Chiral de Rham complex and automorphic forms

Xuanzhong DAI

Kyoto University, RIMS

1st Jul 2023
Representation Theory XVIII, Dubrovnik

Motivation

- The cohomology of the chiral de Rham complex can be identified with the infinite-volume limit of the half-twisted sigma-model defined by E. Witten [Kapustin 2005].
- Speculation by Y. Manin and W. Eholzer that the Rankin-Cohen brackets are related to vertex operator algebras [Zagier 1994].
- To produce a vast generalization of the Rankin-Cohen brackets

Definition of Vertex Algebras

A vertex (super)algebra is the following data $(V, 1, T, Y(\ ,\ ))$:

1. the space of states——a vector space V(with $\mathbb{Z}_2$-gradation $V = V_{\bar{0}} + V_{\bar{1}}$),
2. the vacuum vector——a vector $1 \in V(V_{\bar{0}})$,
3. an (even) endomorphism $T : V \to V$ called the translation operator,
4. the state-field correspondence—— a (parity preserving) linear map

$$Y : V \longrightarrow \mathrm{End}(V)[[z, z^{-1}]],$$
$$a \longmapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}.$$

satisfying some axioms [K].
A conformal vertex algebra (or vertex operator algebra) is a pair (V, ω), where V is a vertex algebra, ω is a Virasora element such that L_0 is semisimple.

Heisenberg and Clifford Vertex Algebra

- Heisenberg algebra $H_N : a^i_n, b^i_n, i = 1, 2, \cdots, N,\ n \in \mathbb{Z}$ and center C

$$[a^i_m, b^j_n] = \delta_{ij}\delta_{m,-n}C$$

V_N is a vacuum representation of H_N, namely a polynomial algebra of variables $b^i_0, b^i_{-1}, \cdots, a^j_{-1}, a^j_{-2}, \cdots$.

$$Y(b^i_0, z) = b^i(z) = \sum_{n\in\mathbb{Z}} b^i_n z^{-n}, Y(a^j_{-1}, z) = a^j(z) = \sum_{n\in\mathbb{Z}} a^j_n z^{-n-1}.$$

- Lie superalgebra $Cl_N : \phi^i_n, \psi^i_n, i = 1, \cdots, N,\ n \in \mathbb{Z}$ and center C

$$\{\phi^i_m, \psi^j_n\} = \delta_{ij}\delta_{m,-n}C$$

Λ_N is an exterior algebra of variables $\phi^i_0, \phi^i_{-1}, \cdots, \psi^j_{-1}, \psi^j_{-2} \cdots$.

Topological Vertex Algebra Ω_N

- $\Omega_N := V_N \otimes \Lambda_N$
- The fields corresponding to the following four elements give a topological vertex algebra structure on Ω_N[MSV, 1999]

$$\omega = \sum (b^i_{-1} a^i + \phi^i_{-1}\psi^i), \qquad J = \sum \phi^i_0 \psi^i,$$
$$Q = \sum a^i_{-1}\phi^i, \qquad G = \sum \psi^i_{-1} b^i_{-1}.$$

- Fermionic charge operator $J_0 = \sum_i \sum_{n\in\mathbb{Z}} : \phi^i_n \psi^i_{-n} :$ counts the number of ϕ minus the number of ψ.

$$J_0 \phi^{i_1}_{-n_1} \cdots \phi^{i_k}_{-n_k} \psi^{j_1}_{-m_1} \cdots \psi^{j_l}_{-m_l} = (k-l)\phi^{i_1}_{-n_1} \cdots \phi^{i_k}_{-n_k} \psi^{j_1}_{-m_1} \cdots \psi^{j_l}_{-m_l}$$

- Chiral de Rham differential $d = Q_0 = \sum_{i,n} : a^i_n \phi^i_{-n} :$ increases the fermionic charge by 1 and $d^2 = 0$.

Extension of Ω_N

- Affine subset $U \subset X$ ($\dim X = N$) with a coordinate system $b = (b^1, \cdots, b^N) : U \longrightarrow \mathbb{C}^N$.
- $\mathcal{O}(U)$: the algebra of complex analytic functions on U.

$$\Omega^{ch}(U) := \Omega_N \otimes_{\mathbb{C}[b_0^1, \cdots, b_0^N]} \mathcal{O}(U).$$

- For $f \in \mathcal{O}(U)$,

$$Y(f(b), z) := \sum_I \frac{1}{I!} \partial^I f(b) \left(\sum_{n \neq 0} b_n^1 z^{-n} \right)^{i_1} \cdots \left(\sum_{n \neq 0} b_n^N z^{-n} \right)^{i_N}$$
$$(\text{``} = \text{''} f(b^1(z), \cdots, b^N(z)).$$

- $\Omega^{ch}(U)$: the topological vertex algebra generated by $a^i(z), b^i(z)$, $\phi^i(z)$, $\psi^i(z)$ and $f(z)$ for $f \in \mathcal{O}(U)$ with the nontrivial OPEs

$$a^i(z) b^j(w) \sim \frac{\delta_{ij}}{z - w}, \quad \phi^i(z) \psi^j(w) \sim \frac{\delta_{i,j}}{z - w}, \quad a^i(z) f(w) \sim \frac{\frac{\partial f}{\partial b^i}(z)}{z - w}.$$

Coordinate transformation

- Consider another coordinates $\tilde{b}^1, \cdots, \tilde{b}^N$ on U with relations

$$\tilde{b}^i = g^i(b^1, \cdots, b^N), \quad b^i = f^i(\tilde{b}^1, \cdots, \tilde{b}^N).$$

- Coordinate changes:

$$\tilde{a}^i = a^j_{-1} \frac{\partial f^j}{\partial \tilde{b}^i}(g(b)) + \phi^r_0 \psi^j_{-1} \frac{\partial^2 f^j}{\partial \tilde{b}^i \partial \tilde{b}^m}(g(b)) \frac{\partial g^m}{\partial b^r},$$

$$\tilde{b}^i_{-1} = b^j_{-1} \frac{\partial g^i}{\partial b^j}, \quad \tilde{\phi}^i = \phi^j_0 \frac{\partial g^i}{\partial b^j}, \quad \tilde{\psi}^i = \psi^j_{-1} \frac{\partial f^j}{\partial \tilde{b}^i}(g(b))$$

- $\tilde{a}^i, \tilde{b}^i, \tilde{\phi}^i, \tilde{\psi}^i$ satisfy the relations of elements without $\sim$.
- The field $L(z)$ is globally defined, i.e. Ω^{ch}_X is canonically a sheaf of conformal vertex algebras.

Recent progress

- When $X = P^n$, the global section is computed as a module over the affine Lie algebra $\widehat{\mathfrak{sl}}_{n+1}$. More explicitly, $\Gamma(\mathbb{CP}^N, \Omega^{ch}_{\mathbb{CP}^N})$ is the maximal $\mathfrak{sl}_{n+1}$-integrable submodule of the generalized Wakimoto module in the sense of Feigin and Frenkel [Malikov-Schectman, 1999].
- When X is a Kummer surface, the global section is isomorphic to an $N = 4$ algebra with central charge 6 [Song, 2016], and when X is a $K3$ surface, it is isomorphic to the simple $N = 4$ algebra with central charge 6 [Song, 2021].
- When X is a compact Ricci-flat Kähler manifold, the global section can be viewed as invariant elements of a $\beta\gamma - bc$ system under the action of certain Lie algebra of Cartan type [Song, 2021].

Chiral de Rham complex on $\mathbb{H}$

- X: the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} | \operatorname{im} \tau > 0\}$.
- $a = a^1, b = b^1, \phi = \phi^1, \psi = \psi^1$.
- The vertex algebra of global sections

$$\Omega^{ch}(\mathbb{H}) = \Omega_1 \otimes_{\mathbb{C}[b_0]} \mathcal{O}(\mathbb{H}),$$

  is generated by $a(z), \phi(z), \psi(z)$ and $f(z)$ for $f \in \mathcal{O}(\mathbb{H})$.
- $E := -a, \quad F := a_{-1} b^2 + 2\phi_0 \psi_{-1} b, \quad H := -2a_{-1} b - 2\phi_0 \psi.$

Theorem (Wakimoto, Feigin-Frenkel, Frenkel)

The coefficients $E_{(n)}, F_{(n)}, H_{(n)}$ of fields $Y(E,z), Y(F,z), Y(H,z)$ satisfy the relations of affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$ of level 0, where E, F, H correspond to matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively.

Notation

- Partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)$ with $\lambda_1 \geq \cdots \geq \lambda_d \geq 1$.
- $p(\lambda) := d$, $|\lambda| := \sum \lambda_i$, and for $X \in \{a, b, \psi\}$
  $X_{-\lambda} := X_{-\lambda_1} X_{-\lambda_2} \cdots X_{-\lambda_d}, \quad \phi_{-\lambda} := \phi_{-\lambda_1+1} \phi_{-\lambda_2+1} \cdots \phi_{-\lambda_d+1}$.
- λ, ν: partitions, μ, χ: partitions with distinct parts,
  $p(\lambda, \mu, \nu, \chi) := -p(\lambda) + p(\mu) - p(\nu) + p(\chi)$.
- $W_n := \mathrm{Span}\{a_{-\lambda}\phi_{-\mu}\psi_{-\nu}b_{-\chi}f(b) \in \Omega^{ch}(\mathbb{H}) | p(\lambda, \mu, \nu, \chi) \geq n\}$

Lemma

For any $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R})$, *and any holomorphic function* f *on* $\mathbb{H}$,

$$\pi(g) a_{-\lambda}\phi_{-\mu}\psi_{-\nu}b_{-\chi}f(b) = a_{-\lambda}\phi_{-\mu}\psi_{-\nu}b_{-\chi}(\gamma b + \delta)^{-2n} f(gb) \mod W_{n+1},$$

where $n = p(\lambda, \mu, \nu, \chi)$.

Short exact sequence

- $\Omega^{ch}(\mathbb{H},\Gamma)$: Γ-invariant vectors in $\Omega^{ch}(\mathbb{H})$ that are holomorphic at all the cusps.
- $W_n(k,l)_0^\Gamma$: conformal weight k and fermionic charge l vectors in $W_n \cap \Omega^{ch}(\mathbb{H},\Gamma)$.
- $I_{k,l}^n$: four-tuples (λ,μ,ν,χ) with conformal weight k and fermionic charge l, such that $p(\lambda,\mu,\nu,\chi) = n$.

Theorem (D, 2022)

We have a short exact sequence:

$$0 \longrightarrow W_{n+1}(k,l)_0^\Gamma \longrightarrow W_n(k,l)_0^\Gamma \xrightarrow{\alpha_n} M_{2n}(\Gamma)^{\oplus |I_{k,l}^n|} \longrightarrow 0.$$

Liftings of modular forms

Theorem (D, 2022)

Let $w = (\lambda, \mu, \nu, \chi)$ be a four-tuple with $p(w) = n$ and $f \in M_{2n}(\Gamma)$.

- *When $n > 0$, then $L(w, f)$ is defined as follows*

$$\sum_{m \geq 0} \frac{(2n-1)!}{m!(m+2n-1)!} D^m(a_{-\lambda}\phi_{-\mu}\psi_{-\nu}b_{-\chi}) f^{(m)}(b) \in \Omega^{ch}(\mathbb{H}, \Gamma).$$

- *When $n = 0$, 1 is a basis of $M_0(\Gamma)$, then $L(w, 1)$ is defined as follows*

$$a_{-\lambda}\phi_{-\mu}\psi_{-\nu}b_{-\chi} + \frac{\pi i}{6} \sum_{n=1}^{\infty} \frac{1}{n!(n-1)!} D^n(a_{-\lambda}\phi_{-\mu}\psi_{-\nu}b_{-\chi}) E_2^{(n-1)}(b)$$

- *The elements $L(w, f)$ when n runs through all non-negative integers, f runs through a basis of $M_{2n}(\Gamma)$, and w runs through all four-tuples with $p(w) = n$ form a linear basis of $\Omega^{ch}(\mathbb{H}, \Gamma)$.*

Topological Vertex Algebra Structure

- $\Omega^{ch}(\mathbb{H})$ has topological vertex algebra structure.
- $\omega = b_{-1}a + \phi_{-1}\psi, G = \psi_{-1}b_{-1} \in \Omega^{ch}(\mathbb{H}, \Gamma)$. But the elements $J = \phi_0\psi$ and $Q = a_{-1}\phi$ are not fixed by Γ.
- J (resp. Q) corresponds to the four-tuple w_J (resp. w_Q).

$$\tilde{J} := L(w_J, 1) = J + \frac{\pi i}{3} b_{-1}E_2(b),$$

$$\tilde{Q} := L(w_Q, 1) = Q - \frac{\pi i}{3}\phi_{-1}E_2(b) - \frac{\pi i}{3}\phi_0 b_{-1}E_2'(b).$$

- $L(z), \tilde{J}(z), \tilde{Q}(z), G(z)$ make $\Omega^{ch}(\mathbb{H}, \Gamma)$ a topological vertex algebra.

Character Formula $tr\ q^{L_0}$

Theorem (D, 2022)

The character formula of $\Omega^{ch}(\mathbb{H},\Gamma)$ *is given by*

$$\sum_{m,n=0}^{\infty}\sum_{u=0}^{n}\sum_{v=0}^{m+n}\dim M_{2m}(\Gamma)q^{m+2n+u(u-1)/2+v(v-3)/2}.$$

$$\prod_{i=1}^{u}\frac{1}{1-q^i}\prod_{j=1}^{v}\frac{1}{1-q^j}\prod_{k=1}^{n-u}\frac{1}{1-q^k}\prod_{l=1}^{m+n-v}\frac{1}{1-q^l}.$$

Theorem (D, 2021)

The character formula of $\mathcal{D}^{ch}(\mathbb{H},\Gamma)$ *is given by*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\dim M_{2m}(\Gamma)q^{2n+m}\prod_{i=1}^{n}\frac{1}{1-q^i}\prod_{j=1}^{m+n}\frac{1}{1-q^j}.$$

The Rankin-Cohen bracket

- Γ: a congruence subgroup of $SL(2,\mathbb{Z})$, $f \in M_k(\Gamma)$, $h \in M_l(\Gamma)$,

$$[f,h]_n := \frac{1}{(2\pi i)^n} \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r} f^{(r)}(\tau) h^{(s)}(\tau) \in M_{k+l+2n}(\Gamma)$$

- $[f,g]_n = (-1)^n [g,f]_n$.
- $[f,g]_0 = fg$, $[f,1]_n = [1,f]_n = 0$, for $n > 0$.
- $[[f,g]_1,h]_1 + [[g,h]_1,f]_1 + [[h,f]_1,g]_1 = 0$.
- $[\ ,\ ]_n$ is the (unique) universal bilinear map

$$M_k(\Gamma) \otimes M_l(\Gamma) \to M_{k+l+2n}(\Gamma)$$

Modularity of the Rankin-Cohen bracket

- The Cohen-Kuznetsov lifting of $f \in M_k(\Gamma)$

$$\tilde{f}(\tau, X) := \sum_{n=0}^{\infty} \frac{f^{(n)}(\tau)}{n!(n+k-1)!} X^n.$$

- $\tilde{f}$ satisfies the transformation law, for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$

$$\tilde{f}\left(g\tau, \frac{X}{(\gamma\tau+\delta)^2}\right) = (\gamma\tau+\delta)^k e^{\gamma X/(\gamma\tau+\delta)} \tilde{f}(\tau, X).$$

- For $f \in M_k(\Gamma)$ and $h \in M_l(\Gamma)$

$$\tilde{f}(\tau, -X)\tilde{h}(\tau, X) = \sum_{n=0}^{\infty} \frac{[f, h]_n(\tau)}{(n+k-1)!(n+l-1)!} (2\pi i X)^n,$$

Generalized Cohen-Kuznetsov lifting

- Define the generalized Cohen-Kuznetsov lifting of 1

$$\tilde{1}(\tau, X) = 1 + \frac{\pi i}{6} \sum_{n=1}^{\infty} \frac{E_2^{(n-1)}(\tau)}{n!(n-1)!} X^n.$$

- $\tilde{1}$ satisfies the transformation law

$$\tilde{1}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{X}{(\gamma\tau+\delta)^2}\right) = e^{\gamma X/(\gamma\tau+\delta)}\tilde{1}(\tau, X), \quad \text{for } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1).$$

- $\tilde{1}(\tau, -X)\tilde{f}(\tau, X)$
- $\tilde{1}(\tau - X)\tilde{1}(\tau, X)$

Modified Rankin-Cohen bracket

For $f \in M_k(\Gamma)$ with $k > 0$ and $n > 0$

$$[1, f]^{\sim}_n := \frac{1}{12(2\pi i)^{n-1}} \sum_{\substack{r+s=n \\ 1 \le r \le n}} (-1)^r \binom{n-1}{s} \binom{n+k-1}{r} E_2^{(r-1)}(\tau) f^{(s)}(\tau)$$

$$+ \frac{1}{n(2\pi i)^n} f^{(n)}(\tau) \in M_{k+2n}(\Gamma),$$

$$[1, 1]^{\sim}_n := \frac{1}{144(2\pi i)^{n-2}} \sum_{\substack{r+s=n \\ 1 \le r \le n-1}} (-1)^r \binom{n-1}{s} \binom{n-1}{r} E_2^{(r-1)}(\tau) E_2^{(s-1)}(\tau)$$

$$+ \frac{(-1)^n + 1}{12n(2\pi i)^{n-1}} E_2^{(n-1)}(\tau) \in M_{2n}(\Gamma(1)).$$

Example: $[1, f]^{\sim}_1 = \frac{1}{2\pi i}(-\frac{\pi i k}{6} E_2(\tau) f(\tau) + f'(\tau)) \in M_{k+2}(\Gamma)$,

$$[1, 1]^{\sim}_{2k+1} = 0, \ [1, 1]^{\sim}_2 = \frac{1}{(2\pi i)^2}(\frac{\pi^2}{36} E_2(\tau)^2 + \frac{\pi i}{6} E_2'(\tau)) \in M_4(\Gamma(1)).$$

Modified Rankin-Cohen bracket

The n-th modified Rankin-Cohen bracket of modular forms is the bilinear operation from $M_*(\Gamma) \otimes M_*(\Gamma)$ to $M_{*+*+2n}(\Gamma)$ defined by [D 2022, Nagatomo-Sakai-Zagier]

$$[f,h]^{\sim}_n := \begin{cases} [f,h]_n, & \text{if both } f \text{ and } h \text{ have positive weights,} \\ [1,h]^{\sim}_n, & \text{if } h \text{ has positive weight, and } f = 1, \\ (-1)^n [1,f]^{\sim}_n, & \text{if } f \text{ has positive weight, and } h = 1, \\ [1,1]^{\sim}_n, & \text{if } f = h = 1. \end{cases}$$

- $[f,g]^{\sim}_n = (-1)^n [g,f]^{\sim}_n$
- $[[f,g]^{\sim}_1, h]^{\sim}_1 + [[g,h]^{\sim}_1, f]^{\sim}_1 + [[h,f]^{\sim}_1, g]^{\sim}_1 = 0.$

Relations with the modified bracket

Theorem (D, 2022)

For any modular forms $f_1 \in M_{2k}(\Gamma), f_2 \in M_{2l}(\Gamma)$, *and any four-tuples* $w = (\lambda_1, \mu_1, \nu_1, \chi_1)$, $v = (\lambda_2, \mu_2, \nu_2, \chi_2)$ *of part* k *and* l *respectively, we have*

$$L(w, f_1)_{(n)} L(v, f_2) = \sum_{u=(\lambda,\mu,\nu,\chi)} c^n_{w,v,u} L(u, [f_1, f_2]^{\sim}_{p(u)-p(w)-p(v)}),$$

where $c^n_{w,v,u}$ *is a constant determined by* n *and the four-tuples* w, v *and* u. *Moreover,* $c^n_{w,v,u}$ *is nonzero only if* $p(u) \geq p(w) + p(v) = k + l$ *and* $|u| = |w| + |v| - n - 1$.

Ideals of $\Omega^{ch}(\mathbb{H},\Gamma)$

- Vertex algebra ideal of $\Omega^{ch}(\mathbb{H},\Gamma)$:

$$L_n := \mathrm{Span}_{\mathbb{C}}\{L(w,f) \in \Omega^{ch}(\mathbb{H},\Gamma) \mid p(w) = k, f \in M_{2k}(\Gamma) \text{ for } k \geq n\}.$$

- $\Omega'_1 = \oplus_{m\in\mathbb{Z}}\Omega'_1[m]$, where $\Omega'_1[m]$ is the m-th eigenspace of $-\frac{1}{2}H_{(0)}$. For arbitrary $n \in \mathbb{Z}$, $\Omega'_1[n]$ is a simple unitary module of $\Omega'_1[0]$.

$$\Omega'_1[0] = \mathrm{Span}_{\mathbb{C}}\{a_{-\lambda}\phi_{-\mu}\psi_{-\nu}b_{-\chi}1 \in \Omega'_1 \mid p(\lambda,\mu,\nu,\chi) = 0\} = KerH_{(0)}$$

  is a simple topological vertex algebra.

- The map

$$\begin{aligned} I : \Omega'_1[0] &\longrightarrow \Omega^{ch}(\mathbb{H},\Gamma)/L_1 \\ v &\longmapsto [L(v,1)] \end{aligned}$$

  is a vertex algebra isomorphism.

- $L_i/L_{i+1} \cong \Omega'_1[i] \otimes M_{2i}(\Gamma)$, as $\Omega'_1[0]$-modules.

Meromorphic sections

- $\mathcal{M}(\mathbb{H},\Gamma)$: Γ-invariant vectors in $\Omega^{ch}(\mathbb{H})$ that are meromorphic at all the cusps.

- $\Omega^{ch}(\mathbb{H},\Gamma) \subset \mathcal{M}(\mathbb{H},\Gamma) \subset \Omega^{ch}(\mathbb{H})$ are vertex operator algebras.

Theorem (D-Song, 2022)

The vertex operator algebra $\mathcal{M}(\mathbb{H},\Gamma)$ is simple.

Sketch of proof for $\Gamma(1)$:

- Multiplying powers of Δ to get modular form.
- Use $[1,\cdot]_1^{\sim}$ and products by E_4, E_6 and Δ to lower the weight.
- translate the operators to vertex algebra side.

Dubrovnik 2023.06.28

Thank you!