Homological link invariants
from
Floer theory

Mina Aganagic
UC Berkeley

Based on: 2305.13480 with M. Rapcak and E. LePage,

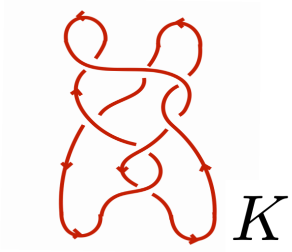
work to appear with: I. Danilenko, Y. Li, V. Shende and P. Zhou

and my earlier work: 2207.14104 , 2105.06039 and 2004.14518

The link categorification problem

was introduced in '98,

by Khovanov, who showed how to associate to a link



a bi-graded homology theory

$$\mathcal{H}_K^{*,*} = \bigoplus_{i,j\in\mathbb{Z}} \mathcal{H}_K^{i,j}$$

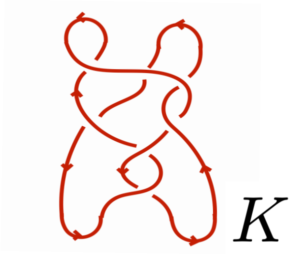
whose equivariant Euler characteristic is the Jones polynomial

$$J_K(\mathfrak{q}) = \sum_{i,j\in\mathbb{Z}} (-1)^i \mathfrak{q}^{j/2} \dim_{\mathbb{C}} \mathcal{H}_K^{i,j}$$

Khovanov's homology groups

$$\mathcal{H}^{*,*}(K) = \bigoplus_{i,j} \mathcal{H}^{i,j}(K),$$

are themselves link invariants,



independent of the link projection he used to define them.

Khovanov's construction is part of the

categorification program

pioneered by Crane and Frenkel.

A simple toy model of categorification comes from a Riemannian manifold M whose Euler characteristic

$$\chi(M) = \sum_{k\in\mathbb{Z}} (-1)^k \mathrm{dim}_{\mathbb{Z}} \mathcal{H}^k(M)$$

is categorified by the cohomology groups

$$\mathcal{H}^k(M) = \ker\, d_k / \mathrm{im}\, d_{k-1}$$

of the de Rham complex

$$C^* = \ldots C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} \ldots .$$

From physics perspective,

the Euler characteristic is the partition function

$$\chi(M) = \mathrm{Tr}(-1)^F e^{-\beta H}$$

of supersymmetric quantum mechanics with M as a target space.

As explained by Witten, the collection of vector spaces

$$C^* = \ldots C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} \ldots.$$

may be provided by

Morse theory approach to supersymmetric quantum mechanics,

as perturbative supersymmetric ground states,

indexed by the fermion number $\boldsymbol{k}$.

The action of the differential

$$\partial = \sum_k d_k$$

which turns the vector spaces into a complex

$$C^* = \ldots C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} \ldots.$$

is generated by solutions of flow equations called

instantons.

The Jones polynomial is a special case of a

quantum group

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

link invariant,

where one takes the Lie algebra to be

$${}^L\mathfrak{g} = \mathfrak{su}_2$$

with link components colored by its fundamental representation.

Unlike in

our toy example of categorification of the Euler characteristic

of a Riemannian manifold

$$\chi(M) = \mathrm{Tr}(-1)^F e^{-\beta H}$$

Khovanov's construction

and its few known generalizations to other

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

did not come from either geometry, or physics in any unified way.

Another famous link invariant is the Alexander polynomial.

It is also a quantum group

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

link invariant,

where one takes the Lie algebra to be

$${}^L\mathfrak{g} = \mathfrak{gl}_{1|1}$$

The Alexander polynomial

also has a known categorification,

but of a very different flavor than Khovanov's.

The categorification of the Alexander polynomial
is based on Floer theory,
which generalizes
supersymmetric quantum mechanics to a theory in one dimension up.
As such, it does come from geometry and physics.

The categorification of the Alexander polynomial

is based on Floer theory,

whose target space is a symmetric product of Heegard surfaces.

The theory, known as Heegard-Floer theory

was discovered in 2003

by Ozsvath and Szabo.

The link categorification problem

is to find a general framework for construction of

link homology groups,

categorifying quantum $U_{\mathfrak{q}}({}^{L}\mathfrak{g})$ link invariants,

that works uniformly for all Lie algebras

which explains what link homology groups are,

and why they exist.

The solution to the problem is based on a theory generalizing

Heegard-Floer theory,

from $\mathfrak{gl}_{1|1}$ to arbitrary Lie algebras ${}^L\mathfrak{g}$

Many special features exist in this family of theories,

which will translate to the fact that

problems whose solutions typically only exist formally,

will now be explicitly solvable.

We will have an explicit description

of the corresponding category

and an algorithm for computing link homologies.

In discovering, formulating and solving the theory,

homological mirror symmetry

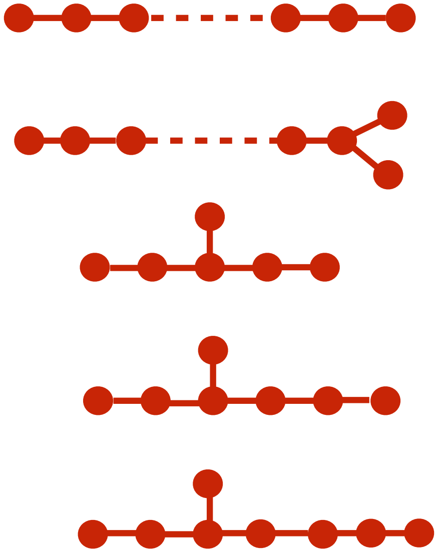
played a key role.

I have described elsewhere, e.g. in the 2022 ICM talk,
how I discovered the theory,
and why it is essentially inevitable.
Here, I will describe the final answer in some detail.

Pick

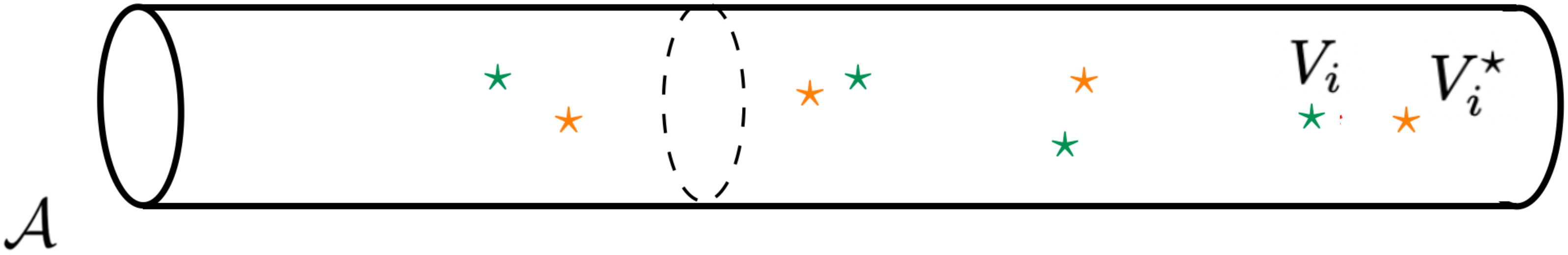
$$^{L}\mathfrak{g}$$

which for simplicity is either a Lie algebra of ADE type,



or is a Lie superalgebra $\mathfrak{gl}_{m|n}$

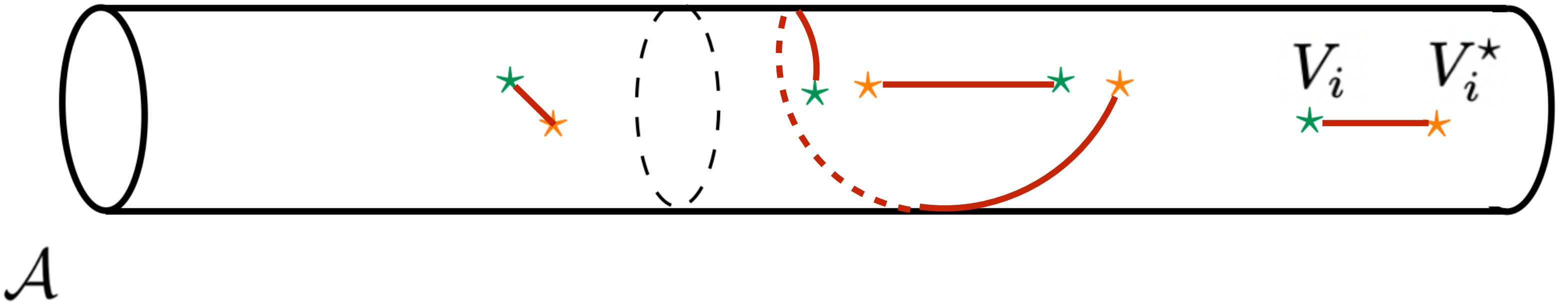
Take a Riemann surface $\mathcal{A}$ which is an infinite complex cylinder



with punctures which come in pairs, labeled by a minuscule representation V_i of ${}^L\mathfrak{g}$ and its conjugate $V_i^\star$

Consider a collection of curves,

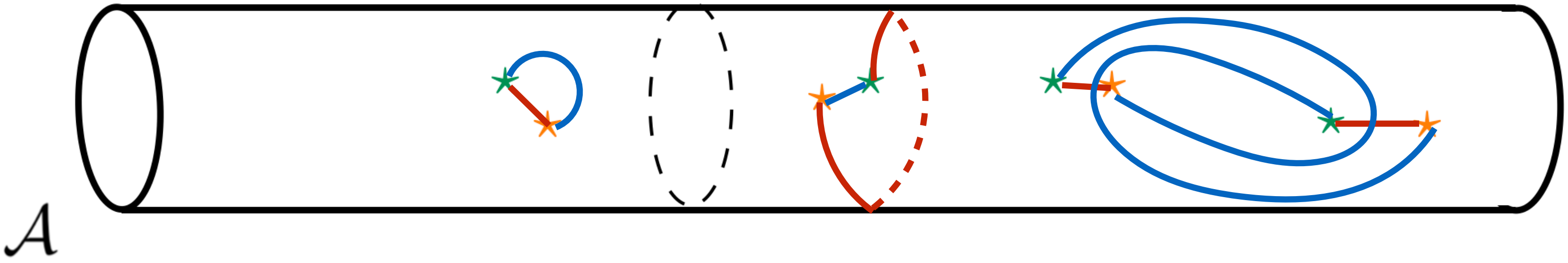
a “matching”



which end on pairs of punctures

colored by complex conjugate representations.

To every pair of such matchings

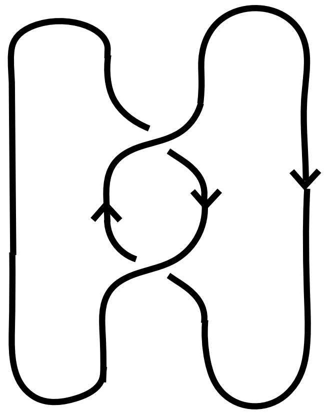


one can associate a link in

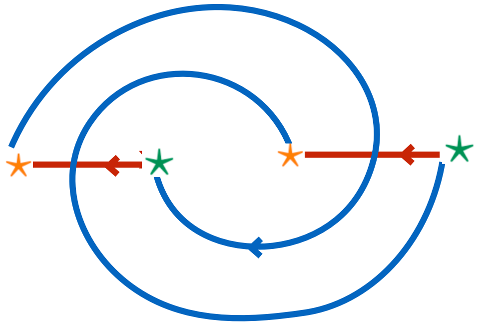
$\mathcal{A} \times \mathbb{R}$

Arbitrary links in

$$\mathcal{A} \times \mathbb{R}$$

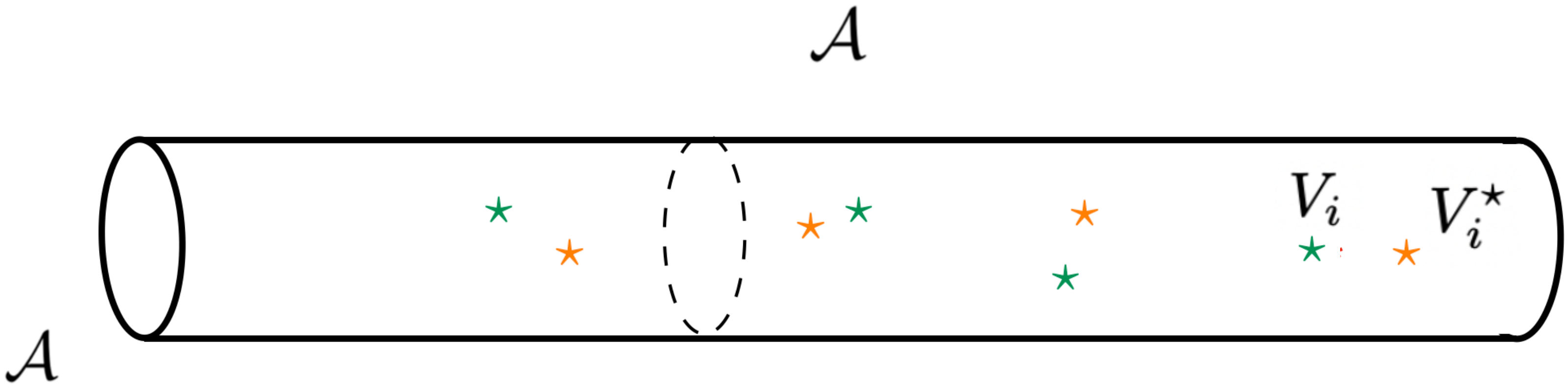


arize from pairs of matchings on $\mathcal{A}$, one colored red, the other blue,



by taking the red to overpass the blue,

To the Riemann surface



with punctures labeled by minuscule representations of

$^L\mathfrak{g}$

we will associate a category.

The category

$$\mathscr{D}_Y$$

is a variant of a derived Fukaya-Seidel category,

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W)).$$

associated with an exact symplectic manifold

$$Y$$

equipped with a potential

$$W : Y \to \mathbb{C}$$

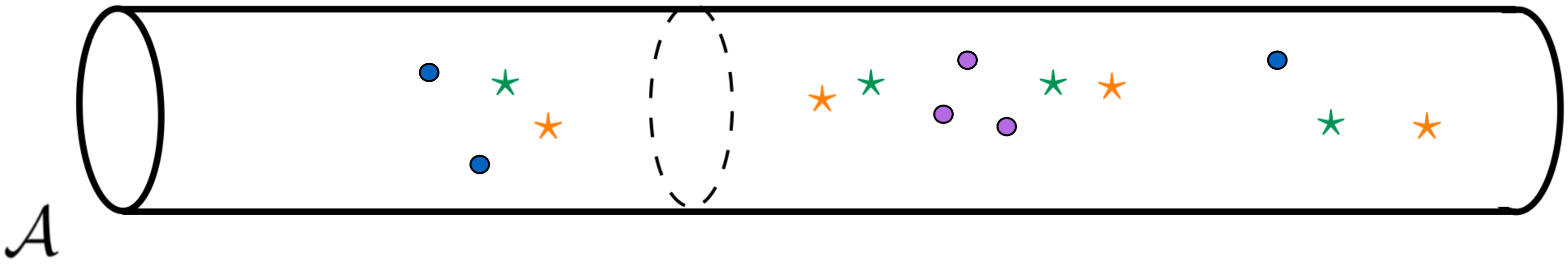
and a top holomorphic form

$$\Omega$$

The manifold Y is

$$Y = \prod_{a=1}^{\mathrm{rk}\,{}^L\mathfrak{g}} Sym^{d_a}(\mathcal{A})$$

A point of Y is a collection of points

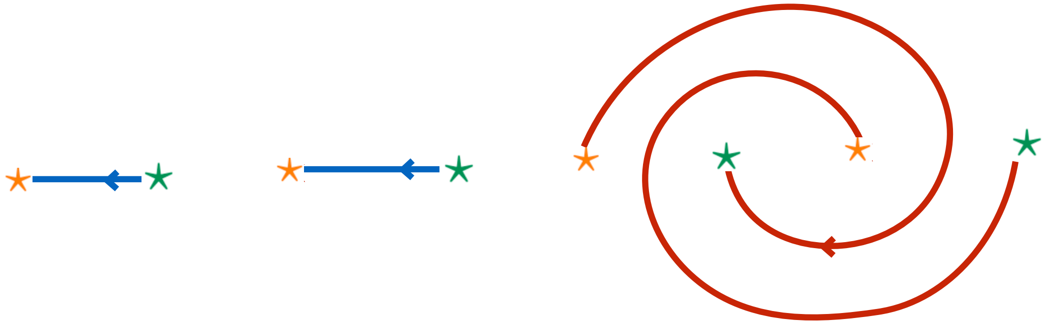


on the punctured Riemann surface $\mathcal{A}$,

colored by simple roots of the Lie algebra

${}^L\mathfrak{g}$

To the pair of red and blue matchings,



and the corresponding link in

$$\mathcal{A} \times \mathbb{R}$$

we will associate a pair of

$$I_{\mathcal{U}}, \quad \mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y .$$

objects, or ``branes'', of our category.

The $U_{\mathfrak{q}}({}^{L}\mathfrak{g})$ homology of the link

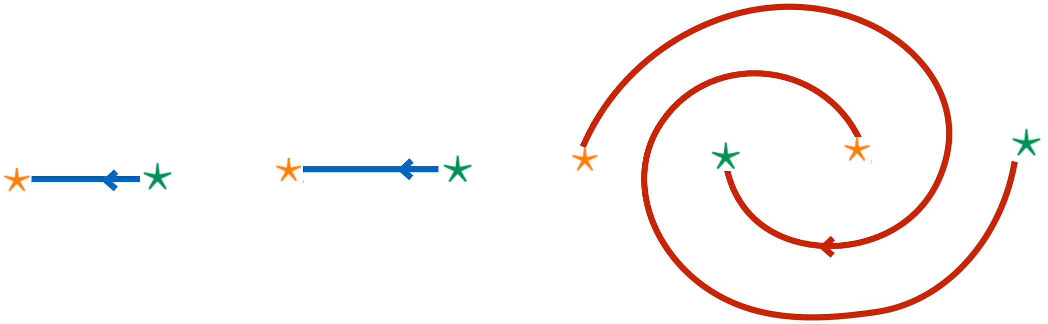
is the graded space of morphisms

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

between the two branes

$$I_{\mathcal{U}}, \quad \mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y .$$

which we assigned to the link starting from the matchings

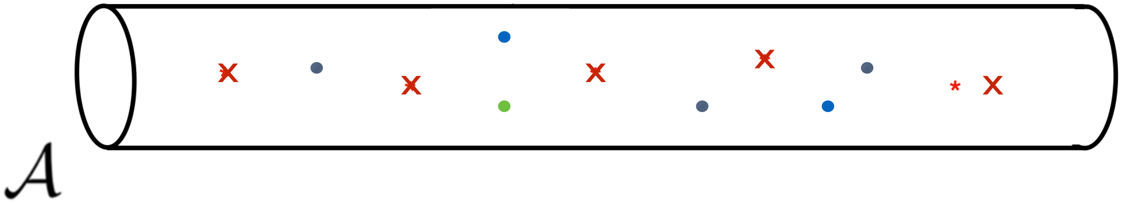


One can describe this category very explicitly

thanks to the fact

$$Y$$

is the configuration space of colored points

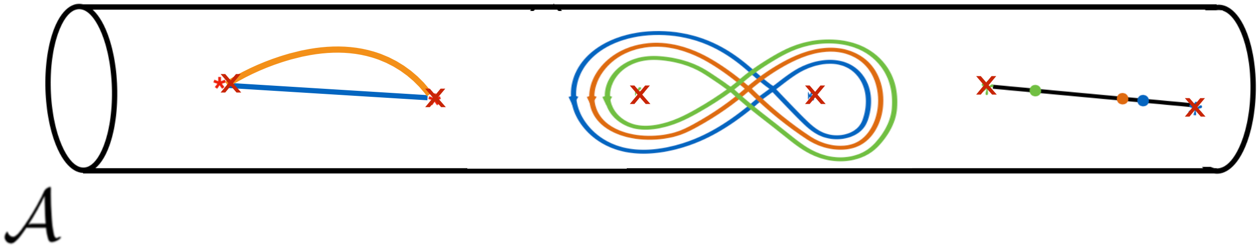


on the punctured Riemann surface.

Objects of the category

$$Y$$

have a description in terms of the Riemann surface,



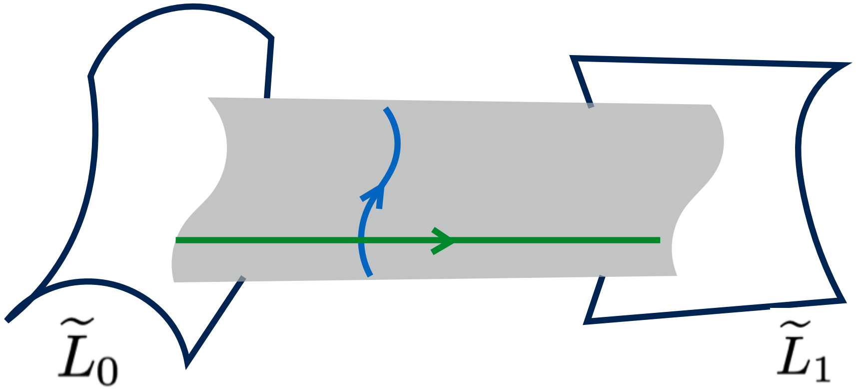
as products of one dimensional curves

colored by simple roots, or generalized intervals.

Spaces of **morphisms** between a pair of objects

$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) = \mathrm{Ker}\, Q/\mathrm{Im}\, Q.$$

are defined by Floer theory,



which is modeled after Morse theory approach

to supersymmetric quantum mechanics,

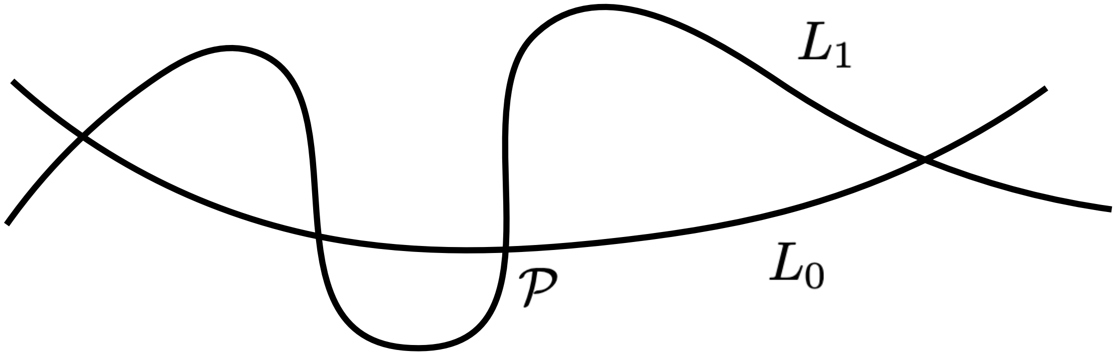
generalized to a theory in one dimension up.

The role of the Morse complex from the beginning of the talk

is taken by the Floer complex, which is a vector space

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

is spanned by the intersection points of the two Lagrangians,



The action of the Floer differential

$$\delta_F$$

which turns the space spanned by intersection points

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

into a complex

$$\delta_F : CF^{*,*}(L_0, L_1) \to CF^{*+1,*}(L_0, L_1)$$

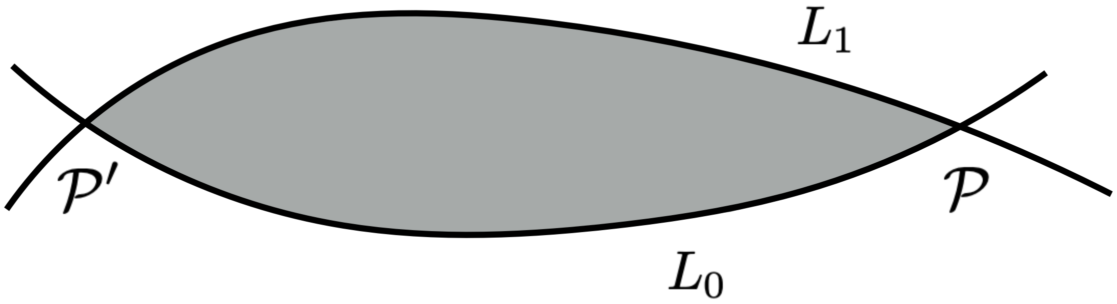
is generated by instantons.

In Floer theory,

the coefficient of $\mathcal{P}'$ in $\delta_F \mathcal{P}$

is obtained by counting holomorphic maps from a strip to Y

$$y : \mathrm{D} \to Y$$



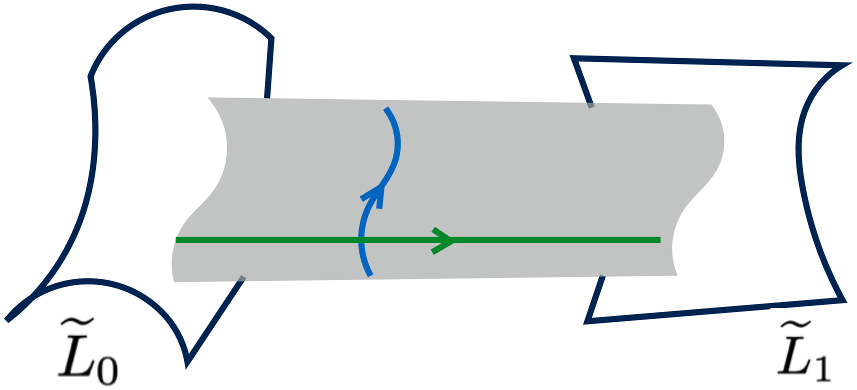
interpolating from $\mathcal{P}$ to $\mathcal{P}'$ of cohomological degree one

and equivariant degree zero.

The cohomology of the resulting complex

$$Hom^{*,*}_{\mathscr{D}_Y}(L_0, L_1) = \mathrm{Ker}\,\delta_F / \mathrm{Im}\,\delta_F$$

is the space of morphisms between the branes in $\mathscr{D}_Y$.



The Euler characteristic of

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

is the equivariant intersection number:

$$\chi(E, \mathscr{B}I) = \sum_{\mathcal{P} \in E \cap \mathscr{B}I} (-1)^{M(\mathcal{P})} \mathfrak{q}^{J(\mathcal{P})}$$

which is the count of intersection points,

$$\mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}$$

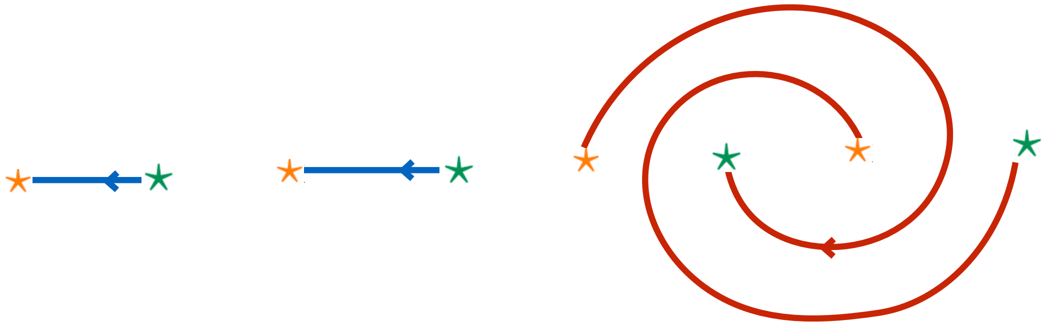
keeping track of gradings.

The fact that the Euler characteristic of

$$Hom^{*,*}_{\mathscr{D}_{\mathrm{Y}}}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

is the quantum $U_{\mathfrak{q}}({}^{L}\mathfrak{g})$ link invariant

is guaranteed to hold by construction.



This follows from Picard-Lefshetz theory, and fact that

the equivariant central charge function

$$\mathcal{Z} : \mathscr{D}_Y \to \mathbb{C}$$

which is given by

$$\mathcal{Z}[L] = \int_L \Omega \, e^{-W},$$

is a close cousin of the conformal block of

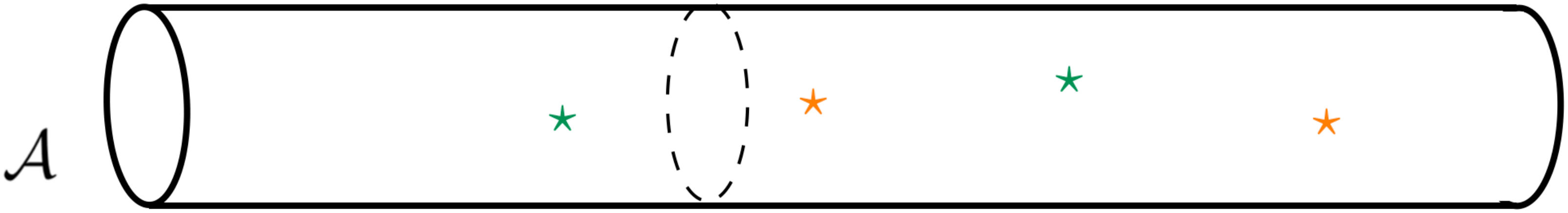
$$\widehat{{}^L\mathfrak{g}}$$

the affine Lie algebra associated to ${}^L\mathfrak{g}$.

The conformal blocks of

$$\widehat{{}^L\mathfrak{g}}$$

on the Riemann surface we started with



have integral formulation as period integrals:

$$\mathcal{V}_\alpha[L] = \int_L \Phi_\alpha \, \Omega \, e^{-W}$$

of Feigin and E.Frenkel, and Schechtman and Varchenko, which differ from

$$\mathcal{Z}[L] = \int_L \Omega \, e^{-W},$$

by insertions which do not affect the monodromy properties.

In what follows, I will describe the category,

$$\mathscr{D}_Y$$

and how to solve the theory exactly.

We will learn how to compute the homology theory

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

for any link,

and why the resulting vector spaces are themselves invariants of links.

We will start with two simplest examples,

when the Lie algebra

$$^{L}\mathfrak{g}$$

is either

$$\mathfrak{su}_2 \qquad\qquad \mathfrak{gl}_{1|1}$$

with links colored by the defining representation and its conjugate.

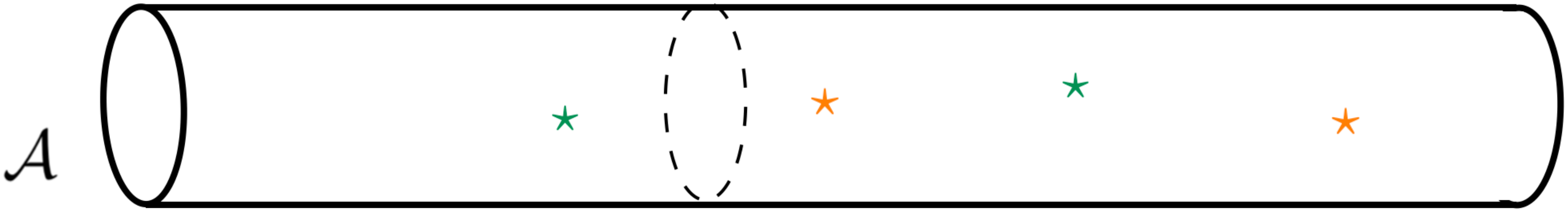
The former theory will categorify the Jones polynomial,

the later the Alexander polynomial.

If we take the Lie algebra to be

$$\mathfrak{gl}_{1|1}$$

our Riemann surface has 2d marked points



the "even" half of which are labeled by the highest weight

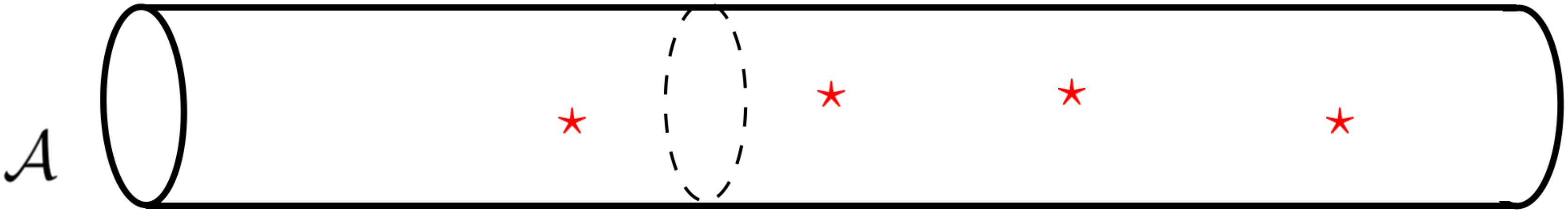
of the fundamental representation

and "odd" half by that of its conjugate.

If we pick the Lie algebra to be

$$\mathfrak{su}_2$$

the 2d marked points are of the same kind



since the fundamental representation is self conjugate.

In both cases, the Lie algebra has a single simple root,

which is fermionic for

$$\mathfrak{gl}_{1|1}$$

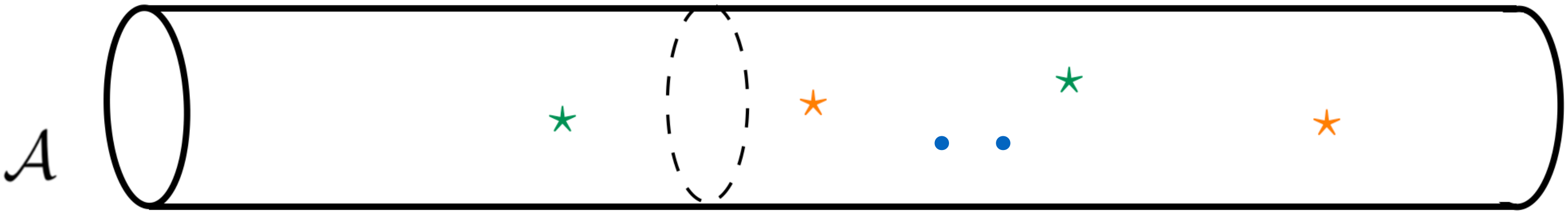
and bosonic for

$$\mathfrak{su}_2$$

The target is

$$Y = Sym^d(\mathcal{A}),$$

the symmetric product of d copies of the Riemann surface,



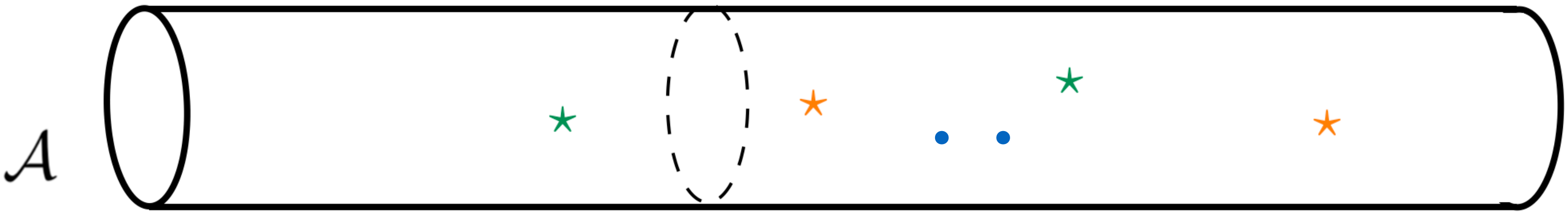
A point in Y is a collection of d unordered ,
not necessarily distinct points on $\mathcal{A}$.

In the $\mathfrak{gl}_{1|1}$ case, the top holomorphic form

$$\Omega_{\mathfrak{gl}_{1|1}}$$

coincides with the standard top holomorphic form on

$$Sym^d(\mathbb{C}^\times)$$



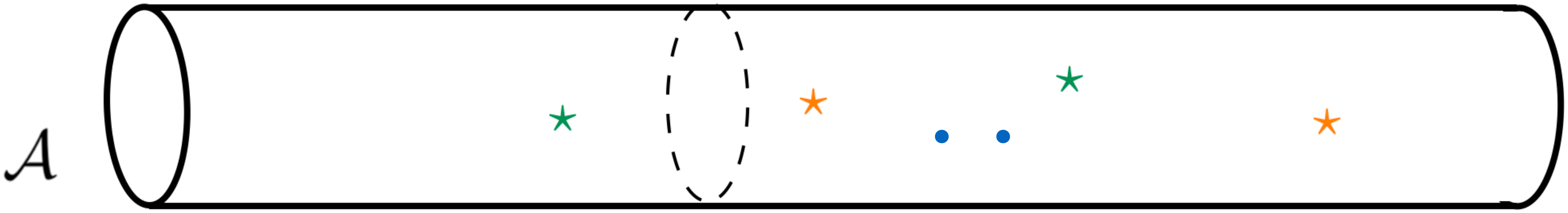
apart from the first order poles along the odd punctures.

In the $\mathfrak{su}_2$ case, the top holomorphic form

$$\Omega_{\mathfrak{su}_2}$$

differs from the standard top holomorphic form on

$$Sym^d(\mathbb{C}^\times)$$



by having a first order poles along the diagonal,

$$\Delta$$

where a pair of points coincide on $\mathcal{A}$.

Objects of

$$\mathscr{D}_Y$$

are products

$$L = L_1 \times L_2 \times \ldots \times L_d$$

of d one-dimensional curves on $\mathcal{A}$

which we take to be non-intersecting.

(More generally, objects of $\mathscr{D}_Y$ are branes supported on

Lagrangians in the symplectic manifold.

Above L is a Lagrangian in $Y = Sym^d(\mathcal{A})$)

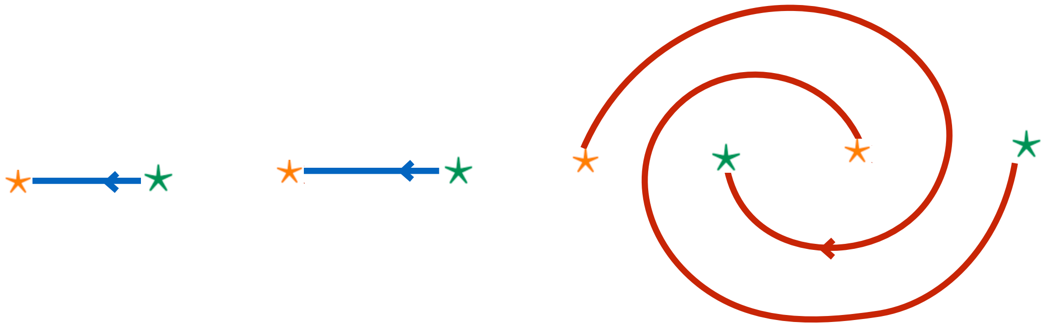
Given a link in

$$\mathcal{A} \times \mathbb{R}$$

we get a pair of objects

$$I_{\mathcal{U}}, \quad \mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_{\mathrm{Y}}.$$

derived from the corresponding the red and blue matchings,



In both theories,

the objects which we will associate to the cups,

will be "I-branes"

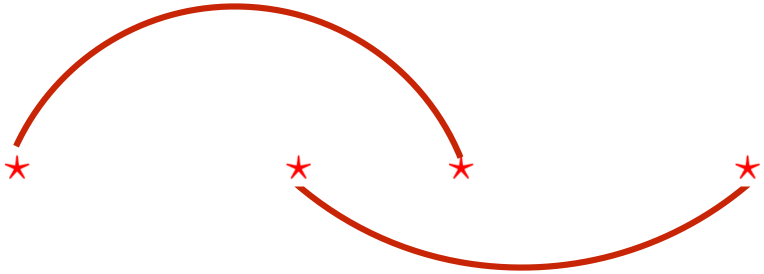
$$I_{\mathcal{U}} = I_1 \times \ldots \times I_d$$

which are the simple products of intervals pictured, The cap branes, which we will denote by:

$$E_{\mathcal{U}} = E_1 \times \ldots \times E_d$$

will be products of closed curves

whose homology class is proportional to the class of



the braided caps.

When the Lie algebra is

$$\mathfrak{su}_2$$

the cap branes

$$E_{\mathcal{U}} = E_1 \times \ldots \times E_d$$

are products of figure eights:

When the Lie algebra is

$$\mathfrak{gl}_{1|1}$$

the cap branes

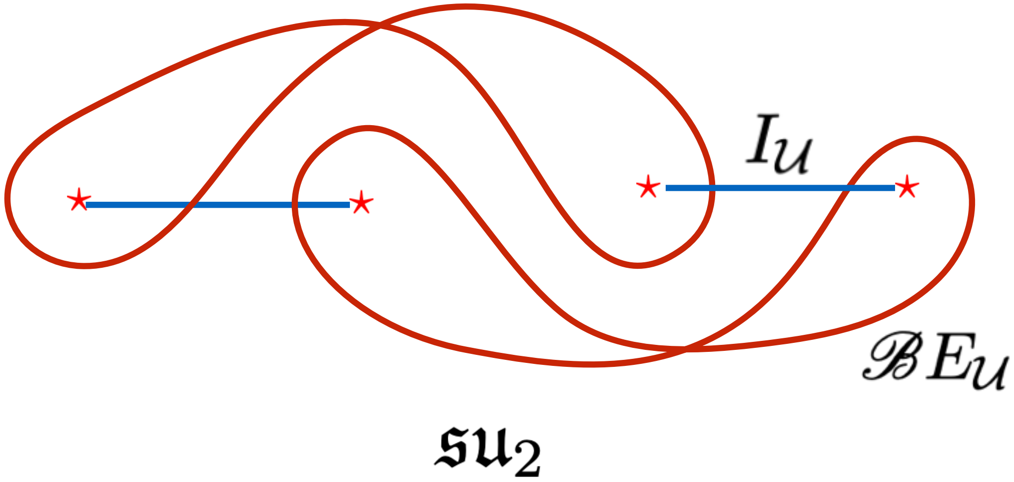
$$E_{\mathcal{U}} = E_1 \times \ldots \times E_d$$

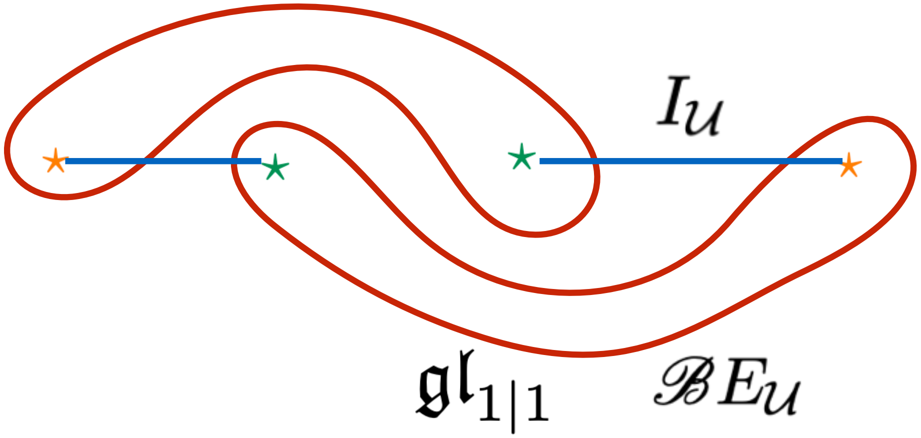
are products of ovals:

The link homologies

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \mathrm{Ker}\,\delta_F/\mathrm{Im}\,\delta_F$$

are the cohomologies of the Floer differential,





acting on the vector space spanned by graded intersection points of the branes.

For

$$\mathfrak{gl}_{1|1}$$

the fact that the Euler characteristic

$$\chi(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{\mathcal{P} \in \mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}} (-1)^{M(\mathcal{P})} \mathfrak{q}^{J(\mathcal{P})},$$

is the Alexander polynomial:

$$\Delta_K(\mathfrak{q})$$

is a corollary of work of Manolescu, Ozsvath, Szabo and others.

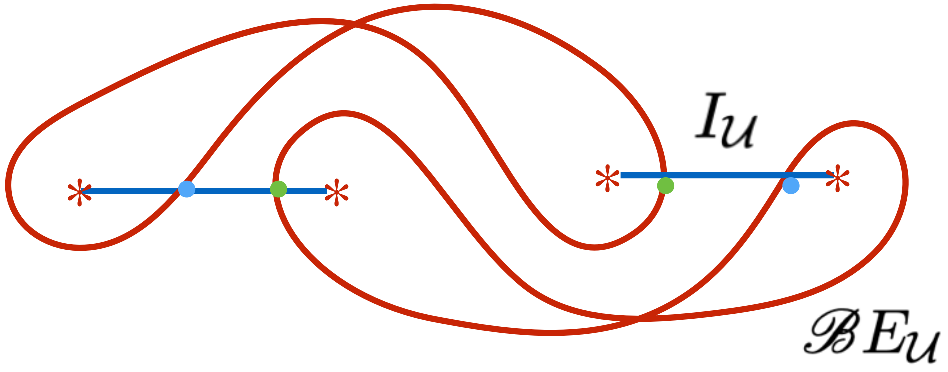
For

$$^{L}\mathfrak{g} = \mathfrak{su}_2$$

the fact that the Euler characteristic

$$\chi(E, \mathscr{B}I) = \sum_{\mathcal{P} \in E \cap \mathscr{B}I} (-1)^{M(\mathcal{P})} \mathfrak{q}^{J(\mathcal{P})}$$

computes the Jones polynomial is a theorem by Bigelow from the '90s.



The description of

categorification of the Alexander polynomial by

$$\mathscr{D}_Y$$

is Heegard-Floer theory,

The theory was discovered in 2003 by Ozsvath and Szabo.

Our particular variant has some minor novel elements,

The approach to solving the theory I'll describe is new.

The categorification of the Jones polynomial

and other $U_{\mathfrak{q}}({}^L\mathfrak{g})$ link invariants by

$\mathscr{D}_Y$

is completely new.

Rather than computing the action of the differential

by counting holomorphic curves,

for which there is no general algorithm,

we will explain how to sum the instantons up.

The categories of branes

$$\mathscr{D}_Y$$

are generated by a finite set.

The generating set of branes

$$T = \bigoplus_{\mathcal{C}} T_{\mathcal{C}}$$

are products of real line Lagrangians,

$$T_{\mathcal{C}} = T_{i_1} \times T_{i_2} \times \ldots \times T_{i_d}$$

colored by simple roots, and taken up to isotopy.

For $^L\mathfrak{g} \neq \mathfrak{gl}_{1|1}$, the branes will get equipped with extra structure, of a local system.

Since the T -brane generates $\mathscr{D}_Y$

$$T = \bigoplus_{\mathcal{C}} T_{\mathcal{C}}$$

we have an equivalence of categories

$$\mathscr{D}_A \cong \mathscr{D}_Y$$

where $\mathscr{D}_A$ is the derived category of modules for

its endomorphism algebra

$$A = hom^{*,*}(T,T) = \bigoplus_{\mathcal{C},\mathcal{C}'} hom^{*,*}(T_{\mathcal{C}}, T_{\mathcal{C}'})$$

in the underlying A_∞ -category, $hom^{*,*}(-,-) = CF^{*,*}(-,-)$

The equivalence of categories

$$\mathscr{D}_A \cong \mathscr{D}_Y$$

comes from the Yoneda functor

$$hom(T,-) : \mathscr{D}_Y \longrightarrow \mathscr{D}_A$$

which maps any A-brane on $L \in \mathscr{D}_Y$ to a

complex of modules for the algebra

$$A = hom^{*,*}(T,T)$$

The complex

$$(\mathscr{B}E(T), \delta)$$

is a direct sum of $T_{\mathcal{C}}$ -branes, together with a differential

$$\delta \in A = hom^{*,*}(T, T)$$

which is a degree one operator

$$\delta : \mathscr{B}E(T) \to \mathscr{B}E(T)[1]$$

that squares to zero in an appropriate sense.

The T-branes

are projective modules for their endomorphism algebra

$$A = hom^{*,*}(T,T)$$

so the complex

$$\mathscr{B}E_{\mathcal{U}} \cong (\mathscr{B}E(T),\delta)$$

is a projective resolution

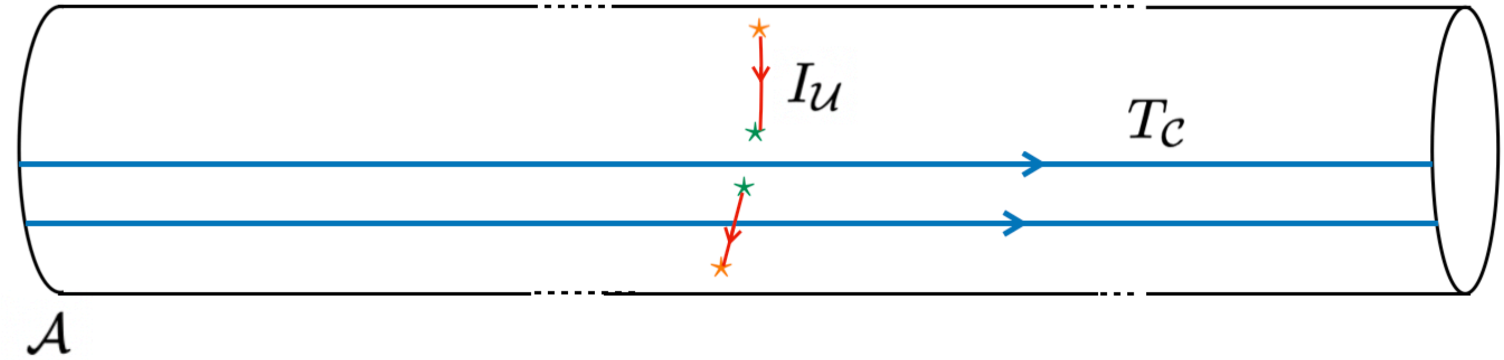
of the A-module corresponding to the brane $\mathscr{B}E_{\mathcal{U}}$

One of the simple, but key properties of this set of generators is that

the cup branes

$$I_{\mathcal{U}} = I_1 \times \ldots \times I_d$$

map to simple modules for the algebra



since the only non-zero morphism is:

$$hom(T_{\mathcal{C}}, I_{\mathcal{U}}) = \mathbb{C}\delta_{\mathcal{C},\mathcal{U}}$$

Given a description of the

braided cap brane as a complex of T-branes

$$\mathscr{B}E_{\mathcal{U}} \cong (\mathscr{B}E(T), \delta)$$

by applying the $hom_A(-, I_{\mathcal{U}})$ -functor,

we get for free

a complex of vector spaces with the action of the differential on it

whose cohomology will be the link homology

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

The complex obtained from

$$\mathscr{B}E_{\mathcal{U}} \cong (\mathscr{B}E(T), \delta)$$

by applying the $hom_A(-, I_{\mathcal{U}})$ -functor,

is the Floer complex

$$\delta_F : CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) \to CF^{*+1,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

where

$$CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{\mathcal{P} \in \mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}} \mathbb{C}\mathcal{P}$$

For

$$\mathfrak{gl}_{1|1}$$

the algebra of endomorphisms of the T-brane,

$$A_{\mathfrak{gl}_{1|1}} = hom^{*,*}(T,T)$$

is the differential graded, associative algebra of

Lipshitz, Ozsvath and Thurston.

For

$\mathfrak{su}_2$

the algebra of endomorphisms of the T-brane,

$$A_{\mathfrak{su}_2} = Hom^*_{\mathscr{D}_Y}(T,T)$$

turns out to be an ordinary associative algebra,

and a close cousin of the KLRW algebra of

Khovanov, Lauda, Rouquier and Webster.

As a vector space,

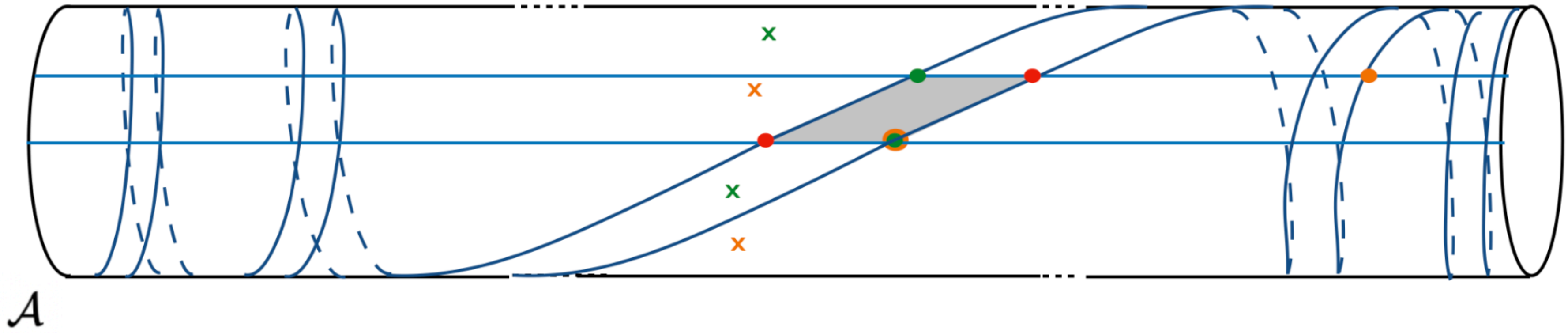
the endomorphism algebra of the T -branes

$$A_{\mathfrak{gl}_{1|1}} = hom^{*,*}(T,T)$$

is generated by the intersection points of

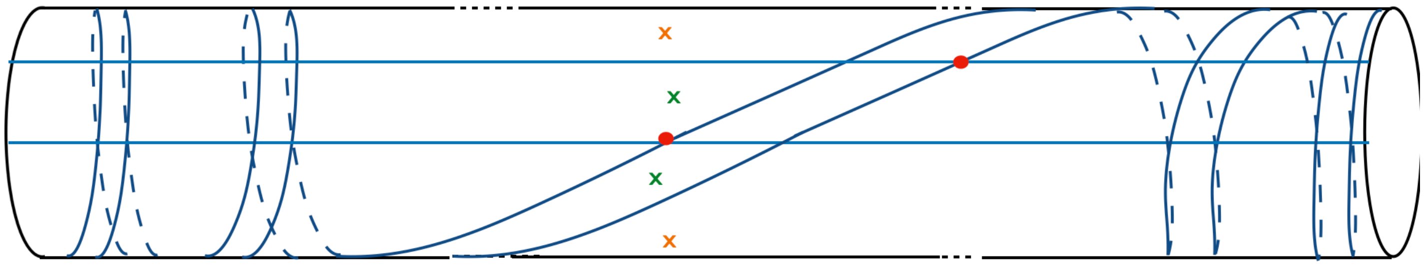
$$T = \bigoplus_{\mathcal{C}} T_{\mathcal{C}}$$

defined by the wrapped Fukaya category:



Thinking of the intersection points

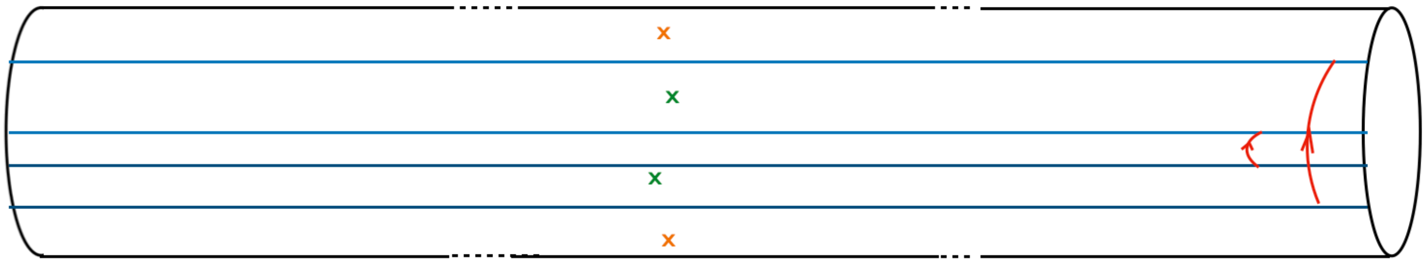
$$\mathcal{P} \in T^{\zeta}_{\mathcal{C}} \cap T_{\mathcal{C}'}$$



as Reeb cords,

(time one flows $\partial_s y = X^{\zeta},$ of a quadratic Hamiltonian

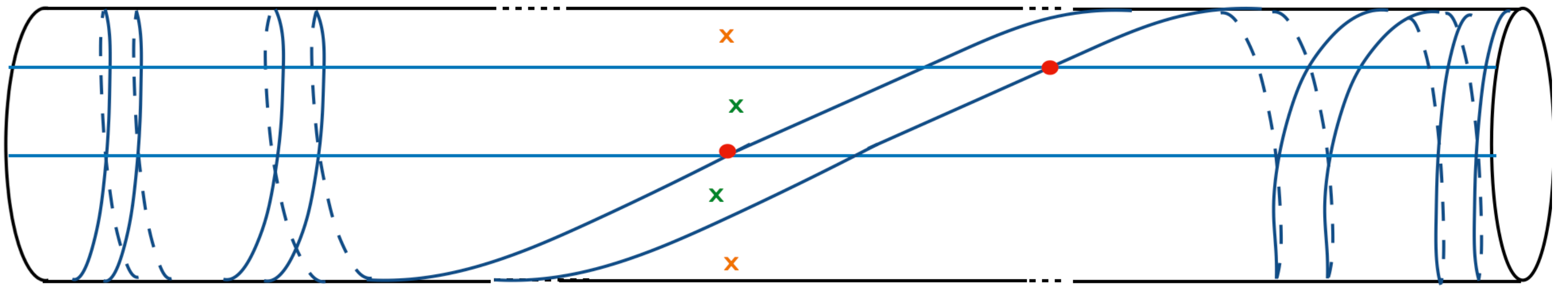
near the two infinities between unperturbed branes)



we get a graphical representation of the algebra elements.

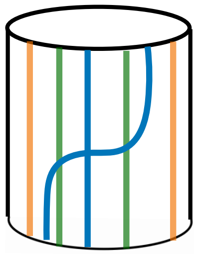
The algebra elements

$$\mathcal{P} \in T_{\mathcal{C}}^{\zeta} \cap T_{\mathcal{C}'}$$



are configurations of d blue strings on a cylinder of unit height

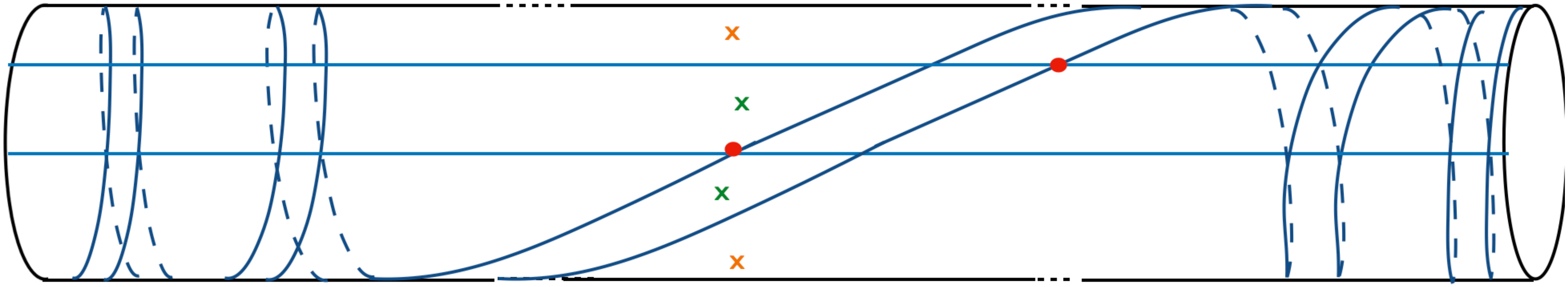
whose vertical direction parameterizes the flow



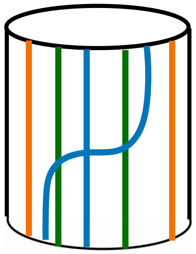
and the horizontal direction represents the S^1 in $\mathcal{A}$

The blue strings encode the intersection point

$$\mathcal{P} \in T^{\zeta}_{\mathcal{C}} \cap T_{\mathcal{C}'}$$



as a path from the position of $T_{\mathcal{C}}$ to $T_{\mathcal{C}'}$ branes on the S^1



The punctures get represented as orange and green strings, constant in ``time”.

The Maslov and J^0 -degree of the intersection points are sums of the degrees of string bits.

$\mathfrak{gl}_{1|1}$

$M = 0, J^0 = 0$

$M = 0, J^0 = 1$

$M = 0, J^0 = 0$

$M = 2, J^0 = -1$

$M = -1, J^0 = 0$

The algebra relations come from the Floer product:

$$m_2 : hom^{*,*}(T_{\mathcal{C}'}, T_{\mathcal{C}''}) \cdot hom^{*,*}(T_{\mathcal{C}}, T_{\mathcal{C}'}) \to hom^{*,*}(T_{\mathcal{C}}, T_{\mathcal{C}''})$$

(twisted by signs)

which translates into stacking cylinders,

and rescaling.

To compute the algebra,

and to describe the resulting category,

we will start not with Y but with

$$Y_0 = Y \backslash \Delta$$

with the divisor of the diagonal

$$\Delta$$

where any two points come together on $\mathcal{A}$ deleted.

Y_0 is the configuration space of d distinct points on $\mathcal{A}$

The two categories

$\mathscr{D}_{Y_0}$ and $\mathscr{D}_{Y}$

have the same objects,

since both are based on Lagrangians that avoid the diagonal,

and the generators of the underlying Floer complexes are the same as well.

Then, we have the equivalence,

$$\mathscr{D}_{A_0} \cong \mathscr{D}_{Y_0}$$

where

$$A_0 = hom^{*,*}_{Y_0}(T,T)$$

is the endomorphism algebra of the same

$$T = \bigoplus_{\mathcal{C}} T_{\mathcal{C}}$$

brane just on $Y_0 = Y \backslash \Delta$ instead of on Y , The theory on

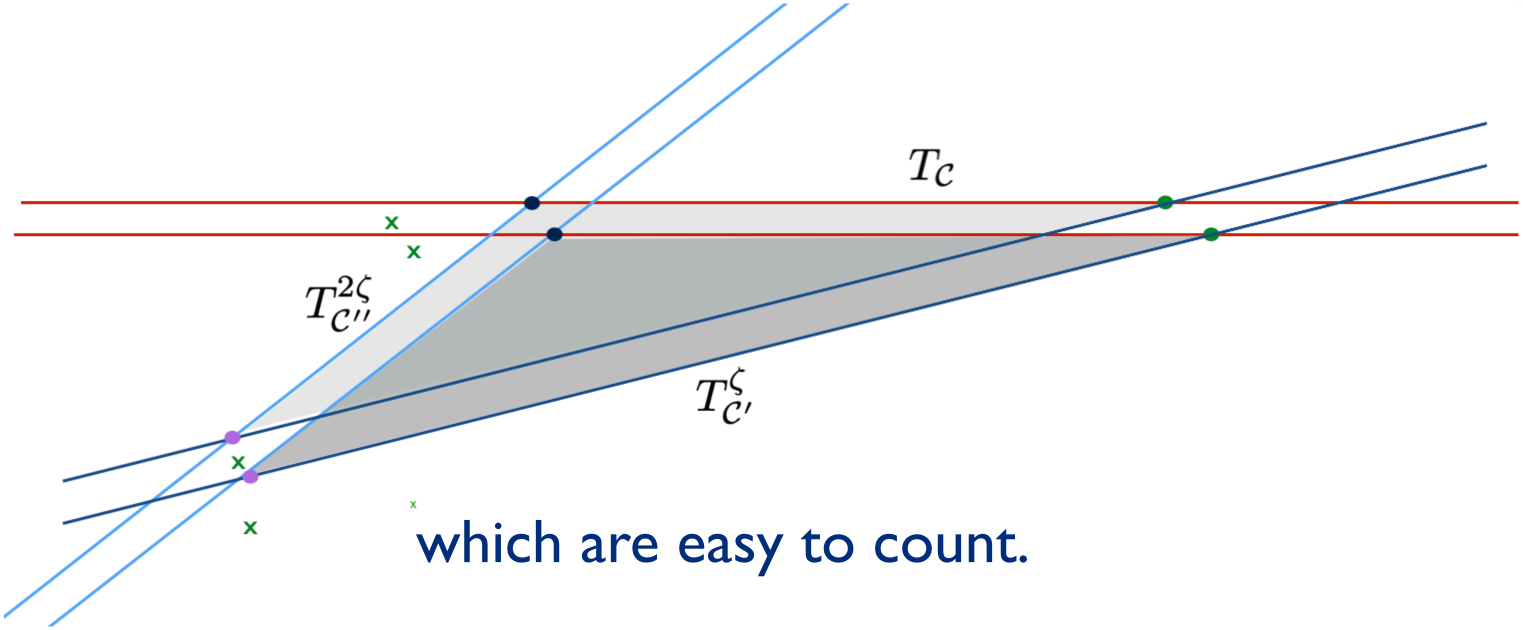
$$Y_0 = Y \backslash \Delta$$

is very simple.

Since we deleted the diagonal in the symmetric product,

the only maps $y : \mathrm{D} \longrightarrow Y = Sym^d(\mathcal{A})$ that survive are

are disconnected products of d one-dimensional maps to $\mathcal{A}$

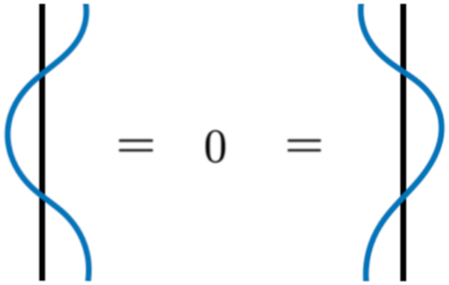


which are easy to count.

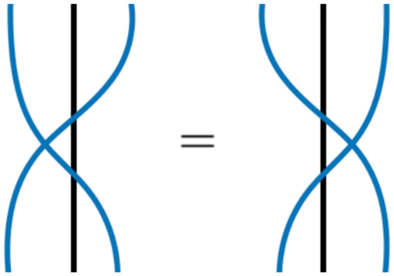
The relations in

$$A_0 = hom^{*,*}_{Y_0}(T,T)$$

say that string diagrams must have no excess intersection,



or the algebra product vanishes, as well as



As a result, A_0 is an associative algebra,

graded by cohomological and equivariant degrees.

The theory on

$$Y$$

is a one parameter deformation of the theory on

$$Y_0 = Y \backslash \Delta$$

corresponding to filling in the diagonal.

The deformation parameter is the instanton counting, or Novikov, parameter

$$\hbar^{\#}$$

that counts the intersection $\#$ with the diagonal Δ

As vector spaces, the algebras

$$A_0 = hom^{*,*}_{Y_0}(T,T)$$

and

$$A_\hbar = hom^{*,*}_{Y}(T,T)$$

are the same, as T -branes avoid the diagonal Δ

$$Y_0 = Y\backslash\Delta$$

only the algebra structure deforms.

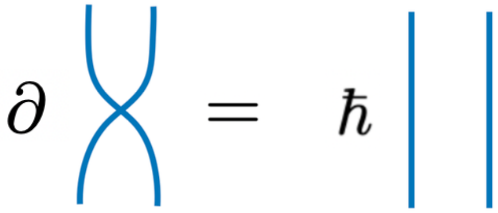
Deformation to

$\hbar \neq 0$

is very simple too.

In the $\mathfrak{gl}_{1|1}$ case, the algebra gains a non-trivial differential ∂

a cohomological degree one operator which acts by



and squares to zero.

For any

$${}^L\mathfrak{g}$$

the same strategy of working in the complement of

$$\Delta = \sum_{a=1}^{\mathrm{rk}\,{}^L\mathfrak{g}} \Delta_a$$

the diagonal divisor in

$$Y = \bigotimes_{a=1}^{\mathrm{rk}\,{}^L\mathfrak{g}} Sym^{d_a}(\mathcal{A})$$

and then filling it back in, to solves the theory, Understanding the

$\mathfrak{gl}_{1|1}$ and $\mathfrak{su}_2$

theories suffices to solve the general case.

In the complement of the diagonal, on

$$Y_0 = Y \backslash \Delta$$

the theories corresponding to

$$\mathfrak{su}_2 \qquad\qquad \mathfrak{gl}_{1|1}$$

are the essentially same, up to regrading.

The key difference between them is how the diagonal gets filled in.

To solve the theory, for any

$$^L\mathfrak{g}$$

just like in the $\mathfrak{gl}_{1|1}$ case, start with

$$Y_0 = Y \backslash \Delta$$

which is

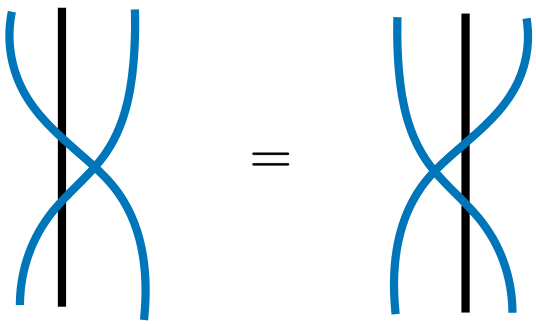
$$Y = \bigotimes_{a=1}^{\mathrm{rk}\,^L\mathfrak{g}} Sym^{d_a}(\mathcal{A})$$

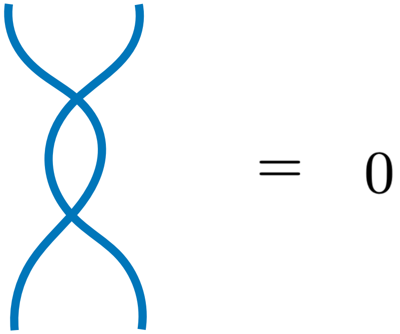
with diagonal $\Delta = \sum_{a=1}^{\mathrm{rk}\,^L\mathfrak{g}} \Delta_a$ removed.

For example, for

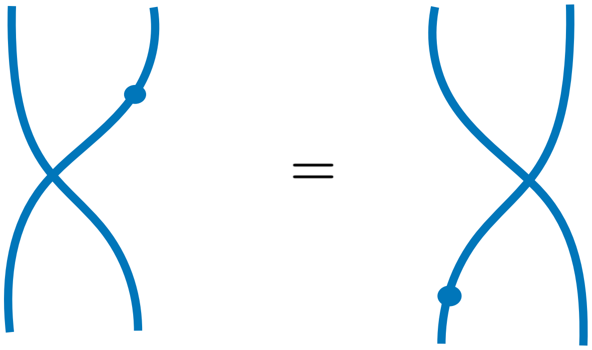
$$ {}^{L}\mathfrak{g} = \mathfrak{su}_2 $$

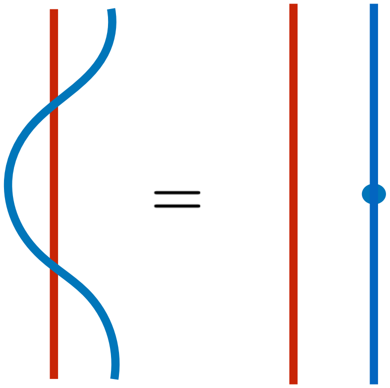
the resulting relations are analogous to the $\mathfrak{gl}_{1|1}$ case





and include two new ones:





Just like in the $\mathfrak{gl}_{1|1}$ case

the degrees of all the intersection points

are sums of degrees of string bits:

$J^0 = 0$

$J^0 = 1$

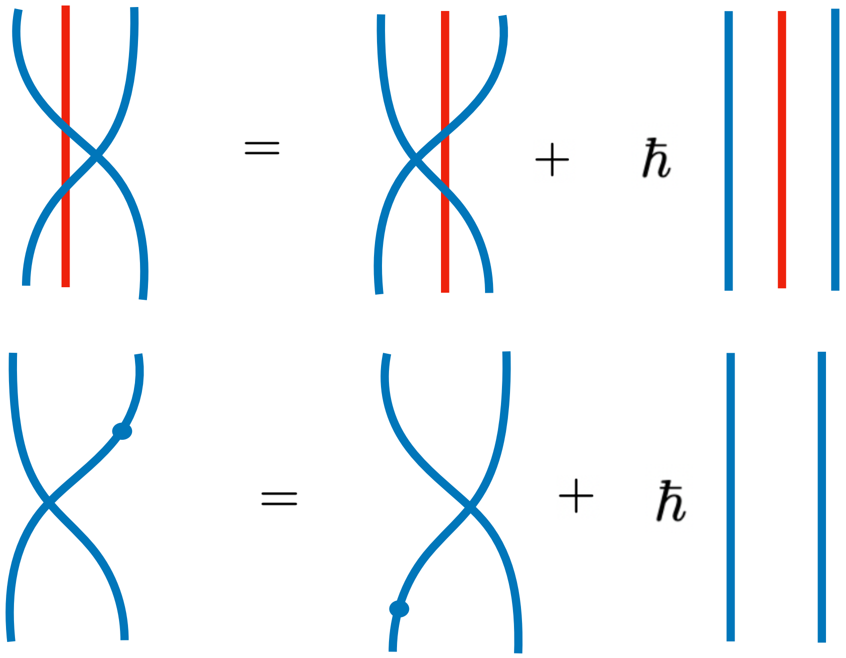
$J^0 = -1$

$J^0 = 1$

Now, they are all in Maslov degree zero.

Filling the diagonal back in,   on degree grounds,

the relations deform:

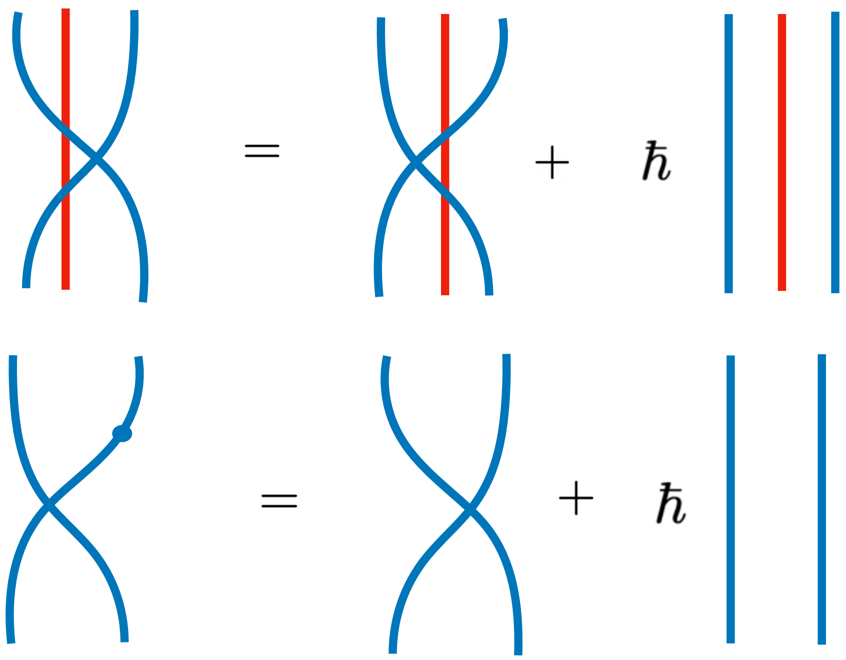


To understand the deformation requires counting a single disk,

everything else is fixed on degree grounds and by associativity.

In fact, the very same count of one single disk

that is necessary to understand



the $\hbar$ -deformation in the $\mathfrak{su}_2$ case,

suffices to compute the algebra for general ${}^L\mathfrak{g}$

Now, we will apply this

to the problem computing the action of the differential

$$\delta_F : CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) \to CF^{*+1,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

on the Floer complex

$$CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{\mathcal{P}\in\mathscr{B}E_{\mathcal{U}}\cap I_{\mathcal{U}}} \mathbb{C}\mathcal{P}$$

whose cohomology should give us link invariants.

While the equivalence

$$\mathscr{D}_A \cong \mathscr{D}_Y$$

guarantees that as any brane in $\mathscr{D}_Y$, the braided cap branes

$$\mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y$$

have a projective resolution as a complex ,

$$(\mathscr{B}E(T), \delta)$$

finding the complex is one of those problems which are

solvable in principle, though not in practice.

To find the resolution

$$(\mathscr{B}E(T), \delta)$$

one has to compute which module of the algebra

$$A = hom^{*,*}(T, T)$$

the brane maps to by the Yoneda functor,

$$hom(T, -) : \mathscr{D}_Y \longrightarrow \mathscr{D}_A$$

and then find the resolution of this module.

I will describe how to solve both of these problems at once.

We can describe the story

for the

$\mathfrak{su}_2$ and $\mathfrak{gl}_{1|1}$

theories largely in parallel.

The key difference is the fact that, in the $\mathfrak{su}_2$ case,

the resulting complexes are ordinary complexes,

whereas for $\mathfrak{gl}_{1|1}$ they are twisted complexes.

As a warmup, we will start by considering

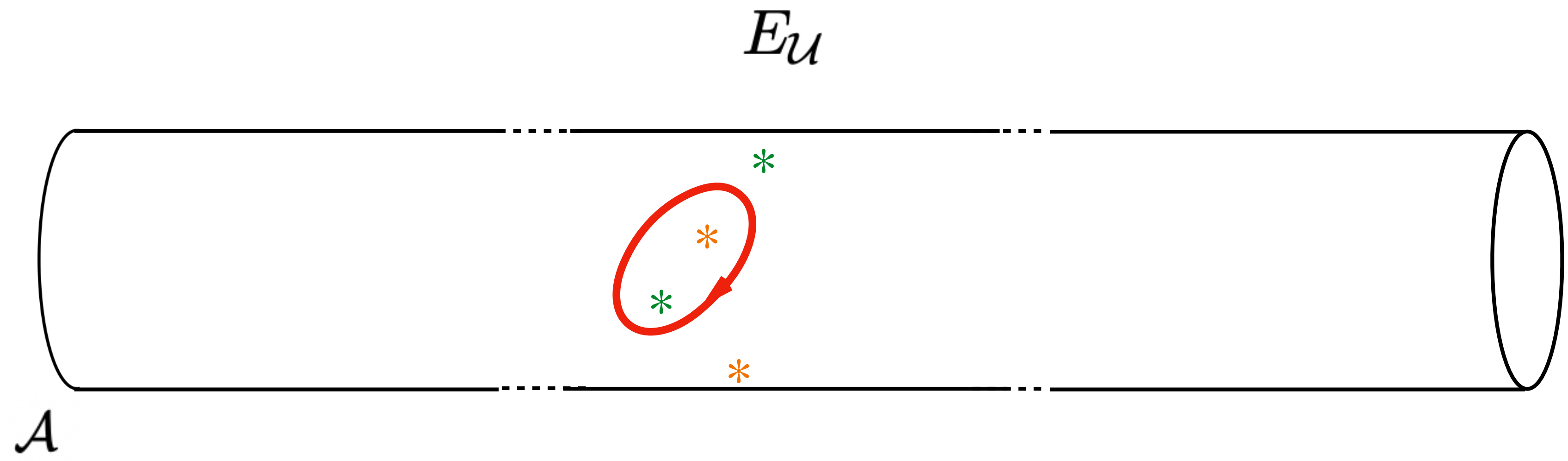
$$d = 1$$

theories, when our target is simply the Riemann surface itself.

$$Y = \mathcal{A}$$

This case is fundamental for all that will follow.

Consider the cap brane

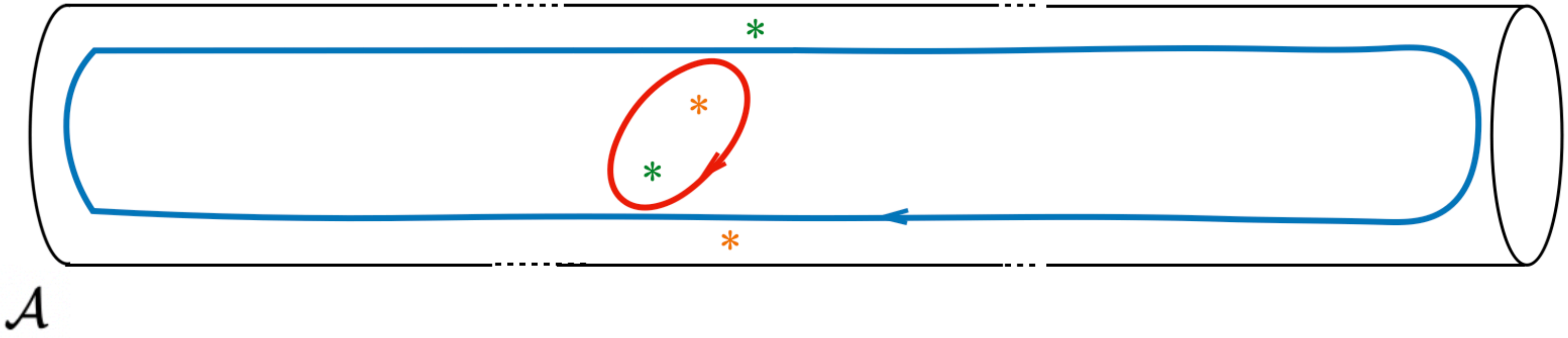


in the one dimensional theory.

The $E_{\mathcal{U}}$ brane is a connected sum of two one-dimensional T-branes

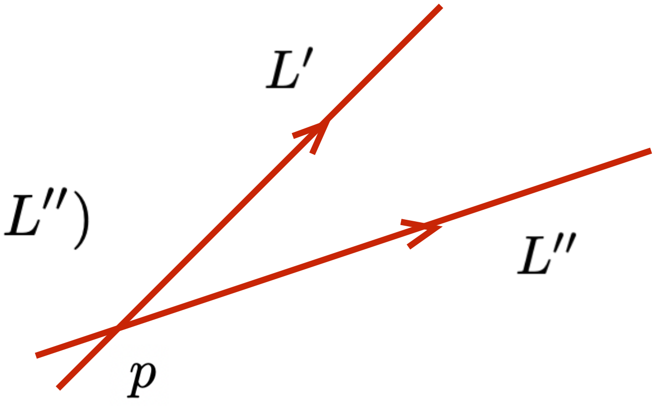
over their intersection points at infinity,

as one can see by isotoping the brane:

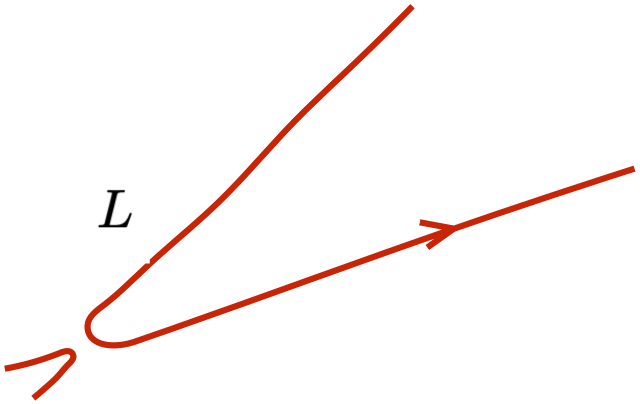


Recall that, if two one dimensional Lagrangians, intersect over a point

$$p \in Hom_{\mathscr{D}_Y}(L', L'')$$



we get a new one dimensional Lagrangian

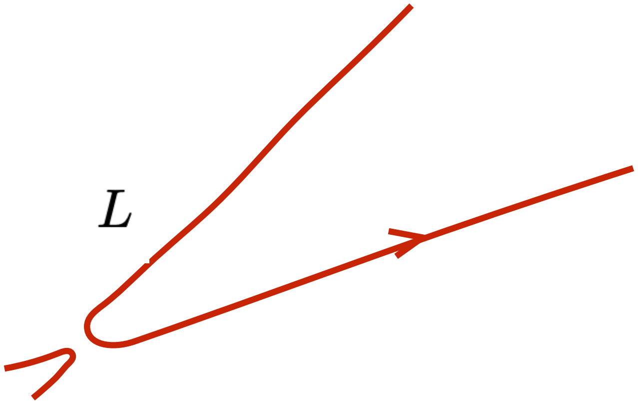


by starting with

$$L'[1] \oplus L''$$

and taking their connected sum at p .

As object of $\mathscr{D}_Y$, the the connected sum brane

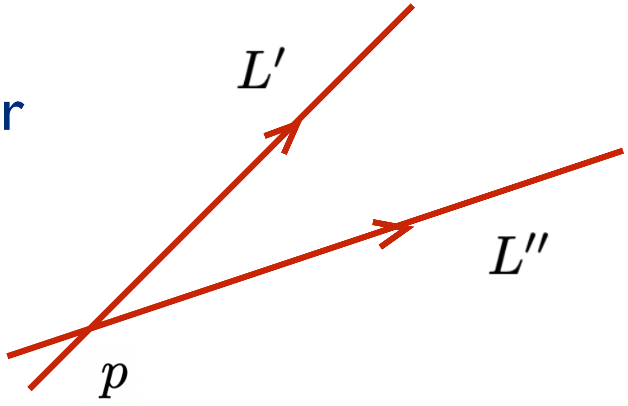


is equivalent, as an object of the derived category to the complex

$$L \cong L' \xrightarrow{p} L''$$

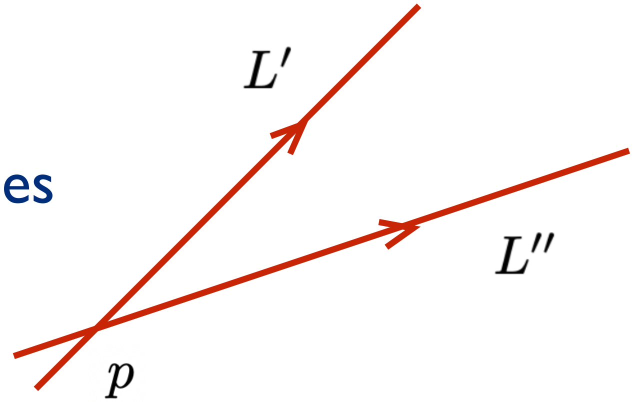
which is known as the cone over

$$p \in Hom_{\mathscr{D}_Y}(L', L'')$$



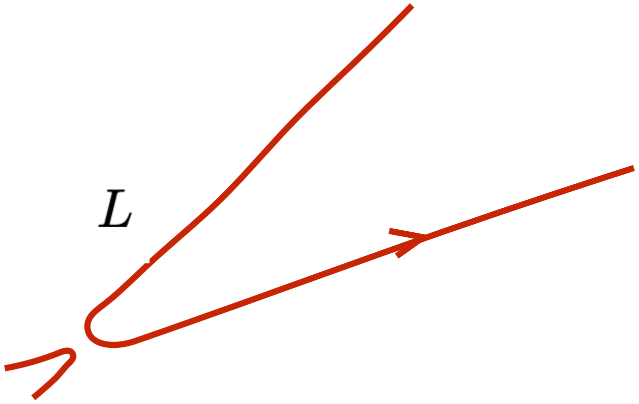
Thus, in one dimension, taking cones

$$L \;\cong\; L' \xrightarrow{p} L''$$



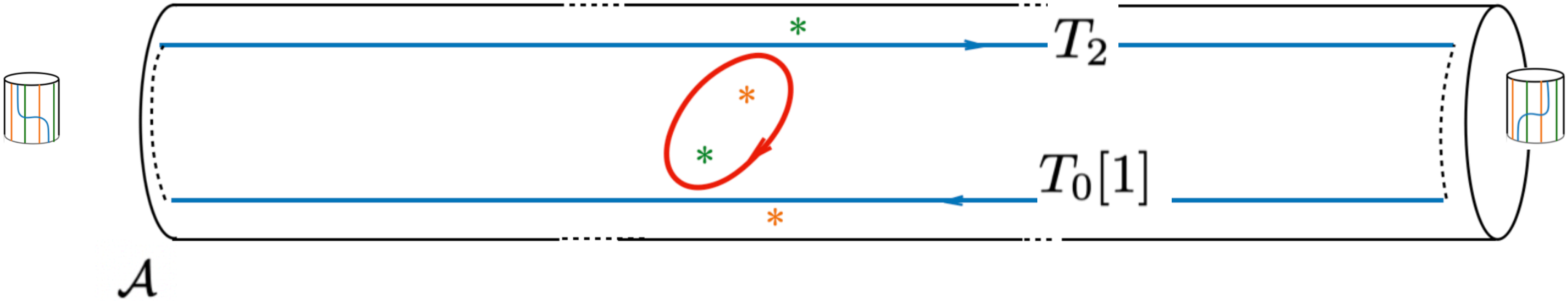
over morphisms in the derived category

has a geometric interpretation of taking connected sums.



The $E_{\mathcal{U}}$ is obtained starting from the direct sum brane:

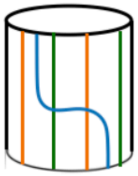
$$E_{\mathcal{U}}(T) = T_2 \oplus T_0[1]$$



and taking connected sums over the intersection points at infinity,

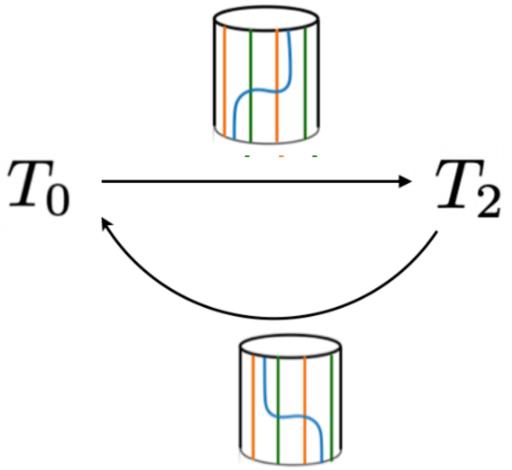
each of which known "name" in the algebra, and degree.

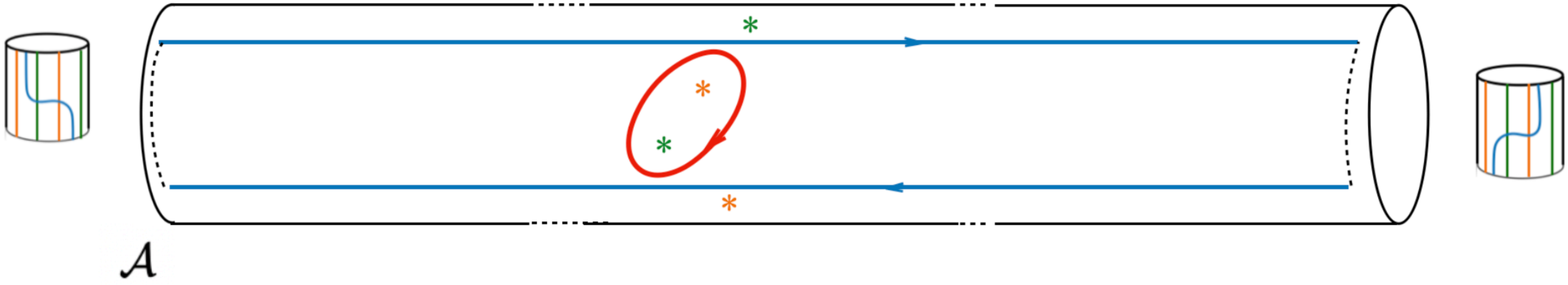
$$\in hom(T_0, T_2) = hom(T_0[1], T_2[0+1]).$$



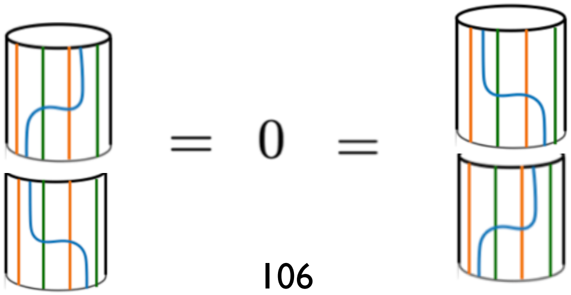
$$\in hom(T_2, T_0[1+1]),$$

The result is the twisted complex:





whose differential "closes", i.e. squares to zero in the algebra:



This is an algorithm for finding projective

resolutions of arbitrary branes of

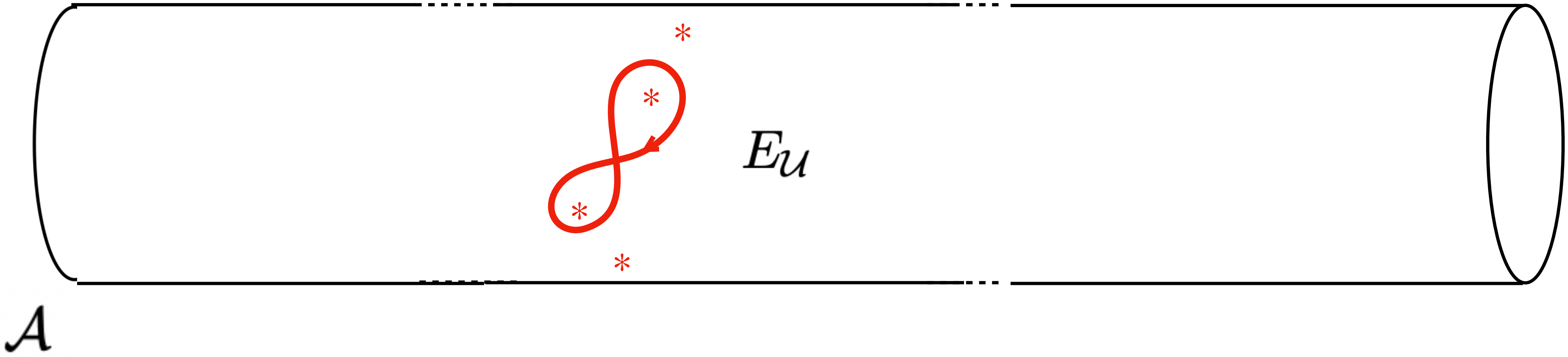
$\mathscr{D}_Y$

for both

$\mathfrak{gl}_{1|1}$ and $\mathfrak{su}_2$

theories as long as the target is (a product of) one dimensional theories

Consider, for example, the cap brane



in the $\mathfrak{su}_2$ theory

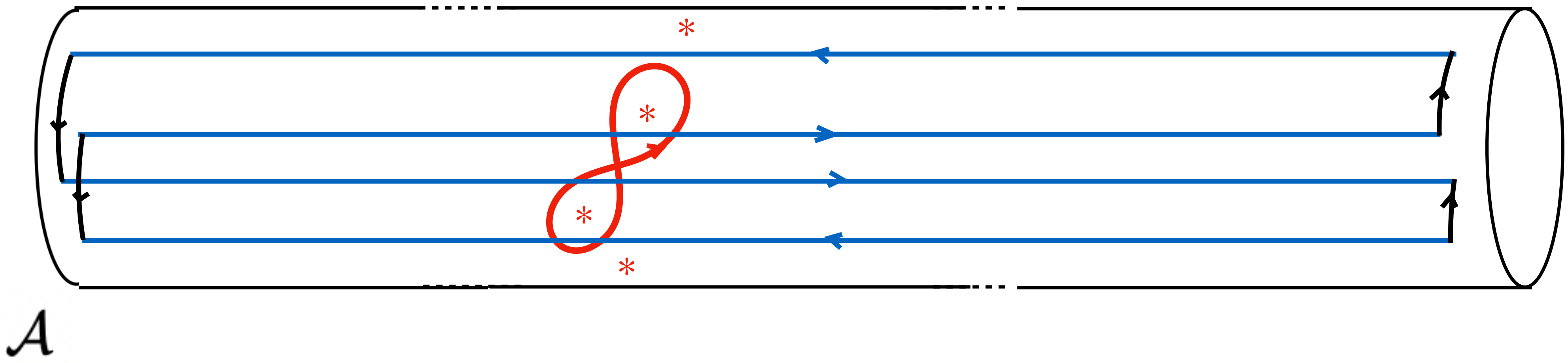
We recover the

$$E_{\mathcal{U}}$$

brane from the direct sum of T- branes

$$E_{\mathcal{U}}(T)$$

by taking connected sums over intersection points at infinity

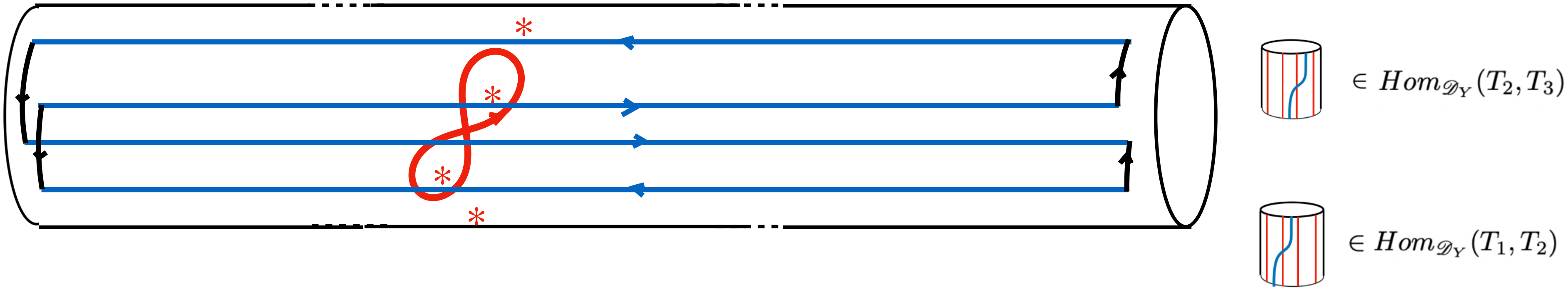


understood as Hom's in the wrapped Fukaya category.

Each intersection point at infinity is a specific

element of the algebra

of definite equivariant degree and zero cohomological degree.



$\in Hom_{\mathscr{D}_Y}(T_2,T_3)$

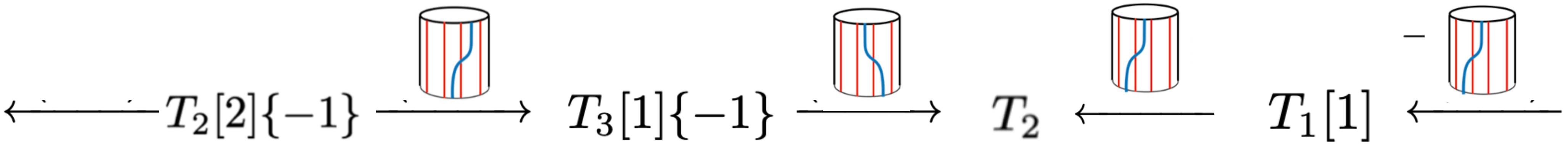
$\in Hom_{\mathscr{D}_Y}(T_1,T_2)$

To write the corresponding complex,

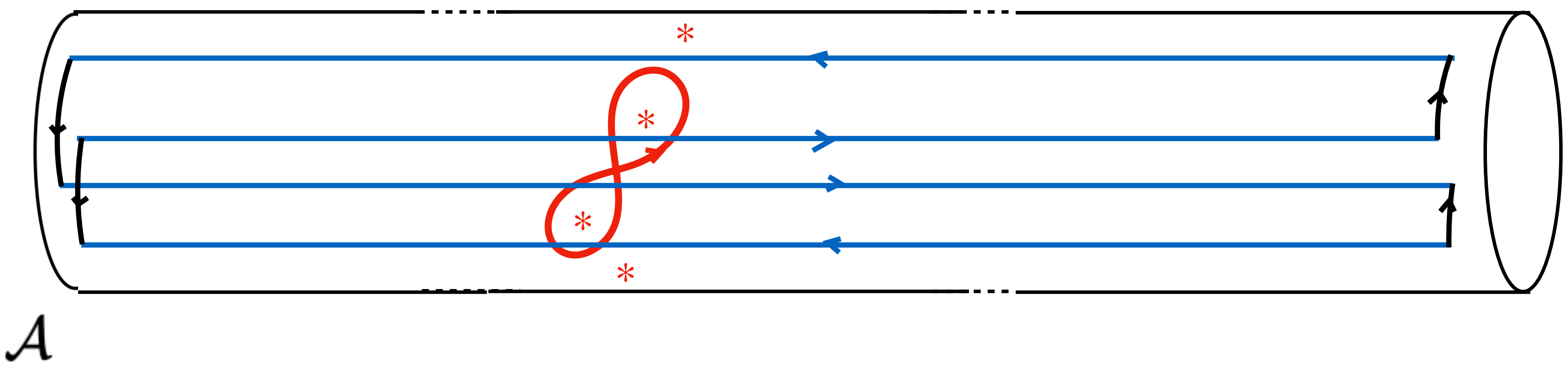
we can start on any one T-brane, and record the Hom's we find,

as we go around the brane,

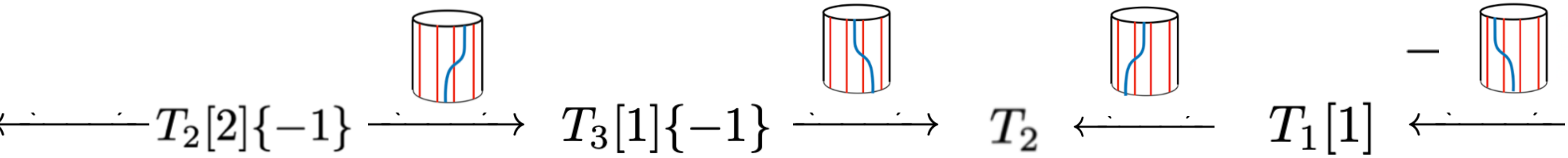
shifting degrees accordingly



$$\begin{tikzcd} {} & T_2[2]\{-1\} \arrow[l] \arrow[r] & T_3[1]\{-1\} \arrow[r] & T_2 & T_1[1] \arrow[l] & {} \arrow[l, "-"'] \end{tikzcd}$$

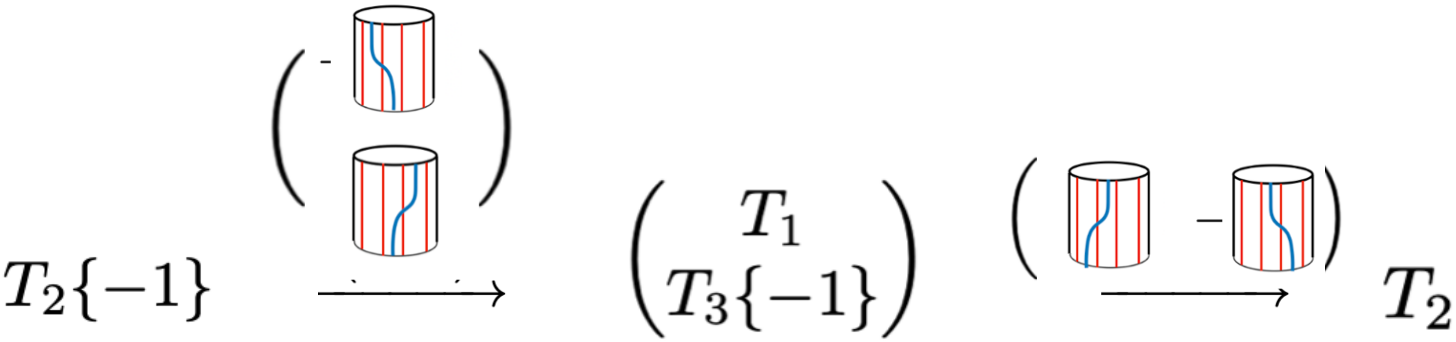


Organizing the result by cohomological degree



$$\begin{tikzcd} {} & T_2[2]\{-1\} \arrow[l] \arrow[r] & T_3[1]\{-1\} \arrow[r] & T_2 & T_1[1] \arrow[l] & {} \arrow[l, "-"'] \end{tikzcd}$$

we get the complex resolving the $E_{\mathcal{U}}$ brane



$$\begin{tikzcd} T_2\{-1\} \arrow[r] & \begin{pmatrix} T_1 \\ T_3\{-1\} \end{pmatrix} \arrow[r] & T_2 \end{tikzcd}$$

With the differential that squares to zero.

Now, consider the theory for general

$$d$$

We will start by describing the complexes resolving the branes not on

$$Y$$

itself, but on

$$Y_0 = Y \backslash \Delta$$

Our branes are products of one dimensional ones,

$$\mathscr{B}E_{\mathcal{U}} = \mathscr{B}E_1 \times \cdots \times \mathscr{B}E_d$$

and by working in

$$Y_0 = Y \backslash \Delta$$

all the holomorphic maps to Y which are not products of one dimensional maps to copies of $\mathcal{A}$ are removed.

As a result, the complex which describes the brane

$$\mathscr{B}E_{\mathcal{U}} = \mathscr{B}E_1 \times \cdots \times \mathscr{B}E_d$$

as an object of the category

$$\mathscr{D}_{Y_0} \cong \mathscr{D}_{A_0}$$

corresponding to

$$Y_0 = Y \backslash \Delta$$

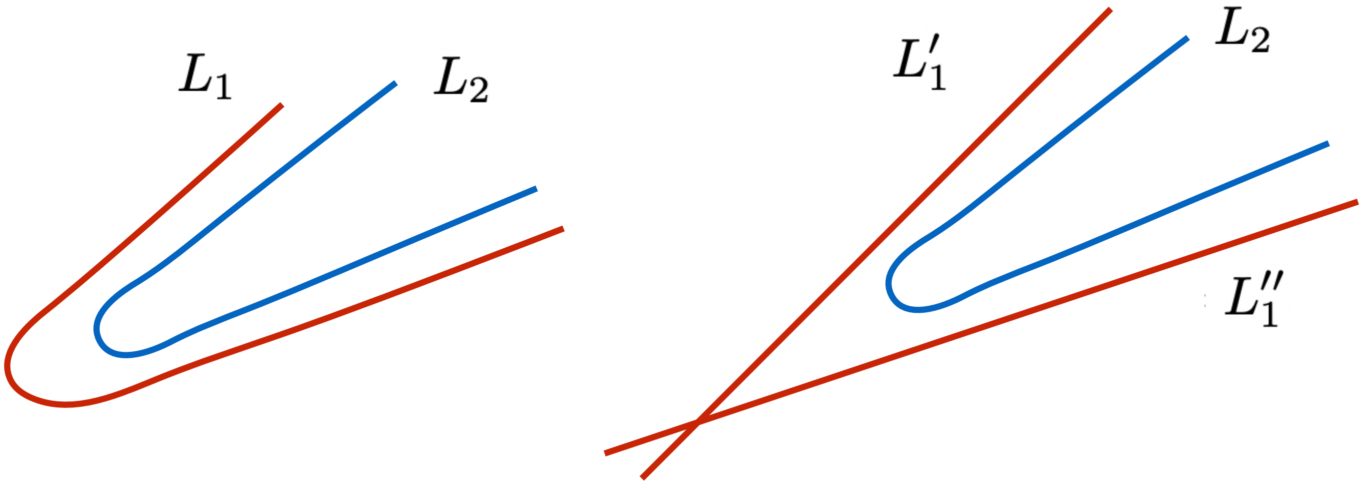
with the divisor of diagonal removed

is a product of one dimensional complexes.

Geometrically, each map in the product complex

is a cone over an intersection point of the form

$$\mathcal{P} = (p, id_{L_2}, ..., id_{L_d})$$



Having found the complex resolving the brane in

$$\mathscr{D}_{Y_0} \cong \mathscr{D}_{A_0}$$

we will use the fact that the theory on

$$Y$$

is the $\hbar$ - deformation of the theory on

$$Y_0 = Y \backslash \Delta$$

to find the complexes resolving the brane in

$$\mathscr{D}_{Y} \cong \mathscr{D}_{A_\hbar}$$

We will start with a very simple example,

in the $\mathfrak{gl}_{1|1}$ theory

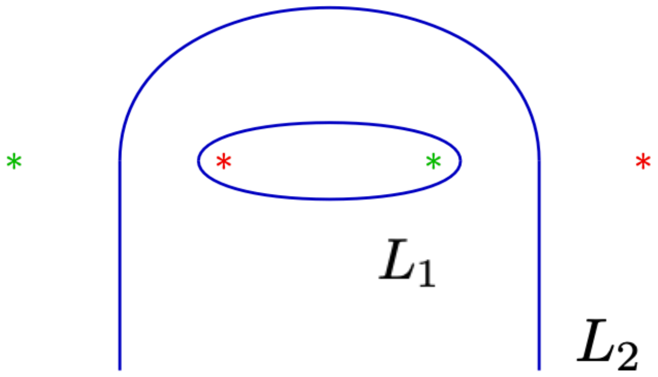
which will also serve to illustrate how,

after the deformation,

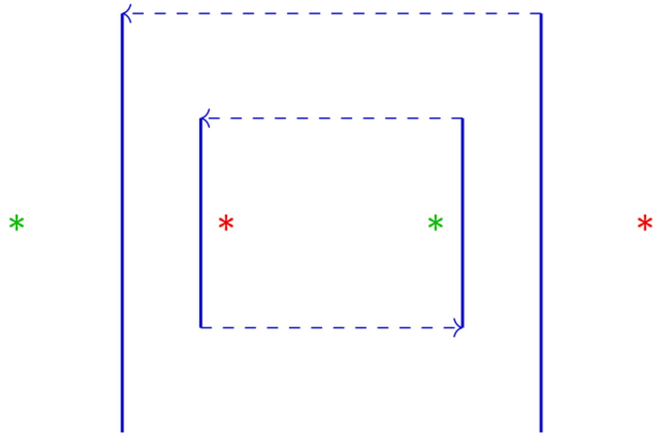
the complexes encode

counts of holomorphic maps passing through the diagonal.

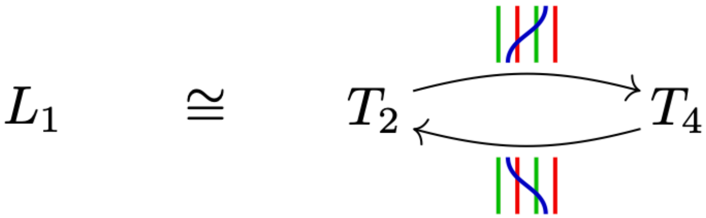
Consider the following brane:



which, broken up into thimbles looks like:

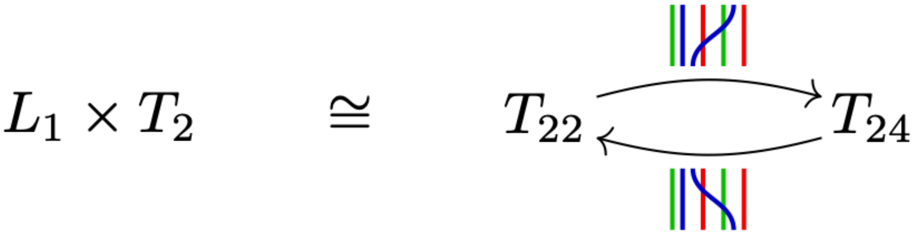


By itself the L_1 brane has resolution:

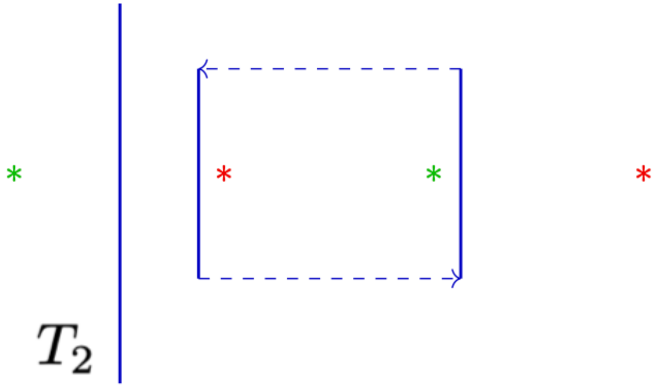


$$\begin{tikzcd} L_1 \arrow[r, phantom, "\cong"] & T_2 \arrow[r, bend left] & T_4 \arrow[l, bend left] \end{tikzcd}$$

Taking a product with one thimble in L_2 , we get:

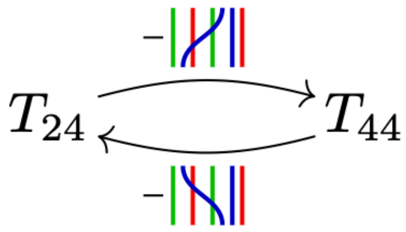


$$\begin{tikzcd} L_1 \times T_2 \arrow[r, phantom, "\cong"] & T_{22} \arrow[r, bend left] & T_{24} \arrow[l, bend left] \end{tikzcd}$$

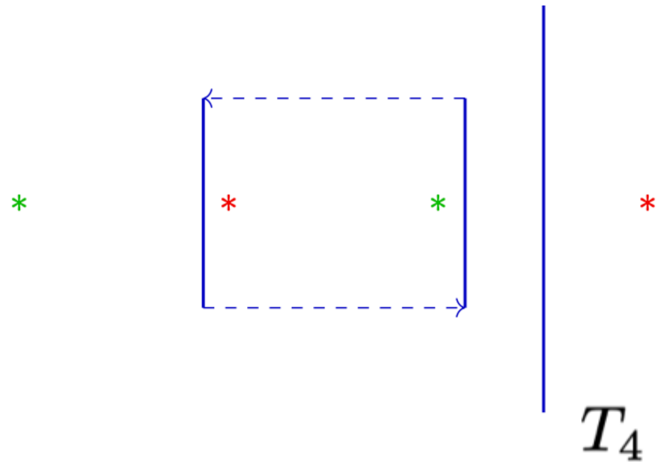


The maps keep track of the relative positions of the branes.

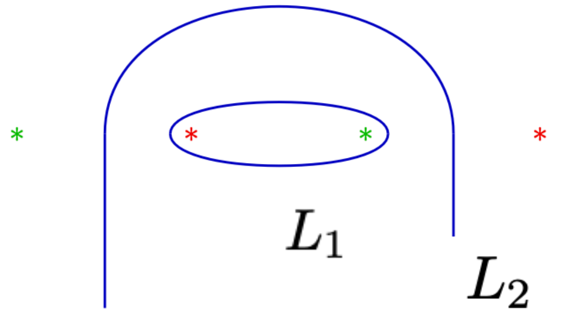
The product with the other thimble gives:



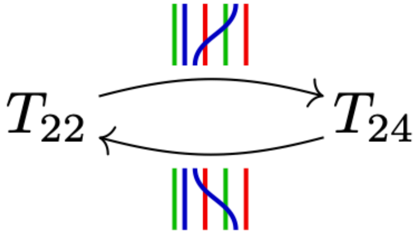
$$\begin{tikzcd} L_1 \times T_4 \arrow[r, phantom, "\cong"] & T_{24} \arrow[r, bend left, "-"] & T_{44} \arrow[l, bend left, "-"] \end{tikzcd}$$



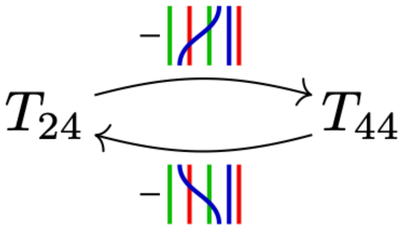
To get the complex corresponding to



we start with the direct sum of the complexes we just got,



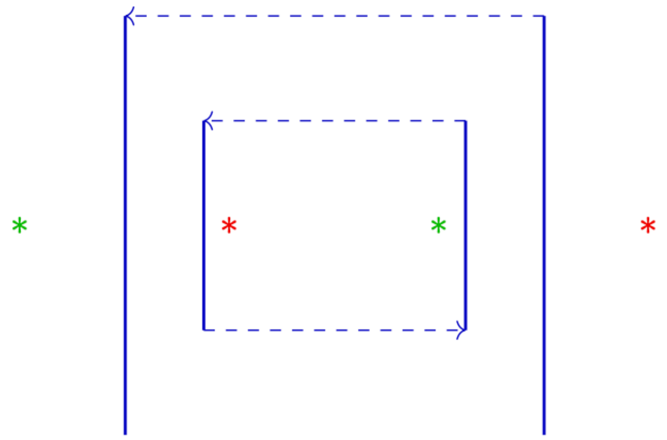
$$\begin{tikzcd} L_1 \times T_2 \arrow[r, phantom, "\cong"] & T_{22} \arrow[r, bend left] & T_{24} \arrow[l, bend left] \end{tikzcd}$$



$$\begin{tikzcd} L_1 \times T_4 \arrow[r, phantom, "\cong"] & T_{24} \arrow[r, bend left, "-"] & T_{44} \arrow[l, bend left, "-"] \end{tikzcd}$$

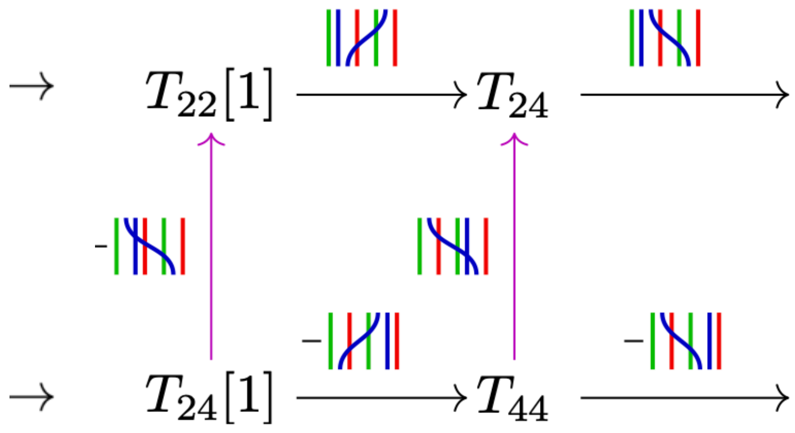
and turn on the maps that glue L_2

The complex corresponding to



lives on, $S^1 \times I$ reflecting the topology of the brane.

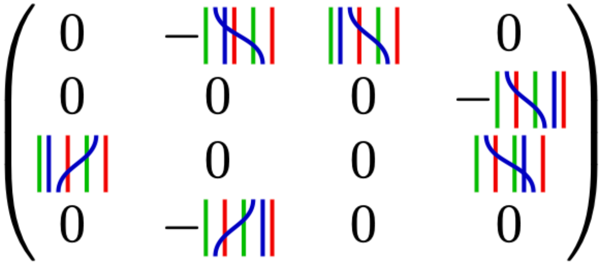
Writing all the Maslov degrees explicitly:



$$\begin{tikzcd}
{} \arrow[r] & T_{22}[1] \arrow[r] & T_{24} \arrow[r] & {} \\
{} \arrow[r] & T_{24}[1] \arrow[u, "-"] \arrow[r, "-"] & T_{44} \arrow[u] \arrow[r, "-"] & {}
\end{tikzcd}$$

The resulting twisted differential

$$\delta_0 =$$



closes in A_0 , and defines an object of

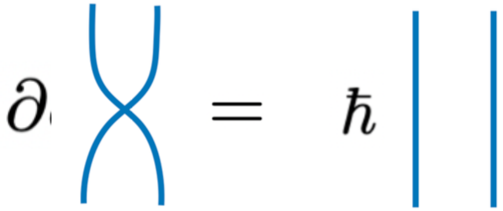
$$\mathscr{D}_{Y_0} \cong \mathscr{D}_{A_0}$$

but it does not close in $A_\hbar$.

For the differential to close in $A_\hbar$ we need that

$$\partial\delta + \delta \cdot \delta = 0.$$

where ∂ is the differential of the algebra itself.



The differential we just found satisfies $\delta_0 \cdot \delta_0 = 0$ but

$$\partial\delta_0 \neq 0$$

To find the differential that squares to zero,

we look for deformations of the differential

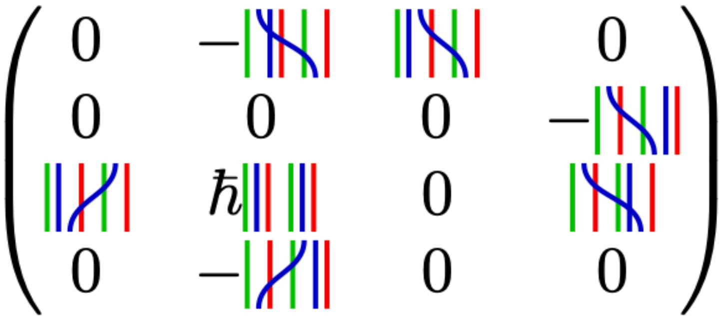
$$\delta = \delta_0 + \hbar\delta_1 + \ldots$$

which are consistent with the Maslov and equivariant gradings

and which closed in the full algebra $A_\hbar$.

There is a single correction allowed by the grading, and the full differential is

$\delta =$



which corresponds to

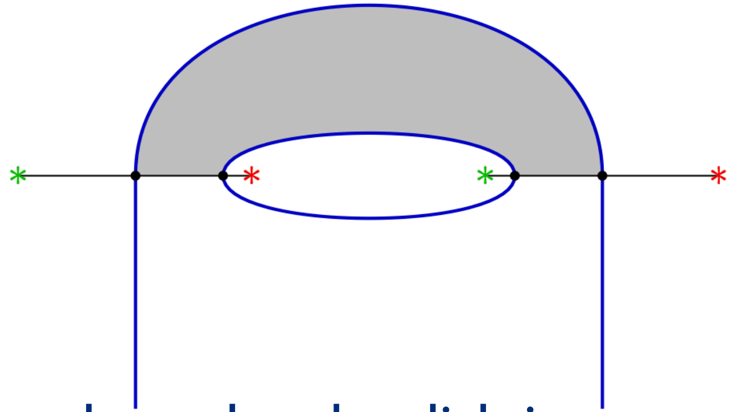
$$\begin{tikzcd}
{} \arrow[r] & T_{22}[1] \arrow[r] & T_{24} \arrow[r] & {} \\
{} \arrow[r] & T_{24}[1] \arrow[r] \arrow[u] \arrow[ur] & T_{44} \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

Starting with the twisted complex,

$$\begin{tikzcd}
{} \arrow[r] & T_{22}[1] \arrow[r] & T_{24} \arrow[r] & {} \\
{} \arrow[r] & T_{24}[1] \arrow[r] \arrow[u] \arrow[ur, "\hbar|||\,|||"] & T_{44} \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

and computing the Hom with

$I_{24} = I_2 \times I_4$ brane dual to $T_{24} = T_2 \times T_4$



The term we just found enodes the disk instanton intersecting the

diagonal Δ once.

The algorithm can be run, in principle, for arbitrarily complicated branes,

starting with the(twisted) products of one dimensional complexes,

which are elementary to write down

and which by construction describe the brane in

$$\mathscr{D}_{Y_0} \cong \mathscr{D}_{A_0}$$

and then finding the deformation of the resulting differential to that in

$$\mathscr{D}_{Y} \cong \mathscr{D}_{A_\hbar}$$

This generalizes to an explicit algorithm for computing arbitrary

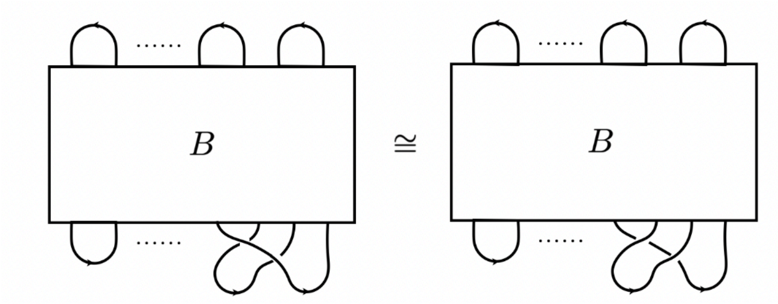
$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

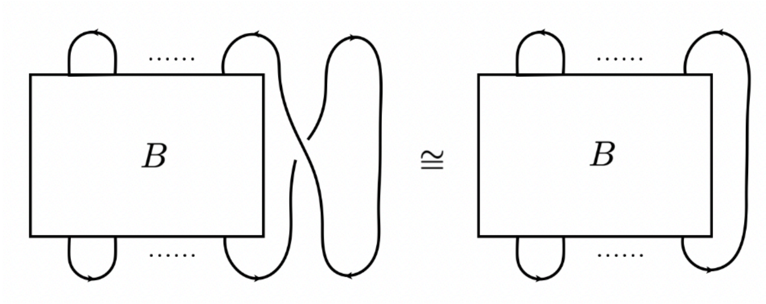
link homologies.

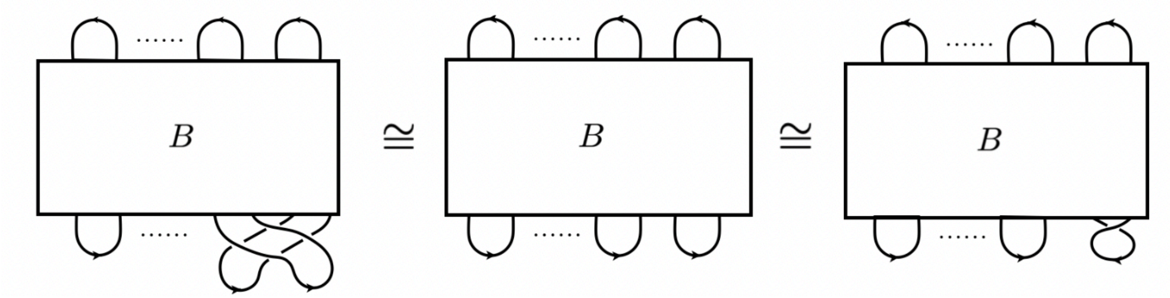
Theorem :

Homology groups $Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$ are invariants of links.

To prove the theorem is to prove that Markov moves for plat closures hold:



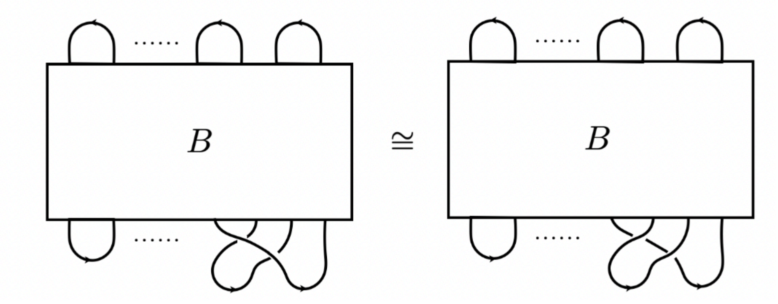




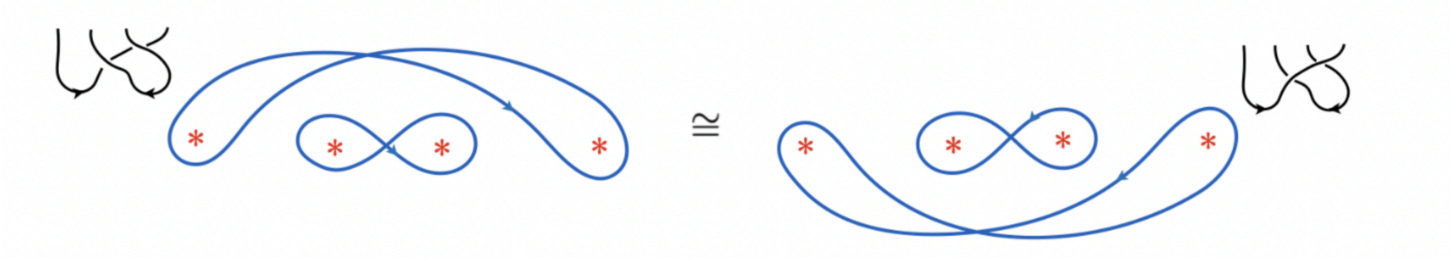
We prove the moves hold as a consequence of equivalences

satisfied by the $\mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y$ branes

The first move

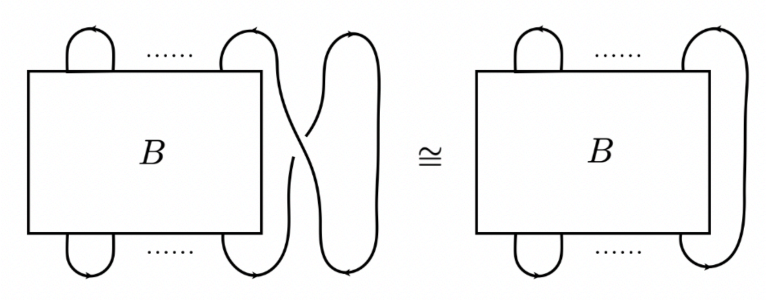


translates, in the $\mathfrak{su}_2$ context to

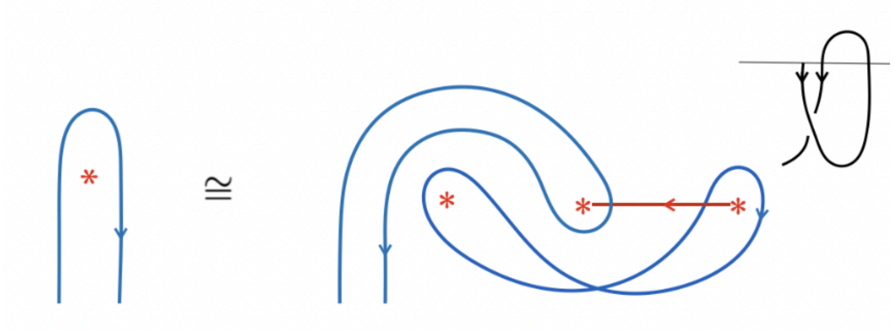


and the analogous statement for $\mathfrak{gl}_{1|1}$

The second

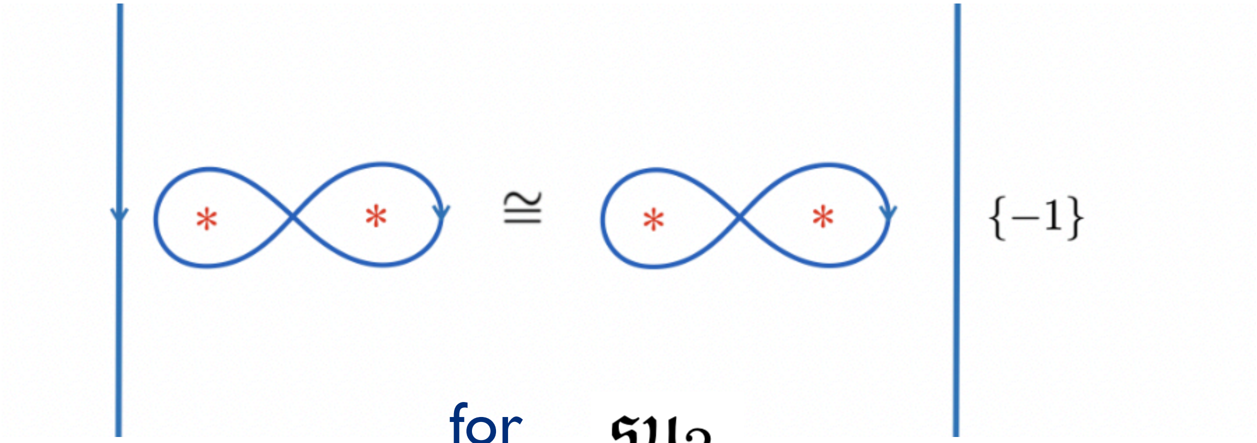


translates to

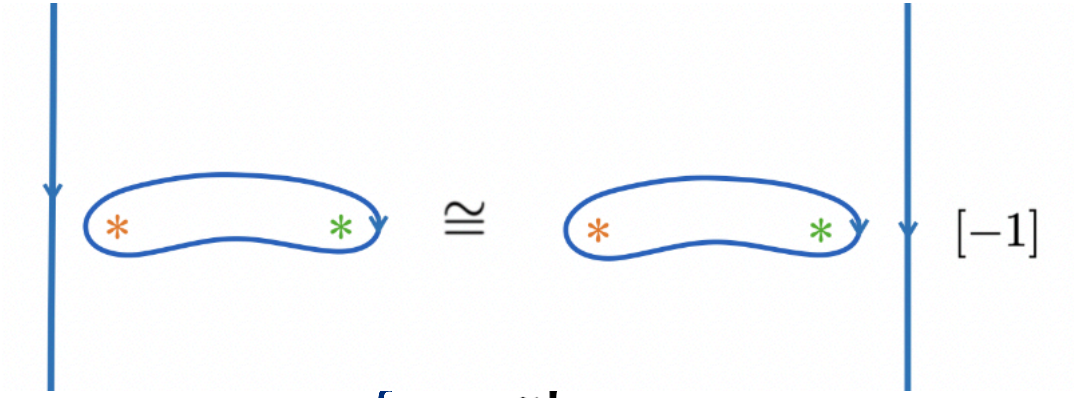


for $\mathfrak{su}_2$ and the analogous statement for $\mathfrak{gl}_{1|1}$

Both, follow, up to degree shifts, from a single statement:



for $\mathfrak{su}_2$

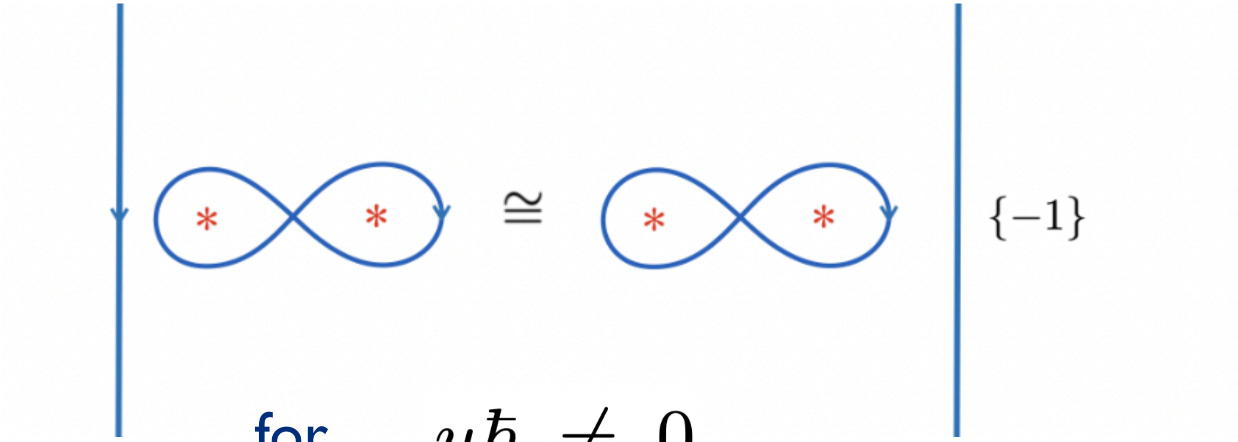


for $\mathfrak{gl}_{1|1}$

Both, equivalences easy to prove in our framework,

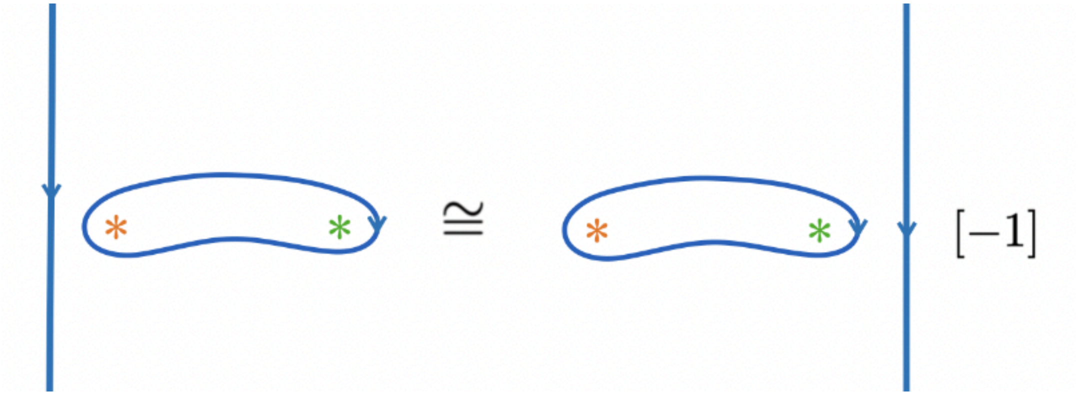
as they are homotopy equivalences of the underlying complexes

$\mathfrak{su}_2$



for $u\hbar \neq 0$

$\mathfrak{gl}_{1|1}$



for $\hbar \neq 0$

We have explicit Mathematica programs

written for the 2-strands algebra, which compute

link homologies of arbitrary 2-bridge knots,

and reduced link homologies of 3-bridge links

in the $\mathfrak{su}_2$ and $\mathfrak{gl}_{1|1}$ theories.

Nothing stands in the way to generalizing them to arbitrary numbers

strands, and of cups and caps.

In the $\mathfrak{su}_2$ case,

in all cases we checked, the theory reproduces Khovanov homology

including torsion.

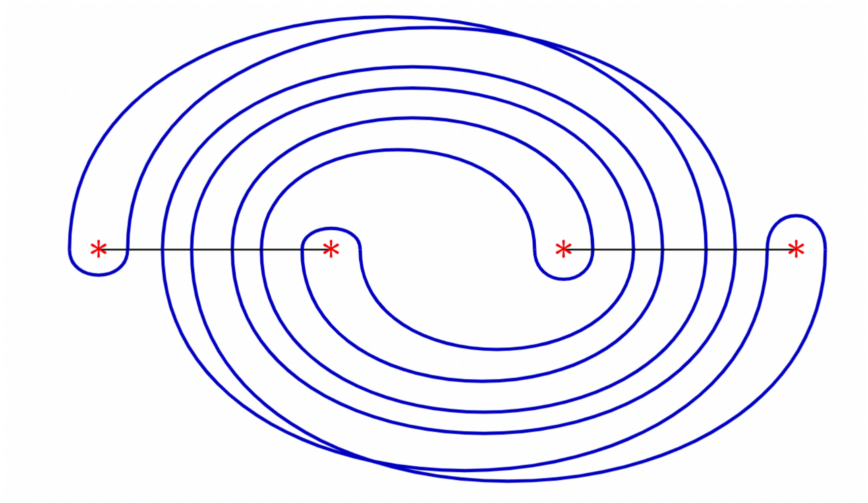
The size of the Floer complex,

whose cohomology is link homology

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \mathrm{Ker}\,\delta_F / \mathrm{Im}\,\delta_F$$

grows polynomially, rather than exponentially in the number of crossings.

For example, for the trefoil, we get the following pair of branes.



The theory computes the Floer complex, over $\mathbb{C}$, as:

$$\mathbb{C} \to \emptyset \to \begin{matrix}\mathbb{C}\{-2\}\\ \mathbb{C}\{-2\}\\ \mathbb{C}\{-1\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\\-1&-1&0\end{pmatrix}} \begin{matrix}\mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-2\}\\ \mathbb{C}\{-2\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\-1&-1&0&0\\0&0&-1&-1\end{pmatrix}} \begin{matrix}\mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-2\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}0&0&0&0\\0&0&0&0\\-1&-1&0&0\\1&1&0&0\end{pmatrix}} \begin{matrix}\mathbb{C}\{-4\}\\ \mathbb{C}\{-4\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}1&1&0&0\\0&0&1&1\end{pmatrix}} \begin{matrix}\mathbb{C}\{-4\}\\ \mathbb{C}\{-3\}\end{matrix}$$

The complex is 18-dimensional,

which should be compared to 30 dimensional complex Khovanov defined.

The same complex

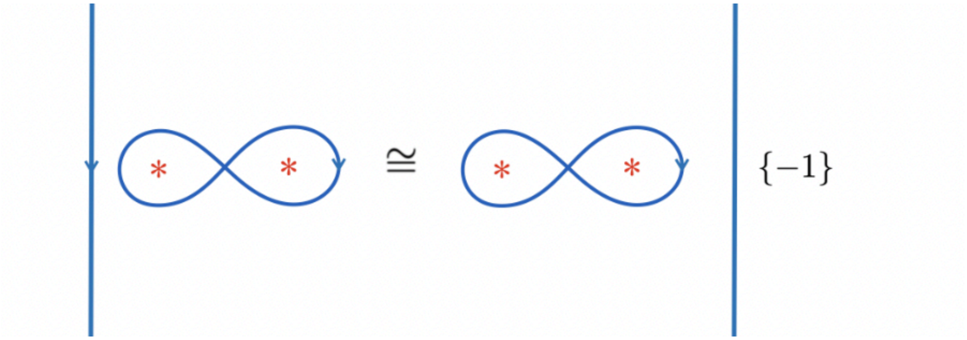
$$\mathbb{C} \to \emptyset \to \begin{matrix}\mathbb{C}\{-2\}\\ \mathbb{C}\{-2\}\\ \mathbb{C}\{-1\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\\-1&-1&0\end{pmatrix}} \begin{matrix}\mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-2\}\\ \mathbb{C}\{-2\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\-1&-1&0&0\\0&0&-1&-1\end{pmatrix}} \begin{matrix}\mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-2\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}0&0&0&0\\0&0&0&0\\-1&-1&0&0\\1&1&0&0\end{pmatrix}} \begin{matrix}\mathbb{C}\{-4\}\\ \mathbb{C}\{-4\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}1&1&0&0\\0&0&1&1\end{pmatrix}} \begin{matrix}\mathbb{C}\{-4\}\\ \mathbb{C}\{-3\}\end{matrix}$$

with $\mathbb{C}$ replaced with $\mathbb{Z}$ reproduces Khovanov homology

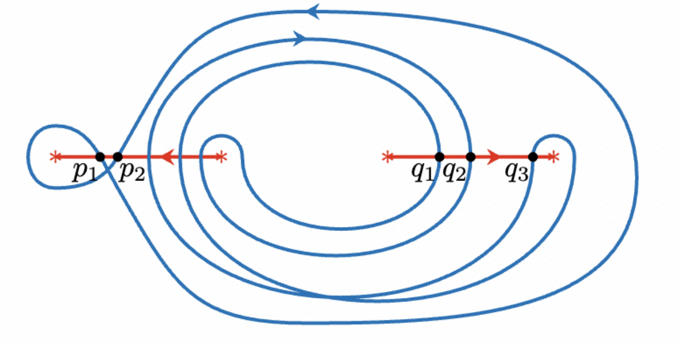
$$\mathbb{Z} \oplus \mathbb{Z}[2]\{-2\} \oplus \mathbb{Z}[2]\{-1\} \oplus \mathbb{Z}_2[4]\{-3\} \oplus \mathbb{Z}[5]\{4\}$$

including torsion.

The swiping move



can be used to simplify the brane to an equivalent one:



Its resolution leads to a complex which is only 6 dimensional

$$\mathbb{C} \to 0 \to \begin{matrix}\mathbb{C}\{-2\}\\ \mathbb{C}\{-1\}\end{matrix} \xrightarrow{0} \mathbb{C}\{-3\} \xrightarrow{-2u^2\hbar^2} \mathbb{C}\{-3\} \xrightarrow{0} \mathbb{C}\{-4\}$$

with homology that is 5-dimensional, including torsion:

$$\mathbb{Z} \oplus \mathbb{Z}[2]\{-2\} \oplus \mathbb{Z}[2]\{-1\} \oplus \mathbb{Z}_2[4]\{-3\} \oplus \mathbb{Z}[5]\{4\}$$