On Higher Order Weierstrass Points on $X_0(N)$ and beyond (joint with D. Mikoč) Representation theory XVIII, Dubrovnik

Goran Muić

June 21, 2023

Notation

Notation

$SL_2(\mathbb{R})$ is defined by

Notation

$SL_2(\mathbb{R})$ is defined by

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ;\ a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

Notation

$SL_2(\mathbb{R})$ is defined by

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

$\mathbb{H}$ is the upper half–plane:

Notation

$SL_2(\mathbb{R})$ is defined by

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

$\mathbb{H}$ is the upper half–plane: $Im(z) > 0$

Notation

$SL_2(\mathbb{R})$ is defined by

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

$\mathbb{H}$ is the upper half–plane: $Im(z) > 0$
$SL_2(\mathbb{R})$ acts on $\mathbb{H}$ in a well–known way

Notation

$SL_2(\mathbb{R})$ is defined by

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

$\mathbb{H}$ is the upper half–plane: $Im(z) > 0$
$SL_2(\mathbb{R})$ acts on $\mathbb{H}$ in a well–known way

$$g.z = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \quad z \in \mathbb{H}$$

Notation

$SL_2(\mathbb{R})$ is defined by

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ;\ a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

$\mathbb{H}$ is the upper half–plane: $Im(z) > 0$
$SL_2(\mathbb{R})$ acts on $\mathbb{H}$ in a well–known way

$$g.z = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \quad z \in \mathbb{H}$$

Notation

$SL_2(\mathbb{R})$ is defined by

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ a,b,c,d \in \mathbb{R}, ad - bc = 1 \right\}$$

$\mathbb{H}$ is the upper half–plane: $Im(z) > 0$
$SL_2(\mathbb{R})$ acts on $\mathbb{H}$ in a well–known way

$$g.z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \quad z \in \mathbb{H}$$

$\mathbb{H}$ can be regarded as a model for the hyperbolic plane with (invariant under $SL_2(\mathbb{R})$)

Notation

$SL_2(\mathbb{R})$ is defined by

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

$\mathbb{H}$ is the upper half–plane: $Im(z) > 0$
$SL_2(\mathbb{R})$ acts on $\mathbb{H}$ in a well–known way

$$g.z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \quad z \in \mathbb{H}$$

$\mathbb{H}$ can be regarded as a model for the hyperbolic plane with (invariant under $SL_2(\mathbb{R})$)

hyperbolic volume: $\frac{dxdy}{y^2}$

Notation

$SL_2(\mathbb{R})$ is defined by

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ;\ a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

$\mathbb{H}$ is the upper half–plane: $Im(z) > 0$
$SL_2(\mathbb{R})$ acts on $\mathbb{H}$ in a well–known way

$$g.z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \quad z \in \mathbb{H}$$

$\mathbb{H}$ can be regarded as a model for the hyperbolic plane with (invariant under $SL_2(\mathbb{R})$)

hyperbolic volume: $\frac{dxdy}{y^2}$

hyperbolic distance: $ds^2 = \frac{dx^2+dy^2}{y^2}$

Fuchsian groups of the first kind

we are interested in discrete subgroups Γ of $SL_2(\mathbb{R})$

Fuchsian groups of the first kind

we are interested in discrete subgroups Γ of $SL_2(\mathbb{R})$ traditionally called **Fuchsian groups**

we are interested in discrete subgroups Γ of $SL_2(\mathbb{R})$ traditionally called **Fuchsian groups**

the hyperbolic geometry can be used to construct nice fundamental domains $\mathcal{F}_\Gamma$ for the action of Γ on $\mathbb{H}$

we are interested in discrete subgroups Γ of $SL_2(\mathbb{R})$ traditionally called **Fuchsian groups**

the hyperbolic geometry can be used to construct nice fundamental domains $\mathcal{F}_\Gamma$ for the action of Γ on $\mathbb{H}$
Γ is a **Fuchsian group of the first kind** if $\iint_{\mathcal{F}_\Gamma} \frac{dxdy}{y^2} < \infty$

Fuchsian groups of the first kind

$\overset{Siegel}{\Longrightarrow}$

$\stackrel{Siegel}{\Longrightarrow}$ $\mathcal{F}_\Gamma$ is a polygon in the hyperbolic plane $\mathbb{H}$ with finitely many vertices: some of them might be at infinity

Fuchsian groups of the first kind

$\overset{Siegel}{\Longrightarrow}$ $\mathcal{F}_\Gamma$ is a polygon in the hyperbolic plane $\mathbb{H}$ with finitely many vertices: some of them might be at infinity $= \mathbb{R} \cup \{\infty\}$

Fuchsian groups of the first kind

$\overset{Siegel}{\Longrightarrow}$ $\mathcal{F}_\Gamma$ is a polygon in the hyperbolic plane $\mathbb{H}$ with finitely many vertices: some of them might be at infinity $= \mathbb{R} \cup \{\infty\}$

a Γ–conjugate of a vertex at infinity is **called cusp** for Γ

In what follows Γ always denotes a Fuchsian group of the first kind

Fuchsian groups of the first kind

$\overset{Siegel}{\Longrightarrow}$ $\mathcal{F}_\Gamma$ is a polygon in the hyperbolic plane $\mathbb{H}$ with finitely many vertices: some of them might be at infinity $= \mathbb{R} \cup \{\infty\}$

a Γ–conjugate of a vertex at infinity is **called cusp** for Γ

In what follows Γ always denotes a Fuchsian group of the first kind
The most important examples are congruence subgroups, and especially among them

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z});\ c \equiv 0 \ (mod \ N) \right\}$$

Fuchsian groups of the first kind

Fuchsian groups of the first kind

Let $\mathbb{H}^*$ be the union of $\mathbb{H}$ and the set of all cusps for Γ

Fuchsian groups of the first kind

Let $\mathbb{H}^*$ be the union of $\mathbb{H}$ and the set of all cusps for Γ the space $\mathfrak{R}_\Gamma$ of Γ–orbits for $\mathbb{H}^*$ has a structure of compact Riemann surface

Let $\mathbb{H}^*$ be the union of $\mathbb{H}$ and the set of all cusps for Γ
the space $\mathfrak{R}_\Gamma$ of Γ–orbits for $\mathbb{H}^*$ has a structure of compact Riemann surface

compact Riemann surface (analysis) is a complete (projective) non–singular irreducible algebraic curve over $\mathbb{C}$ (algebraic geometry)

Fuchsian groups of the first kind

Let $\mathbb{H}^*$ be the union of $\mathbb{H}$ and the set of all cusps for Γ
the space $\mathfrak{R}_\Gamma$ of Γ–orbits for $\mathbb{H}^*$ has a structure of compact Riemann surface

compact Riemann surface (analysis) is a complete (projective) non–singular irreducible algebraic curve over $\mathbb{C}$ (algebraic geometry)

we write $X_0(N)$ for the $\mathfrak{R}_{\Gamma_0(N)}$

Let $\mathbb{H}^*$ be the union of $\mathbb{H}$ and the set of all cusps for Γ
the space $\mathfrak{R}_\Gamma$ of Γ–orbits for $\mathbb{H}^*$ has a structure of compact Riemann surface

compact Riemann surface (analysis) is a complete (projective) non–singular irreducible algebraic curve over $\mathbb{C}$ (algebraic geometry)

we write $X_0(N)$ for the $\mathfrak{R}_{\Gamma_0(N)}$

the set of cusps for congruence subgroups is $\mathbb{Q} \cup \{\infty\}$

let $H^m(\mathfrak{R}_\Gamma)$ be the space of holomorphic differentials of degree m on $\mathfrak{R}_\Gamma$ for each $m \geq 1$

let $H^m(\mathfrak{R}_\Gamma)$ be the space of holomorphic differentials of degree m on $\mathfrak{R}_\Gamma$ for each $m \geq 1$

let $g(\Gamma)$ be the genus of $\mathfrak{R}_\Gamma$

Holomorphic Differentials and m–Weierstrass Points on $\mathfrak{R}_\Gamma$

let $H^m(\mathfrak{R}_\Gamma)$ be the space of holomorphic differentials of degree m on $\mathfrak{R}_\Gamma$ for each $m \geq 1$

let $g(\Gamma)$ be the genus of $\mathfrak{R}_\Gamma$

we have

$$\dim H^m(\mathfrak{R}_\Gamma) = \begin{cases} 0 & \text{if } \; m \geq 1, g(\Gamma) = 0; \\ g(\Gamma) & \text{if } \; m = 1, \; g(\Gamma) \geq 1; \\ g(\Gamma) & \text{if } \; m \geq 2, \; g(\Gamma) = 1; \\ (2m-1)(g(\Gamma)-1) & \text{if } \; m \geq 2, \; g(\Gamma) \geq 2. \end{cases}$$

the canonical class K is the divisor of any non–zero meromorphic form ω on $\mathfrak{R}_\Gamma$

Holomorphic Differentials and m–Weierstrass Points on $\mathfrak{R}_\Gamma$

the canonical class K is the divisor of any non–zero meromorphic form ω on $\mathfrak{R}_\Gamma$

let $f \in \mathbb{C}(\mathfrak{R}_\Gamma)$ be non–zero function in the field of meromorphic functions $\mathbb{C}(\mathfrak{R}_\Gamma)$ on $\mathfrak{R}_\Gamma$

the canonical class K is the divisor of any non–zero meromorphic form ω on $\mathfrak{R}_\Gamma$

let $f \in \mathbb{C}(\mathfrak{R}_\Gamma)$ be non–zero function in the field of meromorphic functions $\mathbb{C}(\mathfrak{R}_\Gamma)$ on $\mathfrak{R}_\Gamma$

then, for $m \geq 1$, $f\omega^m \in H^m(\mathfrak{R}_\Gamma)$ if and only if $f \in L(mK) \overset{def}{=} \{g \in \mathbb{C}(\mathfrak{R}_\Gamma); \;\; g = 0 \text{ or } \mathrm{div}(g) + mK \geq 0\}$

Holomorphic Differentials and m–Weierstrass Points on $\mathfrak{R}_\Gamma$

assume $g(\Gamma) \geq 1$ and $m \geq 1$

Holomorphic Differentials and m–Weierstrass Points on $\mathfrak{R}_\Gamma$

assume $g(\Gamma) \geq 1$ and $m \geq 1 \implies \dim H^m(\mathfrak{R}_\Gamma) \neq 0$

Holomorphic Differentials and m–Weierstrass Points on $\mathfrak{R}_\Gamma$

assume $g(\Gamma) \geq 1$ and $m \geq 1 \implies \dim H^m(\mathfrak{R}_\Gamma) \neq 0$

Let $t = \dim H^m(\mathfrak{R}_\Gamma)$. We fix the basis $\omega_1, \ldots, \omega_t$ of $H^m(\mathfrak{R}_\Gamma)$. Let z be any local coordinate on $\mathfrak{R}_\Gamma$. Then, locally there exists unique holomorphic functions $\varphi_1, \ldots, \varphi_t$ such that $\omega_i = \varphi_i (dz)^m$, for all i. Then, again locally, we can consider the Wronskian W_z defined by

Holomorphic Differentials and m–Weierstrass Points on $\mathfrak{R}_\Gamma$

assume $g(\Gamma) \geq 1$ and $m \geq 1 \implies \dim H^m(\mathfrak{R}_\Gamma) \neq 0$

Let $t = \dim H^m(\mathfrak{R}_\Gamma)$. We fix the basis $\omega_1, \ldots, \omega_t$ of $H^m(\mathfrak{R}_\Gamma)$. Let z be any local coordinate on $\mathfrak{R}_\Gamma$. Then, locally there exists unique holomorphic functions $\varphi_1, \ldots, \varphi_t$ such that $\omega_i = \varphi_i (dz)^m$, for all i. Then, again locally, we can consider the Wronskian W_z defined by

$$W_z(\omega_1, \ldots, \omega_t) \overset{def}{=} \begin{vmatrix} \varphi_1(z) & \cdots & \varphi_t(z) \\ \frac{d\varphi_1(z)}{dz} & \cdots & \frac{d\varphi_t(z)}{dz} \\ & \cdots & \\ \frac{d^{t-1}\varphi_1(z)}{dz^{k-1}} & \cdots & \frac{d^{t-1}\varphi_t(z)}{dz^{t-1}} \end{vmatrix}$$

Holomorphic Differentials and m–Weierstrass Points on $\mathfrak{R}_\Gamma$

the collection of all

$$W_z\left(\omega_1, \ldots, \omega_t\right)(dz)^{\frac{t}{2}(2m-1+t)},$$

defines a non–zero holomorphic differential form

$$W\left(\omega_1, \ldots, \omega_t\right) \in H^{\frac{t}{2}(2m-1+t)}\left(\mathfrak{R}_\Gamma\right).$$

called the Wronskian of the basis $\omega_1, \ldots, \omega_t$

the collection of all

$$W_z\left(\omega_1, \ldots, \omega_t\right)(dz)^{\frac{t}{2}(2m-1+t)},$$

defines a non–zero holomorphic differential form

$$W\left(\omega_1, \ldots, \omega_t\right) \in H^{\frac{t}{2}(2m-1+t)}\left(\mathfrak{R}_\Gamma\right).$$

called the Wronskian of the basis $\omega_1, \ldots, \omega_t$

a different choice of a basis of $H^m\left(\mathfrak{R}_\Gamma\right)$ results in a Wronskian which differ from $W\left(\omega_1, \ldots, \omega_t\right)$ by a multiplication by a non–zero complex number. Also, the degree is given by

$$\deg\left(\mathrm{div}(W\left(\omega_1, \ldots, \omega_t\right))\right) = t\left(2m-1+t\right)\left(g\left(\Gamma\right)-1\right).$$

Holomorphic Differentials and m–Weierstrass Points on $\mathfrak{R}_\Gamma$

Definition

Let $m \geq 1$ be an integer. We say that $\mathfrak{a} \in \mathfrak{R}_\Gamma$ is a m-Weierstrass point if

$$\nu_{\mathfrak{a}} \left(W \left(\omega_1, \ldots, \omega_t \right) \right) \geq 1.$$

When $m = 1$ we speak about classical Weierstrass points. So, 1-Weierstrass points are simply Weierstrass points.

Interpretation in terms of cuspidal modular forms

Let $m \geq 2$ be an even integer

Let $m \geq 2$ be an even integer

Let $S_m(\Gamma)$ be the space of cuspidal modular forms for Γ of weight m

Interpretation in terms of cuspidal modular forms

Let $m \geq 2$ be an even integer

Let $S_m(\Gamma)$ be the space of cuspidal modular forms for Γ of weight m

$f \in S_m(\Gamma)$, $f \neq 0$ has a divisor which has the form

$$\mathrm{div}(f) = (\text{the part independent of } f) + \mathfrak{c}_f,$$

Interpretation in terms of cuspidal modular forms

Let $m \geq 2$ be an even integer

Let $S_m(\Gamma)$ be the space of cuspidal modular forms for Γ of weight m

$f \in S_m(\Gamma)$, $f \neq 0$ has a divisor which has the form

$$\mathrm{div}(f) = (\text{the part independent of } f) + \mathfrak{c}_f,$$

$\mathfrak{c}_f$ is usual effective divisor on the curve $\mathfrak{R}_\Gamma$

Interpretation in terms of cuspidal modular forms

Interpretation in terms of cuspidal modular forms

let $S_m^H(\Gamma)$ be the space of all $f \in S_m(\Gamma)$ either $f = 0$ or f satisfies

$$\mathfrak{c}_f \geq \sum_{\substack{\mathfrak{a} \in \mathfrak{R}_\Gamma, \\ elliptic}} \left[\frac{m}{2}(1 - 1/e_{\mathfrak{a}})\right] \mathfrak{a} + \left(\frac{m}{2} - 1\right) \sum_{\substack{\mathfrak{b} \in \mathfrak{R}_\Gamma, \\ cusp}} \mathfrak{b}$$

Theorem

For each $f \in S_m^H(\Gamma)$, $f \neq 0$, there exists unique $\omega_f \in H^{m/2}(\mathfrak{R}_\Gamma)$ such that

$$\mathrm{div}(\omega_f) = \mathfrak{c}_f - \sum_{\substack{\mathfrak{a} \in \mathfrak{R}_\Gamma, \\ elliptic}} \left[\frac{m}{2}(1 - 1/e_{\mathfrak{a}})\right] \mathfrak{a} - \left(\frac{m}{2} - 1\right) \sum_{\substack{\mathfrak{b} \in \mathfrak{R}_\Gamma, \\ cusp}} \mathfrak{b}.$$

Moreover, the map $f \longmapsto \omega_f$ is an isomorphism of $S_m^H(\Gamma)$ onto $H^{m/2}(\mathfrak{R}_\Gamma)$.

Interpretation in terms of cuspidal modular forms

We remark that when $m = 2$, this reduces to obvious condition $\mathfrak{c}_f \geq 0$. Hence, $S_2^H(\Gamma) = S_2(\Gamma)$ recovering the standard isomorphism of $S_2(\Gamma)$ and $H^1(\mathfrak{R}_\Gamma)$

Interpretation in terms of cuspidal modular forms and $\frac{m}{2}$–Weierstrass points

Theorem

Assume that $g(\Gamma) \geq 2$, and $\mathfrak{a}_\infty$ is a Γ-cusp. Then, there exists a basis $f_1, \ldots f_t$ of $S_m^H(\Gamma)$ such that their q–expansions are of the form

$$f_u = a_u q^{i_u} + \text{higher order terms in } q, \quad 1 \leq u \leq t,$$

where

$$\frac{m}{2} \leq i_1 < i_2 < \cdots < i_t \leq \frac{m}{2} + m\left(g(\Gamma) - 1\right),$$

and

$$a_u \in \mathbb{C}, \quad a_u \neq 0.$$

Interpretation in terms of cuspidal modular forms and $\frac{m}{2}$–Weierstrass points

Theorem

Assume that $g(\Gamma) \geq 2$, and $\mathfrak{a}_\infty$ is a Γ-cusp. Then, $\mathfrak{a}_\infty$ is not *a $\frac{m}{2}$–Weierstrass point if and only if there exists a basis $f_1, \ldots f_t$ of $S_m^H(\Gamma)$ such that their q–expansions are of the form*

$$f_u = a_u q^{u+m/2-1} + \textit{higher order terms in } q, \quad 1 \leq u \leq t,$$

where

$$a_u \in \mathbb{C}, \quad a_u \neq 0.$$

Let $m = 2$. The last theorem gives as a simple and already well–known way of testing if $\mathfrak{a}_\infty$ is usual Weierstrass point for $\Gamma = \Gamma_0(N)$

Let $m = 2$. The last theorem gives as a simple and already well–known way of testing if $\mathfrak{a}_\infty$ is usual Weierstrass point for $\Gamma = \Gamma_0(N)$

List a basis of $S_2(\Gamma_0(N))$ in Sage and see if it is of the form

$$f_u = a_u q^u + \text{higher order terms in } q, \quad 1 \leq u \leq t = g(\Gamma),$$

where

$$a_u \in \mathbb{C}, \quad a_u \neq 0.$$

Let $m = 2$. The last theorem gives as a simple and already well–known way of testing if $\mathfrak{a}_\infty$ is usual Weierstrass point for $\Gamma = \Gamma_0(N)$

List a basis of $S_2(\Gamma_0(N))$ in Sage and see if it is of the form

$$f_u = a_u q^u + \text{higher order terms in } q, \quad 1 \leq u \leq t = g(\Gamma),$$

where

$$a_u \in \mathbb{C}, \quad a_u \neq 0.$$

in what follows we discuss the case $m \geq 4$

Some properties of the space $S_m^H(\Gamma)$ that goes back to the work of Petri

Lemma

Let $m \geq 4$ be an even integer. Let us select a basis $f_0, \ldots, f_{g-1}$, $g = g(\Gamma)$, of $S_2(\Gamma)$. Then, all of $\binom{g+\frac{m}{2}-1}{\frac{m}{2}}$ monomials $f_0^{\alpha_0} f_1^{\alpha_1} \cdots f_{g-1}^{\alpha_{g-1}}$, $\alpha_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=0}^{g-1} \alpha_i = \frac{m}{2}$, belong to $S_m^H(\Gamma)$. We denote by $S_{m,2}^H(\Gamma)$ this subspace of $S_m^H(\Gamma)$.

Lemma

Let $m \geq 4$ be an even integer. Assume that $\mathfrak{R}_\Gamma$ is not hyperelliptic (and $g(\Gamma) \geq 2$). Then, we have $S_{m,2}^H(\Gamma) = S_m^H(\Gamma)$.

Some properties of the space $S_m^H(\Gamma)$ that goes back to the work of Petri

Lemma

Let $m \geq 4$ be an even integer. Let us select a basis $f_0, \ldots, f_{g-1}$, $g = g(\Gamma)$, of $S_2(\Gamma)$. Then, all of $\binom{g+\frac{m}{2}-1}{\frac{m}{2}}$ monomials $f_0^{\alpha_0} f_1^{\alpha_1} \cdots f_{g-1}^{\alpha_{g-1}}$, $\alpha_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=0}^{g-1} \alpha_i = \frac{m}{2}$, belong to $S_m^H(\Gamma)$. We denote by $S_{m,2}^H(\Gamma)$ this subspace of $S_m^H(\Gamma)$.

Lemma

Let $m \geq 4$ be an even integer. Assume that $\mathfrak{R}_\Gamma$ is not hyperelliptic (and $g(\Gamma) \geq 2$). Then, we have $S_{m,2}^H(\Gamma) = S_m^H(\Gamma)$.

the last lemma is crucial for testing if $\mathfrak{a}_\infty$ is a $\frac{m}{2}$–Weierstrass point (combined with earlier criterion)

An Algorithm for $X_0(N)$ using SAGE (second part)

Theorem

Let $m \geq 4$ be an even integer. Assume that $\mathfrak{R}_\Gamma$ is not hyperelliptic. Assume that $\mathfrak{a}_\infty$ is a cusp for Γ. Let us select a basis $f_0, \ldots, f_{g-1}$, $g = g(\Gamma)$, of $S_2(\Gamma)$ (listed by their q–expansions using SAGE system if $\Gamma = \Gamma_0(N)$). Compute q–expansions of all monomials

$$f_0^{\alpha_0} f_1^{\alpha_1} \cdots f_{g-1}^{\alpha_{g-1}}, \quad \alpha_i \in \mathbb{Z}_{\geq 0}, \ \sum_{i=0}^{g-1} \alpha_i = \frac{m}{2}.$$

Then, $\mathfrak{a}_\infty$ is not *a $\frac{m}{2}$–Weierstrass point if and only if there exist $\mathbb{C}$–linear combinations of such monomials, say $F_1, \ldots F_t$, $t = (m-1)(g-1)$, such that their q–expansions are of the form*

$$F_u = a_u q^{u+m/2-1} + \textit{higher order terms in } q, \quad 1 \leq u \leq t,$$

where

$$a_u \in \mathbb{C}, \quad a_u \neq 0.$$

Examples

For $X_0(34)$ the basis of $S_4^H(\Gamma_0(34))$ is given by

$$
\begin{aligned}
f_0^2 &= q^2 - 4q^5 - 4q^6 + 12q^8 + 12q^9 - 2q^{10} \\
f_0 f_1 &= q^3 - q^5 - 2q^6 - 2q^7 + 2q^8 + 5q^9 + 2q^{10} \\
f_0 f_2 &= q^4 - 2q^5 - q^6 - q^7 + 6q^8 + 6q^9 + 2q^{10} \\
-f_1^2 + f_0 f_2 &= -2q^5 + q^6 - q^7 + 5q^8 + 6q^9 + 4q^{10} \\
-f_1^2 + f_0 f_2 + 2f_1 f_2 &= -3q^6 - 5q^7 + 11q^8 + 16q^9 + 2q^{10} \\
-f_1^2 + f_0 f_2 + 2f_1 f_2 + 3f_2^2 &= -17q^7 + 17q^8 + 34q^9 + 17q^{10}
\end{aligned}
$$

Their first exponents are $\frac{m}{2} = 2, 3, 4, 5, 6, \frac{m}{2} + (m-1)(g-1) - 1 = 7$ which shows that $\mathfrak{a}_\infty$ is not 2–Weierstrass point for $X_0(34)$.

Examples

For $X_0(55)$, the basis of $S_4^H(\Gamma_0(55))$ is given by

$$
\begin{aligned}
f_0^2 &= q^2 - 2q^8 + \cdots \\
f_0 f_1 &= q^3 - 2q^7 + \cdots \\
f_0 f_2 &= q^4 - 2q^7 + \cdots \\
f_0 f_3 &= q^5 - 2q^7 + \cdots \\
f_0 f_4 &= q^6 - 2q^{11} + \cdots \\
-f_1 f_2 + f_0 f_3 &= -2q^7 + q^8 + \cdots \\
-f_1 f_2 + f_0 f_3 + 2f_2 f_3 &= q^8 + 2q^9 + \cdots \\
-f_1 f_2 + f_0 f_3 + 2f_2 f_3 - f_3^2 &= 2q^9 - q^{10} + \cdots \\
-f_1 f_2 + f_0 f_3 + 2f_2 f_3 - f_3^2 - 2f_3 f_4 &= -q^{10} + 11q^{12} + \cdots \\
-f_1 f_2 + f_0 f_3 + 2f_2 f_3 - f_3^2 - 2f_3 f_4 + f_4^2 &= 11q^{12} - 11q^{13} + \cdots
\end{aligned}
$$

Examples

$$-f_1 f_2 - f_2^2 + f_0 f_3 + 2f_2 f_3 - f_3^2 + f_0 f_4 - 6f_3 f_4 - f_4^2 = -22q^{13} + 44q^{15} + \cdots$$

$$-f_2^2 + f_3^2 + f_0 f_4 - f_2 f_4 - 4f_3 f_4 + 2f_4^2 = -22q^{14} + 22q^{15} + \cdots$$

The last exponent is $14 > \frac{m}{2} + (m-1)(g-1) - 1 = 13$. So, $\mathfrak{a}_\infty$ is a 2–Weierstrass point for $X_0(55)$.

Thank you!