On the finite irreducible modules over some conformal superalgebras

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Definition (Conformal superalgebra)

A **conformal superalgebra** R is a left $\mathbb{Z}_2-$graded $\mathbb{C}[\partial]-$module endowed with a $\mathbb{C}-$linear map, called $\lambda-$bracket, $R \otimes R \to \mathbb{C}[\lambda] \otimes R$, $a \otimes b \mapsto [a_\lambda b]$, that satisfies the following properties for all $a, b, c \in R$:

1. $[\partial a_\lambda b] = -\lambda [a_\lambda b], \quad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b]$;
2. $[a_\lambda b] = -(-1)^{p(a)p(b)}[b_{-\lambda-\partial} a]$;
3. $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)}[b_\mu [a_\lambda c]]$;

where $p(a)$ denotes the parity of the element $a \in R$ and $p(\partial a) = p(a)$ for all $a \in R$.

A conformal superalgebra can be also defined using the so called $n-$products $(a_{(n)} b)$, where:

$$[a_\lambda b] = \sum_{n \geq 0} \frac{\lambda^n}{n!}(a_{(n)} b).$$

We can define an **ideal** of R. We can define **simple**, **finite** conformal superalgebra, and the **derived** subalgebra.

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Definition

A module M over a conformal superalgebra R is a left $\mathbb{Z}_2-$graded $\mathbb{C}[\partial]-$module endowed with the $\mathbb{C}-$linear map $R \otimes M \to \mathbb{C}[\lambda] \otimes M$, $a \otimes v \mapsto a_\lambda v$ that satisfies the following properties for all $a, b \in R$, $v \in M$:

1. $(\partial a)_\lambda v = [\partial, a_\lambda]v = -\lambda a_\lambda v;$
2. $[a_\lambda, b_\mu]v = [a_\lambda b]_{\lambda+\mu} v.$

A module M is called **finite** if it is a finitely generated $\mathbb{C}[\partial]-$module.

Theorem (Fattori, Kac 2002)

Any finite simple conformal superalgebra R is isomorphic to one of the conformal superalgebras of the following list: Cur $\mathfrak{g}$, *where* $\mathfrak{g}$ *is a simple finite−dimensional Lie superalgebra,* $W_n (n \geq 0)$, $S_{n,b}$, $\tilde{S}_n$ $(n \geq 2,\ b \in \mathbb{C})$, $K_n (n \geq 0,\ n \neq 4)$, K'_4, CK_6.

Annihilation superalgebra

We recall the construction of the annihilation superalgebra associated with R. Let $\widetilde{R} = R[y, y^{-1}]$, $p(y) = 0$, $\widetilde{\partial} = \partial + \partial_y$. We define for all $a, b \in R$, $f, g \in \mathbb{C}[y, y^{-1}]$, $n \geq 0$:

$$(af_{(n)}bg) = \sum_{j \in \mathbb{Z}_+} (a_{(n+j)}b)\Big(\frac{\partial_y^j}{j!} f\Big) g.$$

We observe that $\widetilde{\partial}\widetilde{R}$ is a two sided ideal of $\widetilde{R}$ with respect to the $0-$product. $\mathrm{Lie}\, R := \widetilde{R}/\widetilde{\partial}\widetilde{R}$ is a Lie superalgebra with the bracket induced by the $0-$product.

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The conformal superalgebra of type K

Let $\bigwedge(N)$ be the Grassmann superalgebra in the N odd indeterminates $\xi_1, \dots, \xi_N$. Let t be an even indeterminate and $\bigwedge(1,N) = \mathbb{C}[t,t^{-1}] \otimes \bigwedge(N)$.

$$W(1,N) = \left\{ D = a\partial_t + \sum_{i=1}^{N} a_i \partial_i \mid a, a_i \in \bigwedge(1,N) \right\}.$$

Let $\omega = dt - \sum_{i=1}^{N} \xi_i d\xi_i$.

$$K(1,N) = \{ D \in W(1,N) \mid D\omega = f_D \omega \text{ for some } f_D \in \bigwedge(1,N) \}.$$

We can define $\bigwedge(1,N)_+ = \mathbb{C}[t] \otimes \bigwedge(N)$, $W(1,N)_+$ and $K(1,N)_+$.

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$\bigwedge(1,N)$ has a Lie superalgebra structure as follows: for all $f, g \in \bigwedge(1,N)$

$$[f,g] = \Big(2f - \sum_{i=1}^{N} \xi_i \partial_i f\Big)(\partial_t g) - (\partial_t f)\Big(2g - \sum_{i=1}^{N} \xi_i \partial_i g\Big) + (-1)^{p(f)}\Big(\sum_{i=1}^{N} \partial_i f \partial_i g\Big).$$

$K(1,N) \cong \bigwedge(1,N)$ as Lie superalgebras via:

$$\wedge(1,N) \longrightarrow K(1,N)$$
$$f \longmapsto 2f\partial_t + (-1)^{p(f)} \sum_{i=1}^{N} (\xi_i \partial_t f + \partial_i f)(\xi_i \partial_t + \partial_i).$$

We consider on $K(1,N)$ the standard grading, i.e. $\deg(t^m \xi_{i_1} \cdots \xi_{i_s}) = 2m + s - 2$.

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The conformal superalgebra of type K is defined as

$$K_N := \mathbb{C}[\partial] \otimes \bigwedge(N).$$

For $f = \xi_{i_1} \cdots \xi_{i_r}$ and $g = \xi_{j_1} \cdots \xi_{j_s}$:

$$[f_\lambda g] = \Big((r-2)\partial(fg) + (-1)^r \sum_{i=1}^{N} (\partial_i f)(\partial_i g)\Big) + \lambda(r+s-4)fg.$$

We recall that

$$\mathfrak{g} = \mathcal{A}(K_N) = K(1,N)_+.$$

$\mathfrak{g}$ has depth 2 with respect to the standard grading. $\mathfrak{g}_{-2}$ is one$-$dimensional, we call Θ the generator $-1/2$ of $\mathfrak{g}_{-2}$.

Proposition (Boyallian, Kac, Liberati 2010)

Let $\mathfrak{g}$ be the annihilation superalgebra of a conformal superalgebra R of type K. Finite modules over R correspond to modules V over $\mathfrak{g}$, called **finite conformal***, that satisfy the following properties:*

1. *For every $v \in V$, there exists $j_0 \in \mathbb{Z}$, $j_0 \geq -d$, such that $\mathfrak{g}_j.v = 0$ when $j \geq j_0$;*
2. *V is finitely generated as a $\mathbb{C}[\Theta]-$module.*

Let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be a $\mathbb{Z}-$graded Lie superalgebra. We will use the notation $\mathfrak{g}_+ = \oplus_{i>0}\mathfrak{g}_i$, $\mathfrak{g}_- = \oplus_{i<0}\mathfrak{g}_i$ and $\mathfrak{g}_{\geq 0} = \oplus_{i\geq 0}\mathfrak{g}_i$.

Remark

Let F be a $\mathfrak{g}_{\geq 0}-$module. We denote by $M(F)$ the **generalized Verma module**.
If F is a finite$-$dimensional irreducible $\mathfrak{g}_{\geq 0}-$module, we call $M(F)$ a **finite Verma module**.
If $M(F)$ is not irreducible, we call $M(F)$ **degenerate**.
We have a $\mathbb{Z}_{\geq 0}-$grading on $U(\mathfrak{g}_-)$ and $M(F)$.

Definition

Given a $\mathfrak{g}-$module V, we call **singular vectors** the elements of:

$$\mathrm{Sing}(V) = \{v \in V \mid \mathfrak{g}_+.v = 0\}.$$

If $V = M(F)$, we will call **trivial singular vectors** the singular vectors of degree 0 and **nontrivial singular vectors** the singular vectors of positive degree.

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Theorem (Kac, Rudakov 2002; Cheng, Lam 2001)

Let $\mathfrak{g}$ be the annihilation superalgebra associated with a conformal superalgebra of type K, then

1. *if F is an irreducible finite−dimensional $\mathfrak{g}_{\geq 0}$−module, then $\mathfrak{g}_+$ acts trivially on it and $M(F)$ has a unique maximal submodule;*
2. *the map $F \mapsto \mathrm{I}(F)$, where $\mathrm{I}(F)$ is the quotient of $M(F)$ by the unique maximal submodule, is a bijective map between irreducible finite−dimensional $\mathfrak{g}_0$−modules and irreducible finite conformal $\mathfrak{g}$−modules;*
3. *the $\mathfrak{g}$−module $M(F)$ is irreducible if and only if the $\mathfrak{g}_0$−module F is irreducible and $M(F)$ has no nontrivial singular vectors.*

The conformal superalgebra K_4'

The conformal superalgebra K_N is simple if $N \neq 4$. If $N = 4$:

$$K_4 = K_4' \oplus \mathbb{C}\xi_1\xi_2\xi_3\xi_4$$

Proposition

$\mathcal{A}(K_4')$ is a central extension of $K(1,4)_+$ by a one–dimensional center $\mathbb{C}C$:

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Let $\mathfrak{g} := \mathcal{A}(K_4')$.

$$\begin{aligned}
&\mathfrak{g}_{-2} = \langle 1 \rangle \,, \\
&\mathfrak{g}_{-1} = \langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle \,, \\
&\mathfrak{g}_0 = \langle \{ C, t, \xi_i \xi_j \ 1 \leq i < j \leq 4 \} \rangle \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathbb{C}t \oplus \mathbb{C}C \\
&\cong \langle e_x, f_x, h_x \rangle \oplus \langle e_y, f_y, h_y \rangle \oplus \mathbb{C}t \oplus \mathbb{C}C.
\end{aligned}$$

Remark

$\mathfrak{g}$ has finite depth 2.
The element t is a grading element, i.e. $[t, a] = \deg(a)a$ for all $a \in \mathfrak{g}$.

Similarly to the case of K_N, we study the singular vectors in order to classify finite irreducible modules over K_4'.
We will denote the weights of weight vectors of $\mathfrak{g}_0-$modules as $\mu = (m, n, \mu_t, \mu_c)$ with respect to the action of h_x, h_y, t, C.

Theorem (B., Caselli 2022)

There are no singular vectors of degree greater than 3.

There are four families of highest weight singular vectors of degree 1.

There are four families of highest weight singular vectors of degree 2.

There are exactly two highest weight singular vectors of degree 3.

Let F be an irreducible finite−dimensional $\mathfrak{g}_0$−module with highest weight μ. We call $M(\mu)$ the finite Verma module $M(F)$.

Theorem (B., Caselli 2022)

The finite Verma module $M(m, n, \mu_t, \mu_c)$ is degenerate if and only if (m, n, μ_t, μ_c) is one of the following:

A) $(m, n, -\frac{m+n}{2}, \frac{m-n}{2})$ *with* $m, n \in \mathbb{Z}_{\geq 0}$,

B) $(m, n, 1 + \frac{m-n}{2}, -1 - \frac{m+n}{2})$ *with* $m, n \in \mathbb{Z}_{\geq 0}$,

C) $(m, n, 2 + \frac{m+n}{2}, \frac{n-m}{2})$ *with* $m, n \in \mathbb{Z}_{\geq 0}$, $(m, n) \neq (0, 0)$,

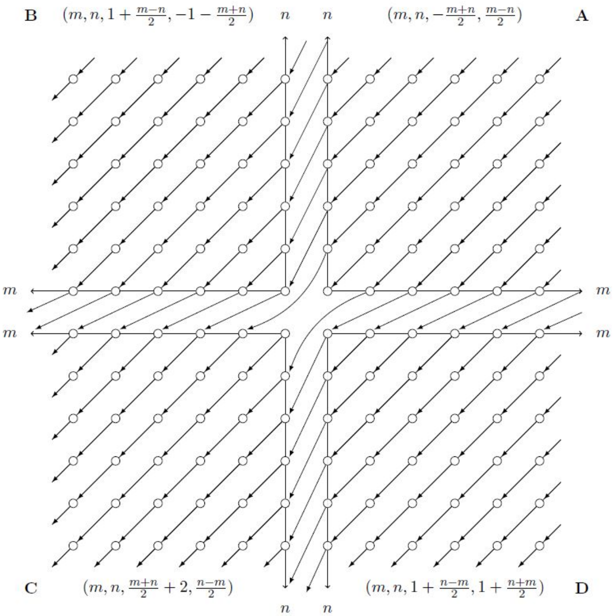
D) $(m, n, 1 + \frac{n-m}{2}, 1 + \frac{m+n}{2})$ *with* $m, n \in \mathbb{Z}_{\geq 0}$.

Remark

Between $M(\mu)$ and $M(\tilde{\mu})$ there exists a morphism of $\mathfrak{g}-$modules if and only if there exists a non trivial singular vector $\vec{m}$ in $M(\tilde{\mu})$ of highest weight μ.

$$\nabla : M(\mu) \longrightarrow M(\tilde{\mu})$$
$$v_\mu \longmapsto \vec{m}$$

If $\vec{m}$ is a singular vector of degree d, we say that ∇ is a morphism of degree d.



B $(m, n, 1 + \frac{m-n}{2}, -1 - \frac{m+n}{2})$
n
n
$(m, n, -\frac{m+n}{2}, \frac{m-n}{2})$ A
m
m
m
m
C $(m, n, \frac{m+n}{2} + 2, \frac{n-m}{2})$
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n
n

Definition

The conformal dual M^* of M is defined by:

$$M^* = \{f_\lambda : M \to \mathbb{C}[\lambda] \mid f_\lambda(\partial m) = \lambda f_\lambda(m), \ \forall m \in M\}.$$

The structure of $\mathbb{C}[\partial]-$module is given by $(\partial f)_\lambda(m) = -\lambda f_\lambda(m)$. The $\lambda-$action of R is given by:

$$(a_\lambda f)_\mu(m) = -(-1)^{p(a)p(f)} f_{\mu-\lambda}(a_\lambda m).$$

Definition

Let $T : M \to N$ be a morphism of $R-$modules. The dual morphism $T^* : N^* \to M^*$ is defined by $[T^*(f)]_\lambda (m) = -f_\lambda (T(m))$.

Remark

A consequence of the main result on conformal duality, showed by Cantarini, Caselli and Kac, is that, for K_4', the conformal dual of $M(F(m, n, \mu_t, \mu_c))$ corresponds to the shifted dual $M(F^\vee)$, where $F^\vee \cong F(m, n, -\mu_t + 2, -\mu_c)$.

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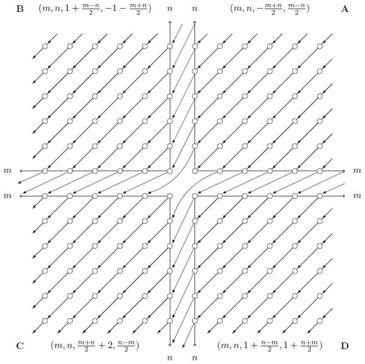
Homology

Theorem (B. 2022)

The sequences in the Figure below are complexes and they are exact in each module except for $M(0,0,0,0)$ *and* $M(1,1,3,0)$*. The homology spaces in* $M(0,0,0,0)$ *and* $M(1,1,3,0)$ *are isomorphic to the trivial representation.*



The conformal superalgebra CK_6

For $\xi_I \in \Lambda(6)$ we define ξ_I^* to be such that $\xi_I \xi_I^* = \xi_1\xi_2\xi_3\xi_4\xi_5\xi_6$.

$$CK_6 = \mathbb{C}[\partial] - \text{span}\left\{ f - i(-1)^{\frac{|f|(|f|+1)}{2}}(-\partial)^{3-|f|} f^*,\ f \in \Lambda(6), 0 \leq |f| \leq 3 \right\}.$$

Remark

We recall that

$$\mathfrak{g} := \mathcal{A}(CK_6) \cong E(1,6).$$

The homogeneous components of non−positive degree of $\mathfrak{g}$ and $K(1,6)_+$ coincide and are:

$$\mathfrak{g}_{-2} = \langle 1 \rangle,$$
$$\mathfrak{g}_{-1} = \langle \xi_1, \xi_2, ..., \xi_6 \rangle,$$
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Remark

Let F be an irreducible finite-dimensional $\mathfrak{g}_0$−module. Then

$$M(F) \cong \mathbb{C}[\Theta] \otimes \Lambda(6) \otimes F.$$

Lemma (B. 2023; statement by Boyallian, Kac, Liberati 2013)

Let $\vec{m} \in M(F)$ is a singular vector. Then the degree of $\vec{m}$ with respect to Θ is at most 2.

From now on we denote the highest weight of an irreducible finite−dimensional $\mathfrak{g}_0$−module as $\mu = (n_1, n_2, n_3, \mu_t)$ where μ_t is the weight with respect to h_1, h_2, h_3 and t, where

$$h_1 = -i\xi_{34} - i\xi_{56}, \quad h_2 = -i\xi_{12} + i\xi_{34}, \quad h_3 = -i\xi_{34} + i\xi_{56}.$$

Boyallian, Kac and Liberati classified all highest weight singular vectors for CK_6 and obtained the following morphisms between degenerate finite Verma modules.

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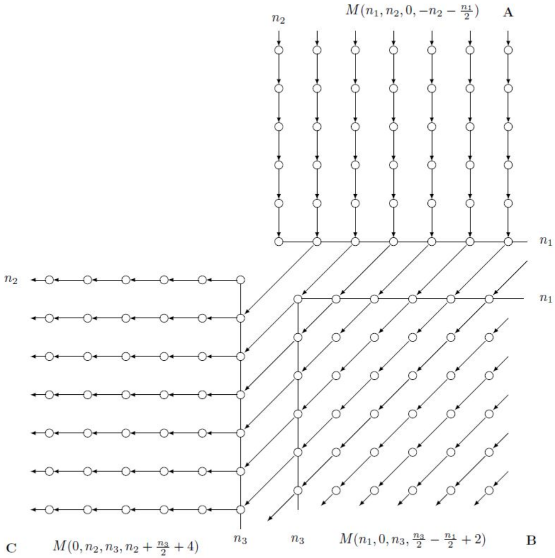

We denote by **F** the dual functor.

Proposition

The functor **F** *is exact if we consider only morphisms* $T : M \to N$*, where* $N/\operatorname{Im} T$ *is a finitely generated torsion free* $\mathbb{C}[\partial]-$*module.*

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A consequence of the main result on conformal duality showed by Cantarini, Caselli and Kac is that, in the case of CK_6, the conformal dual of $M(F(n_1, n_2, n_3, \mu_t))$ corresponds to the shifted dual $M(F^\vee)$, where $F^\vee \cong F(n_3, n_2, n_1, -\mu_t + 4)$.

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Proposition (B. 2022)

As $\mathfrak{g}_0$−modules:

$$H^{n_1,n_2}(M_A) \cong \begin{cases} \mathbb{C} & \text{if } (n_1,n_2)=(0,0), \\ 0 & \text{otherwise.} \end{cases}$$

$$H^{n_2,n_3}(M_C) \cong \begin{cases} \mathbb{C} & \text{if } (n_2,n_3)=(1,0), \\ 0 & \text{otherwise.} \end{cases}$$

Open problems

- It remains to complete the study of the homology of the complexes for the second quadrant of CK_6;
- it would be interesting to understand if it is possible to use similar techniques to compute the homology of the complexes for $E(5, 10)$.