Quantum Groups and VOAs

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Dubrovnik, June 27, 2023

Motivation

Yesterday

- David: Weight categories of affine VOAs are large, not semisimple and there are tools to study its abelian structure.
- The embedding

$$L_k(\mathfrak{sl}_2) \hookrightarrow \mathrm{Vir}_k \otimes \Pi$$

  is a great aid.
- Jinwei: Improved technology to establish tensor category structure and rigidity..
- Simon: good weaker notions than rigidity.

The Question

Given a vertex tensor category of modules of a VOA, is there some simpler structure like a quasi Hopf algebra that has a braided tensor equivalent category?

Kazhdan-Lusztig

- Kazhdan-Lusztig, 1994, proved such a correspondence for $V^k(\mathfrak{g})$ for $\mathfrak{g}$ a simple Lie algebra and most k such that $k + h^\vee \notin \mathbb{Q}_{>0}$ (here $h^\vee$ is the dual Coxeter number).
- The category is $KL_k(\mathfrak{g})$, that is modules whose conformal weight spaces are integrable $\mathfrak{g}$ modules.
- The dual algebra is the quantum group of $\mathfrak{g}$ for $q = \exp\left(\frac{2\pi i}{2r^\vee(k+h^\vee)}\right)$ (here $r^\vee$ is the lacing number).
- The proof is the content of four articles in JAMS and is highly impressive.

Logarithmic Kazhdan-Lusztig correspondences

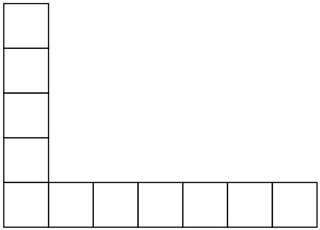
- Lisse VOAs are rare, otherwise categories of VOA modules usually satisfy:
- Uncountable number of isomorphism classes of simple objects
- Not semisimple.
- Often modules lack nice finiteness conditions as C_1-cofiniteness or lower-bounded conformal weight.
- Studying representation theory and in particular tensor category is highly challenging.
- Eventually a correspondence to a quantum group or a similar structure like a quasi Hopf algebra is desired.

Example: superalgebras

- Let $\mathfrak{g}$ be a simple basic, classical Lie algebra (or $\mathfrak{gl}_{n|n}$).
- Consider $KL_k(\mathfrak{g})$, that is modules whose conformal weight spaces are finite dimensional $\mathfrak{g}$ weight modules.
- Let $U_q(\mathfrak{g})$ the usual quantum supergroup of $\mathfrak{g}$ and $\mathcal{U}_q(\mathfrak{g})$ the category of weight modules of $U_q(\mathfrak{g})$.
- Then one expects that $KL_k(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$ are braided tensor equivalent for $q = \exp\left(\frac{2\pi i}{2r^\vee(k+h^\vee)}\right)$ and suitable $r^\vee$.
- Note that $\mathfrak{osp}_{1|2n}$ behaves in many respects like a simple Lie algebra and in particular one can and will prove the correspondence.

Example: Small Hook

- Let $\mathfrak{g} = \mathfrak{sl}_{n+m}$, then nilpotent elements are identified with partitions of $n+m$ or equivalently Young tableaux with $n+m$ boxes.
- A simplest type are small hooks:



- These small hook W-algebras $\mathcal{W}^k(n,m)$ enjoy triality with certain W-superalgebras (TC-Linshaw, 2022)

Example: Small Hook

Triality suggest that $\mathcal{W}^k(n,m)$ has a vertex tensor category of weight modules, call it $\mathcal{C}(n,m)$, that is braided tensor equivalent to a variant of

$$\mathcal{U}_q(\mathfrak{sl}_{n+m}) \boxtimes \mathcal{U}_{\widetilde{q}}(\mathfrak{sl}_{n+m|m}) \boxtimes$$
$$\mathcal{U}_{\widetilde{q}}(\mathfrak{sl}_{m|m-1}) \boxtimes \mathcal{U}_{\widetilde{q}}(\mathfrak{sl}_{m-1|m-2}) \boxtimes \dots \boxtimes \mathcal{U}_{\widetilde{q}}(\mathfrak{sl}_{2|1})$$

for

$$q = \exp\left(\frac{2\pi i}{2(k+n+m)}\right) \qquad \widetilde{q} = \exp\left(\frac{2\pi i(k+n+m)}{2}\right)$$

An algebraic theory of logarithmic Kazhdan-Lusztig correspondences

Joint work with Lentner-Rupert

This Talk

- A good theory is needed to approach these conjectures.
- The key input is that $\mathcal{W}$-algebras and affine vertex superalgebras allow for good free field realizations (Feigin-Frenkel, Genra, Wakimoto, Quella-Schomerus, Adamovic ...)
- By a free field realization I vaguely mean a connection of the algebra of interest to a much simpler structure. Today this will really be a free field VOA, but this is NOT necessary. (See the talks by Fehily and Ridout).
- The idea is that the complicated representation theory is completely specified by a nilpotent algebra inside the category of the free field algebra.

Realizations for $\mathfrak{sl}_2$

- The Cartan subalgebra of $\mathfrak{sl}_2$ is just $\mathbb{C}$ and one can view $\mathfrak{sl}_2$ as an algebra in the category of modules of the Cartan subalgebra.
- In this case the algebra would just be $\mathbb{C}[e]$ with e the usual nilpotent element of $\mathfrak{sl}_2$ (and this would be inside the universal envelopping algebra).
- The Lie algebra $\mathfrak{sl}_2$ acts on the Lie group SL_2 via left (or right) invariant vector fields and so it is a subalgebra of an algebra of differential operators.

Free field realization

- Let V be a VOA, that embeds conformally into a free field VOA A (e.g. Heisenberg and/or fermions).
- Let $\mathcal{U}$ be a braided tensor category of V-modules such that A is an object in $\mathcal{U}$.
- Let $\mathcal{C}$ be the braided tensor category of A-modules that lie in $\mathcal{U}$.
- $\mathcal{C}$ is just the category of vector spaces graded by some abelian group and characterized by a quadratic form. It is as easy as a tensor category can be.
- How much is $\mathcal{U}$ determined by $\mathcal{C}$ and A?

Algebras in Categories

Let $\mathcal{U}$ be a braided tensor category. An **algebra** in $\mathcal{U}$ is an object A in $\mathcal{U}$ together with a multiplication map

$$m : A \otimes A \to A$$

and a unit

$$u : \mathbf{1} \to A$$

such that the multiplication is associative and compatible with left and right multiplication, e.g. the diagram commute:

$$\begin{array}{ccc}
(A \otimes A) \otimes A & \xrightarrow{\alpha^{-1}_{A,A,A}} & A \otimes (A \otimes A) \\
\downarrow{\scriptstyle m \otimes \mathrm{Id}_A} & & \downarrow{\scriptstyle \mathrm{Id}_A \otimes m} \\
A \otimes A & & A \otimes A \\
& \searrow^{m} \quad {}^{m}\swarrow & \\
& A &
\end{array}$$

Commutative Algebras in Categories

The algebra A is called **commutative** if the diagram

$$\begin{array}{ccc} A \boxtimes A & \xrightarrow{c_{A,A}} & A \boxtimes A \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & A & \end{array}$$

commutes.

- There is a category $\mathcal{U}_A$ of A-modules in $\mathcal{U}$.
- $\mathcal{U}_A$ is a tensor category but not braided.
- $\mathcal{U}_A$ has a braided tensor subcategory of local modules.

Back to free field realization

- The free field algebra A translates to a commutative algebra in $\mathcal{U}$. (Huang-Kirillov-Lepowsky).
- The category $\mathcal{C}$ is precisely the tensor category of local A-modules. (TC-Kanade-McRae).
- The first aim is to obtain $\mathcal{U}$ from the knowledge of $\mathcal{U}_A$.

Back to free field realization

- Free field realizations of VOAs are usually characterized as being the joint kernel of certain screening charges.
- The Semikhatov idea is that screening charges can be identified with certain nilpotent algebras (called Nichols algebras) in the category of modules of the free field algebra.
- The aim is to relate a tensor category of Nichols algebra modules to $\mathcal{U}_A$.
- To a Nichols algebra one has an associated category of Yetter-Drinfeld modules and to the latter one can construct an associated quantum group (under some mild degree conditions otherwise one gets an associated quasi Hopf algebra).

Needed assumptions

Assume that $\mathcal{U}$ is a rigid, braided, locally finite abelian category with trivial Müger center, the property that for any object X, there exists a projective object P_X and an injective object I_X with a surjection $\pi_X : P_X \twoheadrightarrow X$ and an embedding $\iota_X : X \hookrightarrow I_X$, and that $A \in \mathcal{U}$ is a commutative and haploid algebra object, such that $\mathcal{U}_A$ is rigid.

Over the last years and jointly with Robert McRae and Jinwei Yang we have concentrated on proving that assumptions of this type hold in important examples.

The Key Theorem

Theorem (TC-Lenter-Rupert)

Suppose $\mathcal{U}$ is a braided tensor category and $A \in \mathcal{U}$ a commutative algebra fulfilling the assumptions. Let $\mathcal{C} = Vect_\Gamma^Q = Rep^{wt}(C)$ and let $X \in \mathcal{C}$ with finite-dimensional Nichols algebra $\mathfrak{N}$ satisfying some mild degree conditions and U_q the corresponding quantum group. Assume that

a) $\mathcal{U} \cong Rep^{wt}(U_q)$ *as abelian categories.*

b) $\mathcal{U}_A^0 \cong Vect_\Gamma^Q$ *as braided tensor categories.*

c) $\mathcal{U}_A \cong Rep(\mathfrak{N})(Vect_\Gamma^Q)$ *as abelian categories, compatible with a certain $\mathcal{C}$-module structure.*

Then in fact these are equivalences of tensor categories and braided tensor categories.

The moral of the key Theorem

The Theorem says that if you can prove

- all necessary assumptions
- suitable abelian equivalence
- a braided equivalence of the much much simpler category $\mathcal{C}$ to suitable graded vector spaces.

Then you already get a braided tensor equivalence.

In our later cases the assumptions can be verified thanks to previous work by Adamovic-Milas, Tsuchiya-Wood, McRae-Yang and others.

The logic of the proof

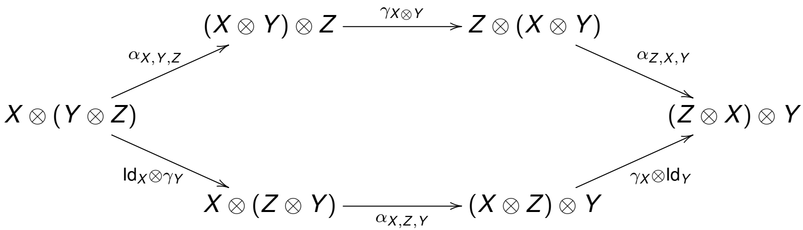
a) Show that $\mathcal{U}$ is a relative Drinfeld center using methods developped by Schauenburg

b) Laugwitz relates relative Drinfeld centers to categories of Yetter-Drinfeld modules

c) A realizing quasi-Hopf algebra is constructed from the category of Yetter-Drinfeld modules. If $\mathcal{C}$ satisfies a condition that we call sufficiently unrolled, then one gets a Hopf algebra, otherwise it is not strict.

Drinfeld Center

The center $\mathcal{Z}(\mathcal{U})$ of a tensor category $\mathcal{U}$ consists of objects (Z, γ) with Z an object in $\mathcal{U}$ and γ a natural family of isomorphisms $\{\gamma_X : X \otimes Z \to Z \otimes X \mid X \in \mathrm{Obj}(\mathcal{U})\}$, satisfying the hexagon diagram



The relative Drinfeld Center

Theorem (TC-Lenter-Rupert)

Under above assumptions and an additional relative finiteness condition

$$\mathcal{U} \cong \mathcal{Z}_{\mathcal{C}}(\mathcal{U}_A)$$

as braided tensor categories, where $\mathcal{Z}_{\mathcal{C}}(\mathcal{U}_A)$ *is the relative Drinfeld Center, that is the centralizer of* $\mathcal{C}^{rev}$ *in* $\mathcal{Z}(\mathcal{U}_A)$.

Examples

a) Let $\mathcal{O}^T_{\mathcal{M}(p)}$ the category of weight modules of the singlet algebra (TC-McRae-Yang) and $\mathrm{Rep}_{\mathrm{wt}} u_q^H(\mathfrak{sl}_2)$ the category of weight modules of the small unrolled quantum group of $\mathfrak{sl}_2$ at $2p$-th root of unity of (Constantino-Geer-Patureau Mirand). Then

$$\mathcal{O}^T_{\mathcal{M}(p)} \cong \mathrm{Rep}_{\mathrm{wt}}(u_q^H({}_2))$$

as braided tensor categories

b) The analogous result for triplet VOA and quasi Hopf modification of the small quantum group.

c) A Hopf algebra whose category of weight modules is braided equivalent to the vertex tensor category of certain principal $\mathcal{W}$-superalgebras of type $\mathfrak{sl}_{n|1}$ that were studied by (TC-McRae-Yang).

More Examples

a) Affine Feigin-Tipunin algebra of $\mathfrak{sl}_2$ at level minus one and $\mathrm{Rep}_{\mathrm{wt}} u^H_{-1}(\mathfrak{sl}_{2|1})$, see Shigenori's talk (TC-Nakatsuka-Sugimoto)

b) A Hopf algebra whose category of weight modules is braided equivalent to the vertex tensor category of certain subregular $\mathcal{W}$-superalgebras of type $\mathfrak{sl}_n$ that were studied by (TC-McRae-Yang). This is work in progress by (Allen-Lentner-Schweigert-Wood).

c) Quantum supergroups associated to the boundary VOAs of abelian three-dimensional $\mathcal{N} = 4$ gauge theories. This is work in progress by (TC-Dimofte-Niu).

Example: $\mathfrak{gl}_{1|1}$

History

- Computes Alexander-Conway polynomials of knots and links (Rozansky-Saleur, 1992).
- A very first example of a logarithmic CFT (Saleur-Schomerus 2005, . . .).
- Used to study disordered systems (LeClair, Ludwig, Saleur, . . .)
- Rigorously only understood now (TC-McRae-Yang 2020) using (TC-Ridout).

The Lie Algebra

Defining representation, even generators

$$N = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

odd generators

$$\psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

Relations

$$[N, \psi^\pm] = \pm\psi^\pm, \quad \{\psi^+, \psi^-\} = E.$$

Verma Modules

- Let $V_{n-\frac{1}{2},e}$ for $n, e \in \mathbb{C}$ be the Verma module generated by a highest-weight vector v such that

$$N \cdot v = nv, \quad E \cdot v = ev, \quad \psi^+ \cdot v = 0.$$

- Since ψ^- squares to zero every Verma module has dimension 2; thus n is the average of the two N-eigenvalues of $V_{n,e}$.
- The Verma module $V_{n,e}$ is irreducible if and only if $e \neq 0$. Atypical when $e = 0$, we denote the 1-dimensional irreducible quotient of $V_{n,e}$ by $A_{n+\frac{1}{2}}$.
- For each $n \in \mathbb{C}$, there is a non-split exact sequence

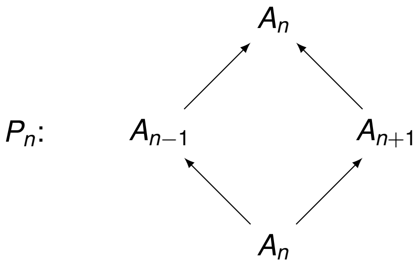
$$0 \to A_{n-\frac{1}{2}} \to V_{n,0} \to A_{n+\frac{1}{2}} \to 0.$$

Projective Modules

- For $n \in \mathbb{C}$, the module P_n has basis $v_n, \psi^{\pm} v_n, \psi^{+}\psi^{-} v_n$, where where $E \cdot v_n = 0$ and $N \cdot v_n = n v_n$.
- The module P_n is indecomposable but reducible and satisfies the non-split exact sequence

$$0 \to V_{n+\frac{1}{2},0} \to P_n \to V_{n-\frac{1}{2},0} \to 0.$$

- It has Loewy diagram



The affine Lie algebra

- $r, s \in \mathbb{Z}$

$$[N_r, E_s] = r\mathbf{k}\delta_{r+s,0}, \qquad [N_r, \psi_s^{\pm}] = \pm\psi_{r+s}^{\pm},$$

$$\{\psi_r^+, \psi_s^-\} = E_{r+s} + r\mathbf{k}\delta_{r+s,0},$$

- $\mathbf{k}$ central
- The zero-mode algebra $< E_0, N_0, \psi_0^{\pm} >$ is isomorphic to $\mathfrak{gl}_{1|1}$

Modules

- Induced module $\widehat{M}$: M a $\mathfrak{gl}_{1|1}$-module, then for $k \in \mathbb{C}$, let $\mathbf{k}$ act by multiplication of k and X_r by zero for $r \in \mathbb{Z}_{>0}$ and X_s freely for $s \in \mathbb{Z}_{<0}$.
- Modules have a similar structure, except that if $e \in k\mathbb{Z}$, then modules are of atypical type, i.e.

$$\widehat{P}^k_{n,m}: \quad \begin{array}{ccccc} & & \widehat{A}^k_{n,m} & & \\ & \nearrow & & \nwarrow & \\ \widehat{A}^k_{n+1,m} & & & & \widehat{A}^k_{n-1,m}. \\ & \nwarrow & & \nearrow & \\ & & \widehat{A}^k_{n,m} & & \end{array}$$

The VOA

- Generating fields $E(z), N(Z), \Psi^{\pm}(z)$.
- Operator products

$$N(z)E(w) \sim \frac{k}{(z-w)^2}, \qquad N(z)\psi^{\pm}(w) \sim \frac{\pm\psi^{\pm}(w)}{(z-w)}$$

$$\psi^{+}(z)\psi^{-}(w) \sim \frac{k}{(z-w)^2} + \frac{E(w)}{(z-w)}$$

The free field realization

- Free bosons $X(z), Y(z)$ and free fermions $b(z), c(z)$
- Operator products

$$X(z)Y(w) \sim \frac{1}{(z-w)^2}, \qquad b(z)c(w) \sim \frac{1}{(z-w)}$$

- The embedding

$$E = kY, \qquad N = X + cb + \frac{1}{2}Y \, ,$$

$$\psi^- = b, \qquad \psi^+ = -k\partial c + kcY \, ,$$

- It is characterized as the kernel of the zero-mode S_0 (screening charge) of the field $S = be^{\int Y}$.

Modules of the free field algebra

- The free fermions are holomorphic, i.e. the VOA is its only simple module
- The free boson simple modules are just Fock modules π_λ for $\lambda \in \mathbb{C}^2$ with fusion rules

$$\pi_\lambda \otimes \pi_\mu \cong \pi_{\lambda+\mu}$$

- The category of modules of the free field algebra is a ribbon category that is equivalent to $\mathcal{C} = \mathrm{Vect}^{Q}_{\mathbb{C}^2} \boxtimes \mathrm{sVect}$ for a certain non-degenerate quadratic form Q.
- This is as easy as a tensor category can be.

The Nichols algebra of screenings

- The screening charge S_0 is naturally associated with the highest-weight vector of a module $x = \pi_\alpha \otimes bc$.
- It satisfies $S_0^2 = 0$.
- It is identified with the algebra $\mathfrak{N} = \mathbb{C}[x]/x^2$, but viewed as an algebra in $\mathcal{C}$.
- $\mathfrak{N}$ is a Hopf algebra (a Nichols algebra) in $\mathcal{C}$ and there is an associated tensor category $\mathrm{Rep}(\mathfrak{N})(\mathcal{C})$.
- Projective modules in this category are of the form

$$0 \to \pi_{\lambda+\alpha} \otimes bc \to P_\lambda \to \pi_\lambda \otimes bc \to 0$$

The category $\mathrm{Rep}(\mathfrak{N})(\mathcal{C})$

- $\mathrm{Rep}(\mathfrak{N})(\mathcal{C})$ is not braided
- The Drinfeld center of a tensor category is always braided, so $\mathcal{Z}(\mathrm{Rep}(\mathfrak{N})(\mathcal{C}))$ is braided.
- $\mathcal{Z}(\mathrm{Rep}(\mathfrak{N})(\mathcal{C}))$ contains $\mathcal{C}^{\mathrm{rev}}$ as a subcategory and its centralizer $\mathcal{Z}_{\mathcal{C}}(\mathrm{Rep}(\mathfrak{N})(\mathcal{C}))$ is braided as well.
- Relative centers $\mathcal{Z}_{\mathcal{C}}(\mathrm{Rep}(\mathfrak{N})(\mathcal{C}))$ can be identified with categories of Yetter-Drinfeld modules ${}^{\mathfrak{N}}_{\mathfrak{N}}\mathcal{YD}(\mathcal{C})$ and the latter allow often for realizing quasi Hopf algebras
- In this case this is $u_q^H(\mathfrak{gl}_{1|1})$ for $q = e^{\pi i/k}$.

The Kazhdan-Lusztig equivalence

Theorem (TC-Lenter-Rupert)

The categories of weight modules of the affine VOA of $\mathfrak{gl}_{1|1}$ *at non-zero level* k *and of* $u_q^H(\mathfrak{gl}_{1|1})$ *are equivalent as braided tensor categories.*

Recall the moral of the key Theorem

The Theorem says that if you can prove

- all necessary assumptions
- suitable abelian equivalences
- a braided equivalence of the much much simpler category $\mathcal{C}$ to suitable graded vector spaces.

Then you already get a braided tensor equivalence.