Combinatorial properties of bases of standard modules for twisted affine Lie algebras

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Hrvatska zaklada za znanost

- Consider twisted affine Lie algebras of type $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$:

$A_{2l-1}^{(2)}$

$D_{l+1}^{(2)}$

- Aim is to construct combinatorial bases of standard modules $L(\Lambda)$ of *rectangular* highest weights, i.e of weights of the form

$$\Lambda = k_0\Lambda_0 + k_j\Lambda_j, \qquad k_0, k_j \in \mathbb{Z}_{\geq 0}, \quad k = k_0 + k_j > 0, \quad \langle \Lambda_j, c \rangle = 1.$$

- Main part is the construction of combinatorial bases of Feigin-Stoyanowski type subspaces of $L(\Lambda)$ using the theory of VOA.

M. B., S. Kožić, *Combinatorial bases of standard modules of twisted affine Lie algebras in types* $A_{2l-1}^{(2)}$ *and* $D_{l+1}^{(2)}$*: rectangular highest weights*, arXiv:2211.05171

Motivation

Theorem (G. Georgiev 1995, M. B., S. Kožić, M. Primc 2021)

Let $\widetilde{\mathfrak{g}}$ be untwisted affine Kac-Moody Lie algebra associated to the simple Lie algebra of type X_l. For any rectangular highest weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ character of parafermionic space is equal to

$$\operatorname{ch} L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{Q(k)} = \sum_{\mathcal{P}} D'_{\mathcal{P}}(q)\, G'_{\mathcal{P}}(q)\, B'_{\mathcal{P}}(q),$$

where the sum goes over all finite sequences $\mathcal{P} = (\mathcal{P}_l, \ldots, \mathcal{P}_1)$ of nonnegative integers such that $\mathcal{P}_i = (p_i^{(1)}, \ldots, p_i^{(k_{\alpha_i}-1)})$ and

$$D'_{\mathcal{P}}(q) = \frac{1}{\prod_{i=1}^{l}\prod_{r=1}^{k_{\alpha_i}-1} (q;q)_{p_i^{(r)}}}, \qquad \nu_i = k_{\alpha_i}/k = 2/\left\langle \alpha_i, \alpha_i \right\rangle,$$

$$G'_{\mathcal{P}}(q) = q^{\frac{1}{2k}\sum_{i,r=1}^{l}\sum_{m=1}^{k_{\alpha_i}-1}\sum_{n=1}^{k_{\alpha_r}-1}\langle\alpha_i,\alpha_r\rangle p_i^{(m)} p_r^{(n)}\left(\min\left\{k_{\alpha_r}m, k_{\alpha_i}n\right\}-mn\right)},$$

$$B'_{\mathcal{P}}(q) = q^{\sum_{t=\nu_j k_0-(\nu_j-1)k_j+1}^{k_{\alpha_j}-1}(t-\nu_j k_0+(\nu_j-1)k_j)p_j^{(t)}}\, q^{-\frac{k_j}{k_{\alpha_j}}\sum_{t=1}^{k_{\alpha_j}-1} t p_j^{(t)}}.$$

Motivation

- Parafermionic character formulas of Gepner, Kuniba, Nakanishi, Suzuki in type $B_2^{(1)}$ for highest weights $\Lambda_0+\Lambda_1$ and $\Lambda_0+\Lambda_2$:

$$\mathrm{ch}(L(\Lambda_0+\Lambda_1)^{\widehat{\mathfrak{h}}^+}_{Q(2)}) = \sum_{p_1^{(1)},p_2^{(1)},p_2^{(2)},p_2^{(3)}\geq 0} \frac{q^{\frac{1}{4}pMp^T}}{(q;q)_{p_1^{(1)}}\prod_{r=1}^{3}(q;q)_{p_2^{(r)}}},$$

$$\mathrm{ch}(L(\Lambda_0+\Lambda_2)^{\widehat{\mathfrak{h}}^+}_{Q(2)}) = \sum_{p_1^{(1)},p_2^{(1)},p_2^{(2)},p_2^{(3)}\geq 0} \frac{q^{\frac{1}{4}pMp^T-\frac{1}{4}(p_2^{(1)}+2p_2^{(2)}+3p_2^{(3)})}}{(q;q)_{p_1^{(1)}}\prod_{r=1}^{3}(q;q)_{p_2^{(r)}}},$$

where

$$M=\begin{bmatrix} 2 & -1 & -2 & -1 \\ -1 & 3 & 2 & 1 \\ -2 & 2 & 4 & 2 \\ -1 & 1 & 2 & 3 \end{bmatrix}$$

and $p=(p_1^{(1)},p_2^{(1)},p_2^{(2)},p_2^{(3)})$.

Parafermionic character formulas

Theorem (M. Okado, R. Takenaka 2022, M. B., S. Kožić, 2023)

For any rectangular highest weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ *of affine Lie algebras* $A_{2l-1}^{(2)}$ *and* $D_{l+1}^{(2)}$ *character of parafermionic space is equal to*

$$\textit{ch}L(\Lambda)_{kQ}^{\widetilde{\mathfrak{h}}[\nu]^+} = \sum_{\mathcal{P}} D_{\mathcal{P}}(q)\, G_{\mathcal{P}}(q)\, B_{\mathcal{P}}(q),$$

where the sum goes over all finite sequences $\mathcal{P} = (\mathcal{P}_l, \ldots, \mathcal{P}_1)$ *of* $l(k-1)$ *nonnegative integers such that* $\mathcal{P}_i = (p_i^{(1)}, \ldots, p_i^{(k-1)})$ *and*

$$D_{\mathcal{P}}(q) = \frac{1}{\prod_{i=1}^{l}\prod_{s=1}^{k-1}(q^{\mu_i}; q^{\mu_i})_{p_i^{(s)}}},$$

$$B_{\mathcal{P}}(q) = q^{\mu_j \sum_{t=k_0+1}^{k-1}(t-k_0)p_j^{(t)}}\, q^{-\mu_j \frac{k_j}{k}\sum_{t=1}^{k-1} t p_j^{(t)}},$$

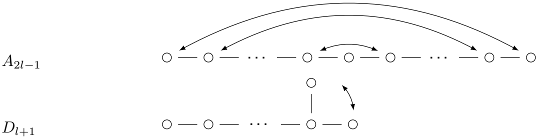
$$G_{\mathcal{P}}(q) = q^{\frac{1}{2}\sum_{i,r=1}^{l}\sum_{m,n=1}^{k-1}\langle \alpha_{i(0)}, \alpha_{r(0)}\rangle\left(\min\{m,n\} - \frac{mn}{k}\right)p_i^{(m)}p_r^{(n)}}.$$

Twisted affine Kac-Moody Lie algebra

$$\widetilde{\mathfrak{g}}[\nu] = \mathfrak{g}_{(0)} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathfrak{g}_{(1)} \otimes t^{1/2}\mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

associated to the triple $(\mathfrak{g}, \langle \cdot, \cdot \rangle, \nu)$, where:

- $\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$ is simple Lie algebra
- $\langle \cdot, \cdot \rangle$ nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$
- ν an automorphism of Dynkin diagram:



- with Lie bracket:

$$[x(j_1), y(j_2)] = [x, y]\,(j_1 + j_2) + \langle x, y \rangle\, j_1 \delta_{j_1+j_2,0} c, [c, x(j)] = 0,$$

$[d, x(j)] = jx(j)$, where $x(j) = x \otimes t^j$ for $x \in \mathfrak{g}$ and $j \in \frac{1}{2}\mathbb{Z}$.

Principal subspace of standard module

- $L(\Lambda)$ - standard $\widetilde{\mathfrak{g}}[\nu]$-module of the highest weight of the form

$$\Lambda = k_0\Lambda_0 + k_j\Lambda_j,$$

The principal subspace of $L(\Lambda)$ is defined as

$$W = U(\overline{\mathfrak{n}}[\nu])v,$$

where

- v is the highest weight vector of $L(\Lambda)$
- $\overline{\mathfrak{n}}[\nu] = \mathfrak{n}_{(0)} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathfrak{n}_{(1)} \otimes t^{1/2}\mathbb{C}[t^{\pm 1}], \qquad \mathfrak{n} = \oplus_{\alpha \in R_+} \mathbb{C}x_\alpha$

Twisted quasi-particles

- Twisted quasi-particles $x^{\nu}_{r\alpha_i}(m)$ of charge r, color i and energy $-m$ is a coefficient of twisted vertex operator

$$x^{\nu}_{r\alpha_i}(z) = (x^{\nu}_{\alpha_i}(z))^r = \sum_{m \in \frac{1}{2}\mathbb{Z}} x^{\nu}_{r\alpha_i}(m) z^{-m-r} \in (\mathrm{End} V)[[z^{\frac{1}{2}}, z^{-\frac{1}{2}}]],$$

where $x^{\nu}(z) = \sum_{n \in \frac{1}{2}\mathbb{Z}} x^{\nu}(n) z^{-n-1} \in (\mathrm{End} V)[[z^{\frac{1}{2}}, z^{-\frac{1}{2}}]]$.

Quasi-particle bases of principal subspaces

Theorem (M. B., C. Sadowski, 2019, M. B., S. Kožić, 2022)

For any rectangular highest weight Λ *the set*

$$\mathfrak{B}_W = \{b \cdot v \,:\, b \in B_W\}$$

forms a basis for the principal subspace W *of twisted affine Lie algebras of types* $A_{2l-1}^{(2)}$ *and* $D_{l+1}^{(2)}$.

- B_W denotes the set of monomials of the form

$$b(\alpha_l) \cdots b(\alpha_1),$$

where

$$b(\alpha_i) = x^{\nu}_{n_{r_i^{(1)},i}\alpha_i}(m_{r_i^{(1)},i}) \cdots x^{\nu}_{n_{2,i}\alpha_i}(m_{2,i}) x^{\nu}_{n_{1,i}\alpha_i}(m_{1,i})$$

- charges of twisted quasi-particles in color $i = 1, \ldots, l$ decrease from right to left,
- energies of quasi-particles satisfy certain conditions.

Quasi-particle bases of principal subspaces-example

- $V = L(\Lambda_0)$, $L(\Lambda_1)$ the standard $A_5^{(2)}$ - modules of level 1
- $x^{\nu}_{\alpha_3}(m_{3,3})x^{\nu}_{\alpha_3}(m_{2,3})x^{\nu}_{\alpha_3}(m_{1,3})x^{\nu}_{\alpha_2}(m_{3,2})x^{\nu}_{\alpha_2}(m_{2,2})x^{\nu}_{\alpha_2}(m_{1,2})$

$$x^{\nu}_{\alpha_1}(m_{3,1})x^{\nu}_{\alpha_1}(m_{2,1})x^{\nu}_{\alpha_1}(m_{1,1})v_V$$

- conditions on energies of color $i = 1$:
  - $W_{L(\Lambda_0)}$:

$$m_{1,1} \leq -1/2$$
$$m_{2,1} \leq -3/2$$
$$m_{3,2} \leq -5/2$$

  - $W_{L(\Lambda_1)}$:

$$m_{1,1} \leq -1$$
$$m_{2,1} \leq -2$$
$$m_{3,1} \leq -3$$

Conditions on energies for color $i = 1$

- conditions on energies for color $i = 1$ follow from the relation on level 1 standard module:

$$x^{\nu}_{2\alpha_1}(z)v_V = \sum_{m \in \frac{1}{2}\mathbb{Z}} \left(\sum_{\substack{m_1, m_2 \in \frac{1}{2}\mathbb{Z} \\ m_1 + m_2 = m}} x^{\nu}_{\alpha_1}(m_1) x^{\nu}_{\alpha_1}(m_2) \right) z^{-m-2} v_V = 0$$

$$\Rightarrow \quad m_2 \leq m_1 - 1$$

Conditions on energies for color $i = 1$

- $p_1^{(1)}$ the number of quasi-particles of color 1 and charge 1 in B_W
- contribution to the character:

$$\sum_{p_1^{(1)} \geq 0} \frac{q^{\frac{1}{2} {p_1^{(1)}}^2}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{p_1^{(1)}}} y^{p_1^{(1)}} \qquad \sum_{p_1^{(1)} \geq 0} \frac{q^{\frac{1}{2} {p_1^{(1)}}^2 - \frac{1}{2} p_1^{(1)}}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{p_1^{(1)}}} y^{p_1^{(1)}}$$

Quasi-particle bases-example

- $x^{\nu}_{\alpha_3}(m_{3,3})x^{\nu}_{\alpha_3}(m_{2,3})x^{\nu}_{\alpha_3}(m_{1,3})x^{\nu}_{\alpha_2}(m_{3,2})x^{\nu}_{\alpha_2}(m_{2,2})x^{\nu}_{\alpha_2}(m_{1,2})b(\alpha_1)v_V$

  - conditions on energies of color $i = 2$:

$$m_{1,2} \leq 1$$
$$m_{2,2} \leq 0$$
$$m_{3,2} \leq -1$$

  - conditions on energies of color $i = 3$:

$$m_{1,3} \leq 2$$
$$m_{2,3} \leq 0$$
$$m_{3,3} \leq -2$$

Conditions on energies for colors $i = 2$ and $i = 3$

- conditions on energies for color $i = 2$ and $i = 3$ follow from the relation on level 1 standard module:

$$(z_1^a - z_2^a) x_\alpha^\nu(z_1) x_\beta^\nu(z_2) = (z_1^a - z_2^a) x_\alpha(z_2) x_\beta(z_1), \ a = \frac{1}{2}\langle \alpha_{i(0)}, \alpha_{i(0)} \rangle$$

$$\Rightarrow \quad x_\alpha(m_2) x_\beta(m_1) \quad \Rightarrow \quad m_2 \leq m_1 + \frac{1}{2}\langle \alpha_{(0)}, \beta_{(0)} \rangle$$

Conditions on energies for colors $i = 2$ and $i = 3$

- character of principal subspace $W_{L(\Lambda_1)}$:

$$\sum_{p_1^{(1)},p_2^{(1)},p_3^{(1)}\geq 0} \frac{q^{\frac{1}{2}{p_1^{(1)}}^2-\frac{1}{2}p_1^{(1)}+\frac{1}{2}{p_2^{(1)}}^2+{p_3^{(1)}}^2-\frac{1}{2}p_1^{(1)}p_2^{(1)}-p_2^{(1)}p_3^{(1)}}}{(q^{\frac{1}{2}};q^{\frac{1}{2}})_{p_1^{(1)}}(q^{\frac{1}{2}};q^{\frac{1}{2}})_{p_2^{(1)}}(q;q)_{p_3^{(1)}}} y_1^{p_1^{(1)}} \prod_{i=2}^{3} y_i^{p_i^{(1)}}$$

$$= \sum_{p_1^{(1)},p_2^{(1)},p_3^{(1)}\geq 0} \frac{q^{\frac{1}{2}(pMp^T-p_1^{(1)})}}{(q^{\frac{1}{2}};q^{\frac{1}{2}})_{p_1^{(1)}}(q^{\frac{1}{2}};q^{\frac{1}{2}})_{p_2^{(1)}}(q;q)_{p_3^{(1)}}} \prod_{i=1}^{3} y_i^{p_i^{(1)}}$$

where

$$M = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

and $p = (p_1^{(1)}, p_2^{(1)}, p_3^{(1)})$.

Conditions on energies for colors $i = 2$ and $i = 3$

- character of principal subspace $W_{L(\Lambda_0+\Lambda_1)}$:

$$\sum_{\mathcal{P}} \frac{q^{\frac{1}{2}(\sum_{i=1}^{2}\sum_{t=1}^{2} p_i^{(t)\,2} - p_1^{(1)} + 2\sum_{t=1}^{2} p_3^{(t)\,2} - \sum_{t=1}^{2} p_1^{(t)} p_2^{(t)} - 2\sum_{t=1}^{2} p_2^{(t)} p_3^{(t)})}}{\prod_{i=1}^{2}\prod_{t=1}^{2}(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{p_i^{(t)}} \prod_{t=1}^{2}(q;q)_{p_3^{(t)}}} \prod_{i=1}^{3} y_i^{n_i}$$

Character of $W_{N(\Lambda)}$

- The principal subspace of $N(\Lambda) = N(k_0\Lambda_0 + k_j\Lambda_j)$ is defined as

$$W_{N(\Lambda)} = U(\overline{\mathfrak{n}}[\nu])v$$

Theorem (M. B., S. Kožić, 2023)

For the affine Lie algebras $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$ we have

$$\frac{1}{\prod_{\alpha\in R_+}(\alpha; q^{\langle\alpha_{(0)},\alpha_{(0)}\rangle/2})_\infty} =$$

$$= \sum_{\mathcal{P}} \frac{q^{\frac{1}{2}\sum_{i,r=1}^{l}\sum_{m,n=1}^{t}\langle\alpha_{i\,(0)},\alpha_{r\,(0)}\rangle \min\{m,n\} p_i^{(m)} p_r^{(n)}}}{\prod_{i=1}^{l}\prod_{s=1}^{t}(q^{\mu_i}; q^{\mu_i})_{p_i^{(s)}}} \prod_{i=1}^{l} y_i^{n_i},$$

where $n_i = \sum_{s=1}^{t} s p_i^{(s)}$, and the sum goes over all finite sequences $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l)$ of nonnegative integers with finite support.

Quasi-particle bases of standard modules

- The root lattice $Q_{(0)} = \sum_{i=0}^{l} \mathbb{Z}\alpha_{i(0)}$ acts on the standard module $L(\Lambda)$ via Weyl translations:

$$e_\alpha = e^{x_{-\alpha}(1)} e^{-x_\alpha(-1)} e^{x_{-\alpha}(1)} e^{x_\alpha(0)} e^{-x_{-\alpha}(0)} e^{x_\alpha(0)} \in \operatorname{End} L(\Lambda).$$

- $B_{U(\widehat{\mathfrak{h}}[\nu]^-)}$ … Poincaré–Birkhoff–Witt-type basis of the universal enveloping algebra of

$$\widehat{\mathfrak{h}}[\nu]^- = \mathfrak{h}_{(0)} \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{h}_{(1)} \otimes t^{1/2}\mathbb{C}[t^{-1}],$$

- $B'_{W_{L(\Lambda)}} = \{b \in B_{W_{L(\Lambda)}} : b$ does not contain q-p's of max. charge $k\}$.

Theorem (M. Okado, R. Takenaka 2022, M. B., S. Kožić, 2023)

For any rectangular highest weight Λ the set

$$\mathcal{B}_{L(\Lambda)} = \left\{ e_\alpha h b v \;\middle|\; \alpha \in Q_{(0)},\, h \in B_{U(\widehat{\mathfrak{h}}[\nu]^-)},\, b \in B'_{W_{L(\Lambda)}} \right\}$$

forms a basis of the standard module $L(\Lambda)$.

Characters of standard modules-example

For the affine Lie algebra $A_5^{(2)}$ we have:

$$\mathit{ch}\, L(\Lambda_1) = \frac{1}{\prod_{i=1}^{3}(q^{\mu_i};q^{\mu_i})_\infty} \sum_{\alpha\in Q} q^{\langle \alpha_{(0)}, \frac{1}{2}\alpha_{(0)}+\Lambda_1\rangle} \prod_{i=1}^{3} y_i^{h_{\lambda_i}(\alpha_{(0)}+\Lambda_1)},$$

$$\mathit{ch}\, L(\Lambda_0+\Lambda_1) = \frac{1}{\prod_{i=1}^{3}(q^{\mu_i};q^{\mu_i})_\infty}$$

$$\sum_{(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)} \frac{q^{\frac{1}{2}(p_1^{(1)\,2}-p_1^{(1)}p_2^{(1)}+p_2^{(1)\,2}-2p_2^{(1)}p_3^{(1)}+2p_3^{(1)\,2})}}{\prod_{i=1}^{3}(q^{\mu_i};q^{\mu_i})_{p_i^{(1)}}}$$

$$\times \sum_{\alpha\in Q} q^{\langle \alpha_{(0)},\alpha_{(0)}+\Lambda_0+\Lambda_1+\sum_{i=1}^{3} p_i^{(1)}\alpha_{i\,(0)}\rangle}$$

$$\prod_{i=1}^{3} y_i^{h_{\lambda_i}(2\alpha_{(0)}+\Lambda_0+\Lambda_1+\sum_{i=1}^{3} p_i^{(1)}\alpha_{i\,(0)})}.$$

Vacuum subspace of the standard module $L(\Lambda)$

- $L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^+}$ - vacuum space of the standard module $L(\Lambda)$

$$L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^+} = \left\{ v \in L(\Lambda) \, : \, \widehat{\mathfrak{h}}^+ \cdot v = 0 \right\},$$

  where

$$\widehat{\mathfrak{h}}[\nu]^\pm = \mathfrak{h}_{(0)} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}] \oplus \mathfrak{h}_{(1)} \otimes t^{\pm 1/2}\mathbb{C}[t^{\pm 1}].$$

- The direct sum decomposition of the standard module,

$$L(\Lambda) = L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^+} \oplus \widehat{\mathfrak{h}}[\nu]^- U(\widehat{\mathfrak{h}}[\nu]^-) \cdot L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^+},$$

  defines the projection

$$\pi^{\widehat{\mathfrak{h}}^+} : L(\Lambda) \to L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^+}.$$

Vacuum subspace basis

- $\mathcal{Z}$-operators:

$$\mathcal{Z}_{n\alpha}(z) = E(\alpha, z)^{n/k} x^{\nu}_{n\alpha}(z) E^{+}(\alpha, z)^{n/k} \in \mathrm{End} L(\Lambda)[[z^{\pm 1/2}]], \quad n \geq 1$$

Theorem (M. Okado, R. Takenaka 2022, M. B., S. Kožić, 2023)

For any rectangular highest weight Λ the vectors

$$e_{\alpha} \cdot \pi^{\widehat{\mathfrak{h}}^{+}}(b \cdot v), \quad \textit{where} \quad \alpha \in Q_{(0)},\, b \in B'_{W_{L(\Lambda)}},$$

form a basis for the vacuum space $L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^{+}}$.

Parafermionic space

The parafermionic space of highest weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ is defined as the quotient

$$L(\Lambda)_{kQ}^{\widehat{\mathfrak{h}}[\nu]^+} = L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^+} \Big/ \text{span}\left\{(\rho(k\alpha)-1)\cdot v \;\middle|\; \alpha \in Q,\, v \in L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^+}\right\},$$

- Denote by $\pi_{kQ}^{\widehat{\mathfrak{h}}[\nu]^+}$ the canonical projection

$$\pi_{kQ}^{\widehat{\mathfrak{h}}[\nu]^+} : L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^+} \to L(\Lambda)_{kQ}^{\widehat{\mathfrak{h}}[\nu]^+}.$$

- We have

$$L(\Lambda)_{kQ}^{\widehat{\mathfrak{h}}[\nu]^+} \cong \coprod_{\mu \in \Lambda + Q/kQ} L(\Lambda)_{\mu}^{\widehat{\mathfrak{h}}[\nu]^+}.$$

Parafermionic space basis

- parafermionic currents of charge $n \geq 1$:

$$\Psi_{n\alpha}(z) = \mathcal{Z}_{n\alpha}(z) z^{n\alpha_{(0)}/k} \epsilon^{n/k} \in z^{n/k} \mathrm{End} L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^+}, \quad n \geq 1$$

Theorem (M. Okado, R. Takenaka 2022, M. B., S. Kožić, 2023)

For any rectangular highest weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ *the vectors*

$$\pi^{\widehat{\mathfrak{h}}^+}_{Q(k)} \left(\pi^{\widehat{\mathfrak{h}}^+} (b \cdot v_\Lambda) \right), \quad \textit{where} \quad b \in B'_{W_{L(\Lambda)}},$$

form a basis for the parafermionic space $L(\Lambda)^{\widehat{\mathfrak{h}}[\nu]^+}_{kQ}$ $A^{(2)}_{2l-1}$ *and* $D^{(2)}_{l+1}$.

Parafermionic character formulas

For the affine Lie algebra $A_5^{(2)}$ *we have*

$$\mathit{ch}\, L(2\Lambda_1)_{2Q}^{\widehat{\mathfrak{h}}[\nu]^+} = \sum_{\mathcal{P}=(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)} \frac{q^{\frac{1}{4}(p_1^{(1)\,2}-p_1^{(1)}p_2^{(1)}+p_2^{(1)\,2}-2p_2^{(1)}p_3^{(1)}+2p_3^{(1)\,2})}}{\prod_{i=1}^{3}(q^{\mu_i};q^{\mu_i})_{p_i^{(1)}}},$$

$$\mathit{ch}\, L(\Lambda_0+\Lambda_1)_{2Q}^{\widehat{\mathfrak{h}}[\nu]^+} = \sum_{\mathcal{P}=(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)} \frac{q^{\frac{1}{4}(p_1^{(1)\,2}-p_1^{(1)}p_2^{(1)}+p_2^{(1)\,2}-2p_2^{(1)}p_3^{(1)}+2p_3^{(1)\,2})-\frac{1}{4}p_1^{(1)}}}{\prod_{i=1}^{3}(q^{\mu_i};q^{\mu_i})_{p_i^{(1)}}}.$$

Conclusion

- Construction of combinatorial bases of the Feigin-Stoyanovsky's type subspaces introduced by M. Primc; this is work in progress (M.B., S. Kožić, M. Primc)

- Fix a minuscule weight ω of simple Lie algebra $\mathfrak{g}$ of type A_{2l-1} and D_{l+1}, such that $\nu(\omega) = \omega$.

- Define a commutative subalgebra $\widetilde{\mathfrak{g}}_1[\nu]$ of $\widetilde{\mathfrak{g}}[\nu]$ by:

$$\widetilde{\mathfrak{g}}_1[\nu] = \text{span}\left\{ x_\gamma^\nu(m) : \gamma \in \Gamma, n \in \frac{\langle \gamma, \gamma \rangle}{2}\mathbb{Z} \right\},$$

where

$$\Gamma = \left\{ \frac{1}{2}(\alpha + \nu\alpha) : \alpha \in R, \langle \alpha, \omega \rangle = 1 \right\}.$$

Conclusion

Feigin-Stoyanovsky's type subspaces W_{FS} *of standard module* $L(\Lambda)$ *is defined as*

$$W_{FS} = U(\widetilde{\mathfrak{g}}_1[\nu])v.$$

- $L(\Lambda) = \bigcup_{m=0}^{\infty} e^{-m} U(\widetilde{\mathfrak{g}}_1[\nu])v$
- In the case of twisted affine Lie algebra $D_{l+1}^{(2)}$, Feigin-Stoyanovsky's type subspace W_{FS} of level one standard module $L(\Lambda_0)$ is spanned by:

$$\Big\{x(\pi)v : x(\pi) = \cdots x^{\nu}_{\gamma_2}(-2)^{a_{l+2}} x^{\nu}_{\gamma_1}(-1)^{a_{l+1}}$$
$$x^{\nu}_{\gamma_{l+1}}(-1)^{a_l} \cdots x^{\nu}_{\gamma_2}(-1)^{a_1} x^{\nu}_{\gamma_1}(-1/2)^{a_0}\Big\},$$

where

$$\gamma_i = \alpha_1 + \cdots + \alpha_{i+1}, \quad i = 1, \ldots, l,$$
$$\gamma_{l+1} = \alpha_1.$$

Conclusion

- $x(\pi)v$ is called admissible for Λ_0 if it satisfies *initial condition*

$$a_0 \leq 1,$$

and *difference conditions*

$$a_j + a_{j+1} + \cdots + a_{l+j+1} \leq 1, \quad j \geq 0.$$

- $x(\pi)v$ is called admissible for Λ_l if it satisfies *initial conditions*

$$a_{l+1} \leq 1, a_j + a_{j+1} \leq 1$$

and *difference conditions*

$$a_j + a_{j+1} + \cdots + a_{l+j+1} \leq 1, \quad j \geq 0.$$

Theorem (M. B., S. Kožić, M. Primc)

Basis for W_{FS} is given by admissible vectors for Λ_0 and Λ_l.

Thank you