The primes of bad reduction of the modular star quotient $X_0(N)^*$

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Modular curves

- For $N \in \mathbb{Z}_{>0}$, the modular curve $Y_1(N)$ classifies pairs (E, P) up to isomorphism of elliptic curves together with a point of order N.
- Similarly, $Y_0(N)$ classifies pairs (E, C_N) up to isomorphism of elliptic curves E together with a cyclic subgroup C_N of order N.
- The point (E, C_N) can equivalently be viewed as (E, ι), where $\iota\colon E \to E/C_N =: E'$ is an isogeny whose kernel is cyclic of order N.
- The curves $X_1(N)$ and $X_0(N)$ are the compactifications of $Y_1(N)$ and $Y_0(N)$, respectively. These curves are defined over $\mathbb{Z}\left[\frac{1}{N}\right]$.

Atkin–Lehner Quotients

Let $d > 1$ be a Hall divisor of N, i.e. $\gcd(d, N/d) = 1$.
We write $d \parallel N$.

The Atkin–Lehner involution w_d is given by

$$w_d \colon (E, C_N) \mapsto (E/C_d, (C_N + E[d])/C_d).$$

Let $W(N) := \langle w_d : d \parallel N \rangle$.

Consider the quotients

$$X_0(N)^+ := X_0(N)/\langle w_N \rangle,$$
$$X_0(N)^* := X_0(N)/W(N).$$

Rational points on $X_0(N)^*$ correspond to $\mathbb{Q}$-curves defined over multi-quadratic extensions of $\mathbb{Q}$.

Motivation

- Rational points on $X_0(N)^*$ correspond to $\mathbb{Q}$-curves defined over multi-quadratic extensions of $\mathbb{Q}$.
- Knowing $X_0(N)^+(\mathbb{Q})$, an "extremely interesting arithmetic question" (Mazur), is helpful in determining all quadratic points on $X_0(N)$.
- Elkies' conjecture: For $N \gg 0$, $X_0(N)^*$ consists only of cusps and CM points.
- Hasegawa proved that there are exactly 64 levels N for which $X_0(N)^*$ is hyperelliptic.

Hyperelliptic $X_0(N)^*$

- Hasegawa proved that there are exactly 64 levels N for which $X_0(N)^*$ is hyperelliptic.
- Adžaga-Chidambaram-Keller-P. completed the computation of $X_0(N)^*(\mathbb{Q})$ for the 15 remaining levels in Hasegawa's table.
- For $N \in \{67, 73, 103, 7 \cdot 19\}$, we notice the following isomorphisms
$$X_0(2N)^* \cong X_0(N)^*.$$
- These isomorphisms are partially explained by the fact that for $N \in \{67, 73, 103, 7 \cdot 19\}$, the curve $X_0(2N)^*$ has good reduction at 2.

Exceptional Isomorphisms

For distinct positive integers N_1, N_2 and Atkin–Lehner subgroups $W_1 \leq W(N_1), W_2 \leq W(N_2)$, if

$$X_0(N_1)/W_1 \cong X_0(N_2)/W_2,$$

then we say that this isomorphism is exceptional.

Question: Are there only finitely many exceptional isomorphisms, and if so, can we give a complete classification?

Infinitely many exceptional isomorphisms

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Answer: No.

Proposition (Hasegawa)

Let $M > 1$ be an odd integer. Let $W = \langle W_4, W_{M_1}, \cdots, W_{M_s} \rangle \leq W(4M)$ with $M_i \parallel M$ and $W' = \langle W_{M_1}, \cdots, W_{M_s} \rangle \leq W(2M)$. Then

$$X_0(4M)/W \cong X_0(2M)/W'.$$

Refined question

Refined question: Are there finitely many isomorphisms

$$X_0(N_1)/W_1 \cong X_0(N_2)/W_2$$

when $\mathrm{rad}(N_1) \neq \mathrm{rad}(N_2)$?

In this case, for at least one of $i \in \{1, 2\}$, $X_0(N_i)/W_i$ would have fewer primes of bad reduction compared to $X_0(N_i)$.

We approach the question by considering the primes of bad reduction of $X_0(N)^*$.

Primes of bad reduction for $X_0(N)^*$

We give a positive answer to the refined question.

Theorem (P.-Voight 2023)

Let $N > 1$ be a square-free integer. The set of primes of bad reduction of $X_0(N)^*$ equals the set of prime divisors of N except for a finite, explicitly computable set of levels N.
This list classifies all exceptional isomorphisms

$$X_0(N_1)/W_1 \cong X_0(N_2)/W_2$$

for distinct and squre-free N_1,N_2 and $W_1 \leq W(N_1)$, $W_2 \leq W(N_2)$.

Outline of proof

Main ingredients of the proof:

- Atkin–Lehner–Li theory (newforms and oldforms), and
- Genus formula for $X_0(N)^*$.

Towards the proof

Consider

$$J_0(N)^* := \mathrm{Jac}(X_0(N)^*) \subset J_0(N).$$

We have that

$$J_0(N)^* \subseteq J_0(N) \sim \bigoplus_f A_f^{m_f},$$

where the sum is taken over a set of representatives $f \in S_2(\Gamma_0(M_f))$ at levels $M_f | N$, and each $m_f = \sigma_0(N/M_f)$.

Consider the subvariety

$$J_0(N)^{*,\mathrm{new}} \subset J_0(N)^*,$$

which is the union of isogeny factors corresponding to newforms of level N.

A sufficient condition

The condition

$$\dim(J_0(N)^{*,\mathrm{new}}) > 0$$

is sufficient (but not necessary) for $X_0(N)^*$ to have as primes of bad reduction all prime divisors of N.

Thus we would like to prove that

$$\dim(J_0(N)^{*,\mathrm{new}}) > 0$$

for $N \gg 0$.

In fact, we show that

$$\lim_{N\to\infty} \dim(J_0(N)^{*,\mathrm{new}}) = \infty.$$

We begin by computing dimensions of spaces of newforms.

Newforms

Let $N = p_1 \cdots p_k$ be square-free. Consider first $k = 2$. We have two embeddings

$$S_2(\Gamma_0(p_1)) \rightrightarrows S_2(\Gamma_0(p_1 p_2)).$$

Then $\dim(S_2(\Gamma_0(p_1 p_2))^{p_1-\text{new}, p_2-\text{new}})$ equals

$$\dim(S_2(\Gamma_0(p_1 p_2))) - 2\dim(S_2(\Gamma_0(p_1))) - 2\dim(S_2(\Gamma_0(p_2))).$$

Lemma

The dimension of $(S_2(\Gamma_0(p_1 p_2))^*)^{p_1-\text{new}, p_2-\text{new}}$ *equals*

$$\dim(S_2(\Gamma_0(p_1 p_2))^*) - \dim(S_2(\Gamma_0(p_1))^*) - \dim(S_2(\Gamma_0(p_2))^*).$$

We can generalize the computation for $k \geq 3$. Using the inclusion-exclusion principle, the coefficients are the same as before, and we obtain that

$$\dim(J_0(N)^{\text{new}}) = \sum_{\substack{d|N \\ d<N}} \mu(d) 2^{\omega(d)} g_{N/d},$$

while

$$\dim(J_0(N)^{*,\text{new}}) = \sum_{\substack{d|N \\ d<N}} \mu(d) g^*_{N/d}.$$

Computing genera of $X_0(N)$ and $X_0(N)^*$

Let g_N be the genus of $X_0(N)$. Then

$$g_N = 1 + \frac{\psi(N)}{12} - \frac{\nu_2(N)}{4} - \frac{\nu_3(N)}{3} - \frac{\nu_\infty(N)}{2}, \text{ where}$$

$$\nu_2(N) := \#\{x \in \mathbb{Z}/N\mathbb{Z} : x^2 + 1 = 0\}, \qquad \nu_3(N) := \#\{x \in \mathbb{Z}/N\mathbb{Z} : x^2 + x + 1 = 0\},$$

$$\psi(N) = N \prod_{p \mid N} \left(1 + \tfrac{1}{p}\right), \quad \nu_\infty(N) = \sum_{d \mid N} \varphi(\gcd(d, N/d)).$$

There is a similar formula for the genus g_N^* of $X_0(N)^*$:

$$g_N^* = 1 + \frac{g_N - 1}{2^{\omega(N)}} - \frac{1}{2^{\omega(N)+1}} \sum_{1 < d \| N} \nu(N, d),$$

where $\nu(N, d)$ denotes the number of fixed points of the involution w_d of $X_0(N)$.

Square root beats log

After considering each summand and manipulating (in)equalities, we obtain that

$$\dim(J_0(N)^{*,\mathrm{new}}) > \frac{1}{12}\prod_{p|N}\frac{p-1}{2} - \frac{7}{2\pi}\prod_{p|N}\left(\frac{\sqrt{p}\log(p)}{2}+1\right) > 0,$$

for $N \gg 0$.

Thus there are only finitely many square-free levels N for which $X_0(N)$ and $X_0(N)^*$ do not have the same set of primes of bad reduction.

Work in progress

- Provide complete table of exceptional isomorphisms explained by "loss" of bad primes and upload it to the LMFDB
- Remove the square-free condition
- Extend the result to more general modular curves and to quotients of Shimura curves

Thank You for Your Attention!