Hilbert scheme of the points in the place and quasi-lisse vertex algebra with $\mathcal{N} = 4$ symmetry

REPRESENTATION THEORY XVIII
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Tomoyuki Arakawa

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RIMS, Kyoto University

joint work with Toshiro Kuwabara (Tsukuba) and Sven Möller (Hamburg)

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i) *There exists only finitely many simple ordinary representations of V;*
ii) *For an ordinary representation M, $\text{tr}_M(q^{L_0 - c_{(V)}/24})$ converges to a holomorphic function on the upper half place. Moreover, $\{\text{tr}_M(q^{L_0 - c_{(V)}/24}) \mid M \text{ ordinary } \}$ is a subspace of the space of the solutions of a modular linear differential equation (MLDE).*

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i) ([AK18]) $L_k(\mathfrak{g})$ with $\mathfrak{g} = D_4, E_6, E_7, E_8,\ k = -h^\vee/6 - 1$. (cf. Kawien's talk).

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ii) (Conjectured by [Milas'22]) Class $\mathcal{S}$ chiral algebras ([A]) "Generalized multiple q-zeta values"

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Theorem (A.-Moreau'21)

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(cf. Rastelli's conjecture below)

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Theorem

Let V be a quotient of a universal affine vertex algebra $V^k(\mathfrak{g})$ or a universal W-algebras $\mathcal{W}^k(\mathfrak{g}, f)$. Then the following conditions are equivalent.

i) V is quasi-lisse.

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Let X be a symplectic singularity, G finite group of automorphisms of X preserving the symplectic form ω on X_{reg}.
$\Rightarrow X/G$ is a symplectic singularity.

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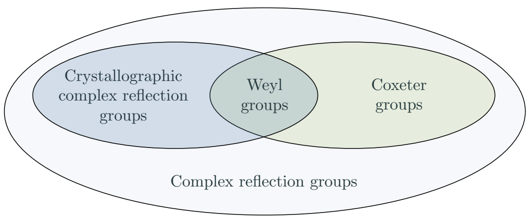

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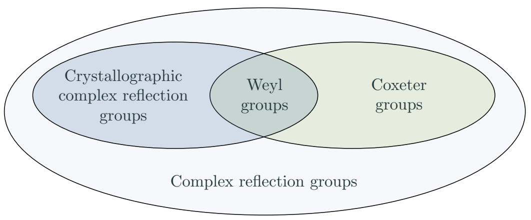

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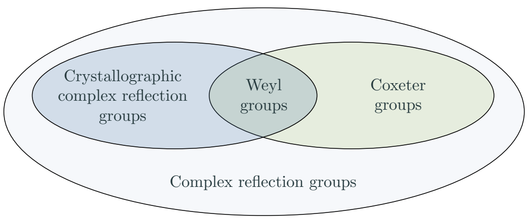

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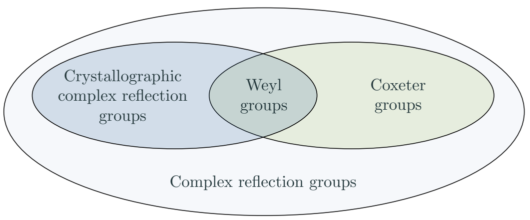

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$$\mathcal{M}_\Gamma := T^*V_\Gamma/\Gamma = \operatorname{Spec}\mathbb{C}[T^*V_\Gamma]^\Gamma$$

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iii) $\mathbf{V}_\Gamma$ admits a free field realization $\mathbf{V}_\Gamma \hookrightarrow (\beta\gamma bc)^{\otimes \mathrm{rank}\,\Gamma}$.

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ii) By the 3D mirror symmetry, $\mathrm{Higgs}(\mathcal{T}) = \mathrm{Higgs}(\mathcal{T}_{3D}) \cong \mathrm{Coulomb}(\check{\mathcal{T}}_{3D})$, where $\mathcal{T}_{3D}$ is the 3D theory obtained from $\mathcal{T}$ by S^1-compaticification.

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- When Γ is a crystallographic complex reflection group that is not a Coxeter group, then it is expected that $\mathbf{V}_{\Gamma} = \mathbb{V}(\mathcal{T}_{\Gamma})$ for some the 4-dimensional theory $\mathcal{T}_{\Gamma}$ with $\mathcal{N} = 3$ supersymmetry.

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$\exists \Omega^{ch}_{\mathbb{P}^1,\alpha}$, $\alpha \in \frac{1}{2}\mathbb{Z}$, twisted chiral de Rham complex of $\mathbb{P}_1$ ([Gorbounov, Malikov, Schechtman '05])

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Kashirara-Rouquier quantization: replace T^*V by its quantization, i.e., the Weyl algebra, and do the quantized Hamiltonian reduction.

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!!! Bonetti-Meneghelli-Rastelli suggest to construct a sheaf of vertex superalgebras.

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In other words, we replace $\mathrm{Hilb}^{\mathrm{N}}(\mathbb{C}^2)$ by the supervariety $\mathrm{Hilb}^{\mathrm{N}}(\mathbb{C}^2)_{\mathcal{L}_{tot}}$ with the underlying topological space $\mathrm{Hilb}^{\mathrm{N}}(\mathbb{C}^2)$ and the structure sheaf $\mathcal{O}_{\mathrm{Hilb}^{\mathrm{N}}(\mathbb{C}^2)} \oplus \mathcal{O}_{\Pi\mathcal{L}_{tot}}$, where $\Pi\mathcal{L}_{tot}$ is the odd tautological line bundle of $\mathrm{Hilb}^{\mathrm{N}}(\mathbb{C}^2)$.

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For $N = 2M + 1$,

$\chi_V(q) = (-1)^M \sum_{k=0}^{M} c_k \tilde{\mathbb{E}}_{2k}$

$\tilde{\mathbb{E}}_0 = 1$, $\tilde{\mathbb{E}}_{2k} = \sum_{\substack{\vec{n} \\ \sum_{j\geq 1} j n_j = k}} \prod_{p\geq 1} \frac{1}{n_p!}\left(-\frac{1}{2p} E_{2p}\right)^{n_p}$, $c_k \in \mathbb{C}$

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The 3D theory obtained from $\mathrm{SYM}_{\mathfrak{sl}_N}$ by S^1-compatification is known to be self-dual, and so its Higgs branch and Coulomb branch are the same. Then the conjecture of [Costello-Creutzig-Gaiotto'18] says

$$\mathrm{Ext}^\bullet(\mathbf{V}_N, \mathbf{V}_N) \cong \mathbb{C}[\mathcal{M}_{\mathfrak{S}_N}]$$

Thank you!