixed point resolution problem in general categorical coset constructions under the assumption that KW-set is a group. This assumption is true for many diagonal coset constructions

5.Main results

Theorem

Assume

- V: simple regular vertex operator algebra of CFT type
- U, U^c : simple regular vertex operator subalgebras of V of CFT type such that $(U^c)^c = U$
- the conformal weights of irreducible weights of U, U^c are positive except U, U^c

Then

- $\mathcal{C}_1 = \mathcal{C}_U, \mathcal{C}_2 = \mathcal{C}_{U^c}$ are modular tensor categories which satisfy the conditions given in the categorical coset setting with $A = V$
- All results for categorical coset constructions hold for VOA coset constructions

5. Main results

Parafermion-like coset constructions

- Let V, U be as before. But we do not assume that U^c is rational or C_2-cofinite
- $\mathbf{O}(\mathcal{C}_U)$ is a group

Theorem

1. U^c is rational and C_2-cofinite
2. $|J_i| = |J_1|$ for any $i \in I$
3. (i, α) is irreducible, and $(i, \alpha) \cong (i, \beta)$ iff there exits $(k, \gamma) \in \mathrm{KW}$ such that $W^\alpha = W^\beta \boxtimes W^\gamma$
4. $\mathrm{qdim}_{U^c}(i, \alpha) = \mathrm{qdim}_V M^i$
5. $|\mathrm{irr}(U^c)| = \frac{|\Lambda_1||\mathrm{irr}(V)|}{|\mathrm{KW}|}$, $|\mathrm{irr}(U)| = |\mathrm{KW}||\Lambda_1|$
6. $M^i \boxtimes M^j = \sum_{k\in I} N_{i,j}^k M^k$ and $W^\alpha \boxtimes W^\beta = W^\gamma$ for $i, j \in I$, $\alpha \in I_i, \beta \in I_j$, then $M^{(i,\alpha)} \boxtimes M^{(j,\beta)} = \sum_k N_{i,j}^k M^{(k,\gamma)}$

THANKS