Characterization of quadratic $\varepsilon-$CNS polynomials and determination of all $\varepsilon-$CNS bases in quadratic number fields

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Introduction

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Definition (Canonical number system in number field)

Let $\mathbb{Z}_K$ be the ring of integers of the algebraic number field K and let $\alpha \in \mathbb{Z}_K$ with $|Norm_{K/\mathbb{Q}}(\alpha)| \geq 2$. A pair $(\alpha, \mathcal{N}_0(\alpha))$ where

$$\mathcal{N}_0(\alpha) = \{0, ..., |Norm_{K/\mathbb{Q}}(\alpha)| - 1\}$$

is called a canonical number system (in short CNS) in $\mathbb{Z}_K$ (or in K), if every $\gamma \in \mathbb{Z}_K$ has a (unique) representation of the form

$$\gamma = \sum_{j=0}^{l-1} d_j \alpha^j, \text{ with } d_0, ..., d_{l-1} \in \mathcal{N}_0(\alpha), \ d_{l-1} \neq 0.$$

α is called basis of this CNS, $\mathcal{N}_0(\alpha)$ is called its set of digits.

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- Elements of a general theory (for an arbitrary number field K) is due to Kovács (1981) as well as Kovács and Pethő (1991, 1992).

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Definition (Number system)

Let $P \in \mathbb{Z}[x]$ be a monic polynomial and let $\mathcal{N}$ be a complete residue system of $\mathbb{Z}$ modulo $P(0)$ containing 0. The pair $(P, \mathcal{N})$ is called a number system *if for each $a \in \mathbb{Z}[x]$ there exist unique integers $l \in \mathbb{N}$, $d_0, \ldots, d_{l-1} \in \mathcal{N}$, $d_{l-1} \neq 0$ such that*

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- If $(P, \mathcal{N})$ is a number system, then each coset $A \in \mathcal{R} := \mathbb{Z}[x] / P(x)\mathbb{Z}[x]$ contains a polynomial with coefficients belonging to $\mathcal{N}$.

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- If $(P, \mathcal{N})$ is a number system, then each coset $A \in \mathcal{R} := \mathbb{Z}[x] / P(x)\mathbb{Z}[x]$ contains a polynomial with coefficients belonging to $\mathcal{N}$.
- Choosing the set of digits $\mathcal{N}_0 = \{0, 1, \ldots, |P(0)| - 1\}$, the number system $(P, \mathcal{N}_0)$ is called a canonical number system (CNS) and P is called a *CNS basis* or *CNS polynomial.*

Canonical number systems $(P, \mathcal{N}_0)$, where $P \in \mathbb{Z}[x]$ is any monic polynomial and $\mathcal{N}_0 = \{0, 1, ..., |P(0)| - 1\}$ have been extensively studied.

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- To give the characterization of the CNS polynomials by considering only its coefficients. Until now the complete description (characterization) of CNS polynomials remains an open problem even for polynomials of small degree.

It turns out that these two problems are closely related to a dynamical systems, so called *shift radix systems (SRS)*.

Definition (SRS)

Let $d \geq 1$ be an integer. To $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ we associate the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ in the following way: For $\mathbf{z} = (z_0, \ldots, z_{d-1}) \in \mathbb{Z}^d$ let

$$\tau_{\mathbf{r}}(\mathbf{z}) = (z_1, \ldots, z_{d-1}, -\lfloor \mathbf{rz} \rfloor)$$

where $\mathbf{rz} = r_1 z_1 + ... + r_d z_d$, i.e. $\mathbf{rz}$ is the inner product of the vectors $\mathbf{r}$ and $\mathbf{z}$. We call $\tau_{\mathbf{r}}$ a shift radix system (SRS) if for all $\mathbf{z} \in \mathbb{Z}^d$ there exists a $k \in \mathbb{N}$ such that $\tau_{\mathbf{r}}^k(\mathbf{z}) = \mathbf{0}$.

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We define two sets related to the behavior of the periods of $\tau_{\mathbf{r}}$. Let

$$\begin{aligned} \mathcal{D}_d &= \left\{ \mathbf{r} \in \mathbb{R}^d : (\tau_{\mathbf{r}}^n(\mathbf{z}))_{n \in \mathbb{N}} \text{ is ultimately periodic for all } \mathbf{z} \in \mathbb{Z}^d \right\} \\ \mathcal{D}_d^0 &= \left\{ \mathbf{r} \in \mathbb{R}^d : \tau_{\mathbf{r}} \text{ is SRS} \right\} \end{aligned}$$

Clearly, we have

$$\mathcal{D}_d^0 \subseteq \mathcal{D}_d.$$

- Shift radix systems were introduced in 2005 by S. Akiyama, T. Borbély, H. Brunotte, A. Pethő, and J. M. Thuswaldner in the first part of the series of their four papers: *Generalized radix representations and dynamical systems I-IV*.

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- In this series of papers Akiyama et al. described the basic properties of SRS as well as their relations to $\beta-$expansions and canonical number systems. Namely, for certain parameters $\mathbf{r} \in \mathcal{D}_d^0$ corresponding SRS are closely related to canonical number systems.

- If P is a CNS polynomial, then each coset $A \in \mathcal{R} := \mathbb{Z}[x] / P(x) \mathbb{Z}[x]$ contains a polynomial with coefficients belonging to $\mathcal{N}_0$, which we call the *CNS representation of this coset.*

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- Let $P(x) = x^d + p_{d-1}x^{d-1} + ... + p_1 x + p_0 \in \mathbb{Z}[x]$. Since P is monic it is clear that every coset $A \in \mathcal{R}$ has a unique element of degree at most $d-1$:

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- Let $\mathbb{Z}'[x] = \{a \in \mathbb{Z}[x] : \deg a < d\}$ and let us define the *backward division mapping* $T_P : \mathbb{Z}'[x] \to \mathbb{Z}'[x]$ by

$$a = \sum_{j=0}^{d-1} A_j x^j \quad \longrightarrow \quad T_P(a) = \sum_{j=0}^{d-1} \left(A_{j+1} - q p_{j+1}\right) x^j,$$

where $q = \left\lfloor \frac{a_0}{p_0} \right\rfloor$ and $A_d = 0$.

- Then

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- If there exists an integer $k \in \mathbb{N}$ such that $T_P^{(k)}\left(a\right) = 0$, then after k iterates a, $T_p\left(a\right)$, $T_p^2\left(a\right)$, ...we obtain CNS representation of coset $A = \left[\sum_{j=0}^{k-1} d_j x^j\right]$ $(d_j \in \mathcal{N}_0)$

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- Consequently, a polynomial P is a CNS polynomial if and only if **for any** $a \in \mathbb{Z}'\left[x\right]$ there exists an integer $k \in \mathbb{N}$ such that $T_P^{(k)}\left(a\right) = 0$.

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- It turns out that the CNS property of a given monic polynomial P is algorithmically decidable.
- One has to apply "backward division mapping" to all polynomials satisfying $\deg a < \deg P$ and of the bounded size (height $H(a) \leq C$), iteratively. The efficiency of the algorithm depends on the size of C.

- Brunotte (2001), (and independently Scheicher and Thuswaldner (2004)) observed that the suitable basis transformation of $\mathbb{Z}' [x]$

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$$a = \sum_{j=0}^{d-1} a_j \omega_j$$

then $T_P : \mathbb{Z}' [x] \to \mathbb{Z}' [x]$ implies the mapping $\tau_P : \mathbb{Z}^d \to \mathbb{Z}^d$ given by

$$\begin{aligned} \mathbf{a} &= (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d \longmapsto \\ \tau_P(\mathbf{a}) &= (a_1, \ldots, a_{d-1}, -\left\lfloor \frac{1}{p_0} a_1 + \frac{p_{d-1}}{p_0} a_2 ... + \frac{p_1}{p_0} a_d \right\rfloor). \end{aligned}$$

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- Consequently, a polynomial P is a CNS polynomial if and only if for each $\mathbf{a} = (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$ there exists an integer $k \in \mathbb{N}$ such that $\tau_P^{(k)}(\mathbf{a}) = \mathbf{0}$.

- Note that

$$\tau_P(\mathbf{a}) = (a_1, \dots, a_{d-1}, -\lfloor \mathbf{ar} \rfloor) = \tau_{\mathbf{r}}(\mathbf{a}), \quad \text{where } \mathbf{r} = \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, .., \frac{p_1}{p_0} \right).$$

  Thus, P is a CNS polynomial if and only if $\tau_{\mathbf{r}}$ is SRS for $\mathbf{r} = \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, .., \frac{p_1}{p_0} \right)$.

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Thus, P is a CNS polynomial if and only if $\tau_{\mathbf{r}}$ is SRS for $\mathbf{r} = \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, .., \frac{p_1}{p_0} \right)$. Or, in other words:

Theorem

A polynomial $P(x) = x^d + p_{d-1}x^{d-1} + ... + p_1 x + p_0 \in \mathbb{Z}[x]$ is a CNS polynomial if and only if $\left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, .., \frac{p_1}{p_0} \right) \in \mathcal{D}_d^0$.

Some generalizations

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Various variants of number systems and canonical number systems $(P, \mathcal{N})$ have been studied in the literature. For example:

- *Symmetric canonical number systems (SCNS)* are number systems $(P, \mathcal{N})$ where $P \in \mathbb{Z}[x]$ is any monic polynomial and

$$\mathcal{N} = \left[-\frac{|P(0)|}{2}, \frac{|P(0)|}{2}\right) \cap \mathbb{Z}.$$

  These number systems were studied for instance by Akiyama and Scheicher (2007), Brunotte (2009), Kátai (1995) and Scheicher, Surer, Thuswaldner and van de Woestijne (2014).

- Generalizations to larger ground rings R, i.e. when $P \in R[x]$
  - Jacob and Reveilles (1995), Brunotte, Kirschenhofer and Thuswaldner (2011): $R = \mathbb{Z}[i]$;
  - Scheicher, Surer, Thuswaldner and van de Woestijne (2014): R - commutative ring;
  - Pethő and Varga (2017): $R = \mathbb{E}_d$ - ring of integers of Euclidean imaginary quadratic number fields.
  - Pethő and Thuswaldner (2019): $R = \mathcal{O}$ - any order in the number field $\mathbb{L}$.

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  - Pethő and Thuswaldner (2019): $R = \mathcal{O}$ - any order in the number field $\mathbb{L}$.
- In this talk we deal with $\varepsilon-$*canonical number system.*

Definition (ε-CNS)

Let $P(x) = x^d + p_{d-1}x^{d-1} + ... + p_1 x + p_0 \in \mathbb{Z}[x]$, $\varepsilon \in [0, 1)$, *and let*

$$\mathcal{N}_\varepsilon = [-\varepsilon |p_0|, (1-\varepsilon)|p_0|) \cap \mathbb{Z}.$$

The par $(P, \mathcal{N}_\varepsilon)$ *is called an* $\varepsilon-$*canonical number system (short* ε*-CNS) if for each* $a \in \mathbb{Z}[x]$ *there exist unique integers* $l \in \mathbb{N}$, $d_0, ..., d_{l-1} \in \mathcal{N}_\varepsilon$, $d_{l-1} \neq 0$ *such that*

$$a \equiv \sum_{j=0}^{l-1} d_j x^j \pmod{P}$$

P *is called base of the* $\varepsilon-$*CNS or* $\varepsilon-$*CNS polynomial. To* $\mathcal{N}_\varepsilon$ *we refer as the* $\varepsilon-$*set of digits.*

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Note that:

- Set $\mathcal{N}_\varepsilon$ consists of $|p_0|$ consecutive integers and contains 0.
- The case $\varepsilon = 0$ corresponds to usual CNS while $\varepsilon = \frac{1}{2}$ corresponds to symmetric canonical number system (SCNS).

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- It turns out that problem of characterization of ε-CNS polynomials is closely related to a dynamical systems, so called ε-*shift radix systems*.
- The concept of shift radix systems (SRS) was introduced Akiyama et al.(2005). Akiyama and Scheicher (2007) presented a slight modification of SRS, so called symmetric shift radix systems (SSRS). P. Surer (2009) constructed a following new generalization:

Definition (ε-SRS)

Let $d \geq 1$ be an integer and $\varepsilon \in [0,1)$. To $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ we associate the mapping $\tau_{\mathbf{r},\varepsilon} : \mathbb{Z}^d \to \mathbb{Z}^d$ in the following way: For $\mathbf{z} = (z_0, \ldots, z_{d-1}) \in \mathbb{Z}^d$ let

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where $\mathbf{rz} = r_1 z_1 + \ldots + r_d z_d$, i.e. $\mathbf{rz}$ is the inner product of the vectors $\mathbf{r}$ and $\mathbf{z}$. The mapping $\tau_{\mathbf{r},\varepsilon}$ is called an ε-shift radix system (ε-SRS) if for any $\mathbf{z} \in \mathbb{Z}^d$ there exists $k \in \mathbb{N}$ such that $\tau^k_{\mathbf{r},\varepsilon}(\mathbf{z}) = \mathbf{0}$.

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where $\mathbf{rz} = r_1 z_1 + \dots + r_d z_d$, i.e. $\mathbf{rz}$ is the inner product of the vectors $\mathbf{r}$ and $\mathbf{z}$. The mapping $\tau_{\mathbf{r},\varepsilon}$ is called an ε-shift radix system (ε-SRS) if for any $\mathbf{z} \in \mathbb{Z}^d$ there exists $k \in \mathbb{N}$ such that $\tau^k_{\mathbf{r},\varepsilon}(\mathbf{z}) = \mathbf{0}$.

We define two sets related to the behavior of the periods of $\tau_{\mathbf{r},\varepsilon}$. Let

$$\begin{aligned} \mathcal{D}_{d,\varepsilon} &= \left\{ \mathbf{r} \in \mathbb{R}^d : \left(\tau^n_{\mathbf{r},\varepsilon}(\mathbf{z})\right)_{n\in\mathbb{N}} \text{ is ultimately periodic for all } \mathbf{z} \in \mathbb{Z} \right\} \\ \mathcal{D}^0_{d,\varepsilon} &= \left\{ \mathbf{r} \in \mathbb{R}^d : \tau_{\mathbf{r},\varepsilon} \text{ is } \varepsilon\text{-SRS} \right\} \end{aligned}$$

We have

$$\mathcal{D}^0_{d,\varepsilon} \subseteq \mathcal{D}_{d,\varepsilon}.$$

Lots of basic properties and notations concerning $\mathcal{D}_{d,\varepsilon}$ and $\mathcal{D}^0_{d,\varepsilon}$, P. Surer (2009) directly adopted from the well analyzed case $\varepsilon = 0$ and the case $\varepsilon = \frac{1}{2}$ since 0-SRS corresponds to classical SRS while $\frac{1}{2}$-SRS corresponds to SSRS.

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Theorem (1)

Let $\varepsilon \in [0, 1)$ and $P(x) = x^d + p_{d-1}x^{d-1} + ... + p_1 x + p_0 \in \mathbb{Z}[x]$. Then P is $\varepsilon-$CNS polynomial if and only if $\left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, ..., \frac{p_1}{p_0}\right) \in \mathcal{D}^0_{d,\varepsilon}$.

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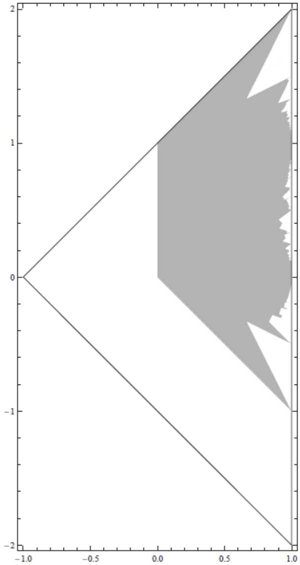
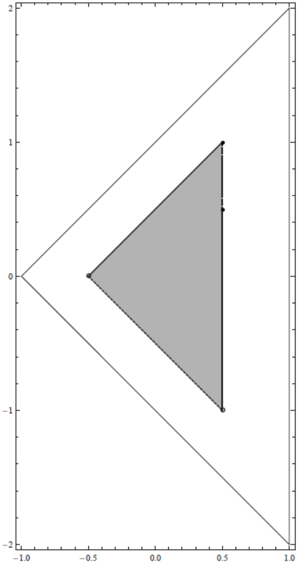
  On the other hand, it turned out that the set $\mathcal{D}^0_{2,\varepsilon} \subset \mathbb{R}^2$ for $\varepsilon = \frac{1}{2}$ has a very simple structure, so it can be characterized completely (Akiyama and Scheicher (2007)).



1) An approximation of $\mathcal{D}^0_{2,0}$



2) The set $\mathcal{D}^0_{2,\frac{1}{2}}$

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$$\mathcal{E}_d \subset \mathcal{D}_{d,\varepsilon} \subset \overline{\mathcal{E}}_d,$$

where $\mathcal{E}_d$ is open bounded set characterized by several strict inequalities (sometimes referred to as the Schur-Takagi region). For example, we have

$$\mathcal{E}_2 = \left\{(x,y) \in \mathbb{R}^2 : |x| < 1, \ \ |y| < x+1\right\}.$$

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- Set
$$\mathcal{D}_{d,\varepsilon} \cap \partial \mathcal{D}_{d,\varepsilon} = \mathcal{D}_{d,\varepsilon} \diagdown \mathcal{E}_d$$
is very hard to describe and probably depends on ε (for example there exist only partial results for $\mathcal{D}_{2,0} \diagdown \mathcal{E}_2$).

- Further, P. Surer (2009) showed that $\mathcal{D}^0_{d,\varepsilon}$, for $\varepsilon \in (0,1) \setminus \left\{ \frac{1}{2} \right\}$ can be gained by cutting out polyhedra from $\mathcal{D}_{d,\varepsilon}$ and presented a method to obtain these polyhedra (method adopted from the case $\varepsilon = 0$ and slightly modified).

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- He also has shown that $\mathcal{D}^0_{d,\varepsilon}$ is closely related to $\mathcal{D}^0_{d,1-\varepsilon}$ for $\varepsilon \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. Precisely, the sets $\mathcal{D}^0_{d,\varepsilon}$ and $\mathcal{D}^0_{d,1-\varepsilon}$ are equal up to the boundary and the boundaries are reversed.

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- He stated several characterization results for the two dimensional case. Namely, for each $\varepsilon \in (0,1) \diagdown \left\{ \frac{1}{2} \right\}$ he has found explicitly given set $D^* (\varepsilon)$ with

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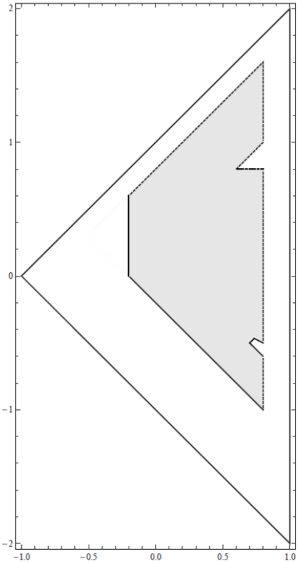
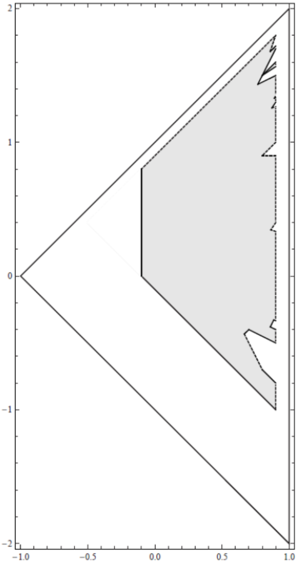
  and showed that $\mathcal{D}^0_{2,\varepsilon}$ can be can be obtained from $D^*(\varepsilon)$ by cutting out finitely many polygons.
- Surer use these results to give explicit characterizations of $\mathcal{D}^0_{2,\varepsilon}$ for some particular values of $\varepsilon \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.



1) The set $\mathcal{D}^0_{2,\frac{1}{5}}$



2) The set $\mathcal{D}^0_{2,\frac{1}{10}}$

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- This result is a consequence of our new characterization results of $\varepsilon-$shift radix systems ($\varepsilon-$SRS) in the two-dimensional case and their relation to quadratic $\varepsilon-$CNS polynomials (Theorem 1).

Theorem (1-dim2)

Let $\varepsilon \in [0,1)$ and $P(x) = x^2 + p_1 x + p_0 \in \mathbb{Z}[x]$. Then P is $\varepsilon-$CNS polynomial if and only if $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in \mathcal{D}^0_{2,\varepsilon}$.

The characterization of the classical quadratic CNS polynomials ($\varepsilon = 0$) is already given in several papers in several ways:

Theorem (2)

Let $P(x) = x^2 + p_1 x + p_0 \in \mathbb{Z}[x]$. Then P is a $0-$CNS polynomial if and only if $-1 \leq p_1 \leq p_0$ and $p_0 \geq 2$.

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Akiyama and Scheicher (2007) gave a complete description of the set $\mathcal{D}^0_{2,\frac{1}{2}}$.

Proposition (1)

$$\mathcal{D}^0_{2,\frac{1}{2}} = \left\{(x,y) \in \mathbb{R}^2 : |x| < \frac{1}{2},\ -x - \frac{1}{2} < y \le x + \frac{1}{2}\right\} \cup \left\{\left(\frac{1}{2}, y\right) \in \mathbb{R}^2 : -1 < y \le \frac{1}{2} \text{ or } y = 1\right\}$$

As a consequence of the Proposition 1 and Theorem 1-dim2, Akiyama and Scheicher (2007) obtained characterization of quadratic SCNS polynomials.

The statement of the following corollary is a slight modification of the statement of the Akiyama and Scheicher's corollary adapted to our needs.

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Corollary (1)

Let $P(x) = x^2 + p_1 x + p_0 \in \mathbb{Z}[x]$*. Then P is a* $\frac{1}{2}-$*CNS polynomial if and only if*

$$|p_1| < sgn(p_0) + \frac{|p_0|}{2} \;\; or \;\; p_1 = 1 + \frac{p_0}{2}, \; |p_0| > 2$$

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Precisely, P *is a* $\frac{1}{2}-$*CNS polynomial if and only if*

$$|p_1| \le \operatorname{sgn}(p_0) + \frac{|p_0| - 1}{2},\ |p_0| \ge 3,\ \text{if } |p_0| \text{ is odd}$$

$$|p_1| \le \operatorname{sgn}(p_0) - 1 + \frac{|p_0|}{2} \text{ or } p_1 = 1 + \frac{p_0}{2},\ |p_0| \ge 4,\ \text{if } |p_0| \text{ is even}$$

$$-1 \le p_1 \le 2,\ p_0 = 2.$$

Our main result is given in the following theorem.

Theorem (3, J. and Miletić)

Let $P(x) = x^2 + p_1 x + p_0 \in \mathbb{Z}[x]$, $|p_0| \geq 2$, $\varepsilon \in [0,1)$, *and let* $k = \lfloor \varepsilon |p_0| \rfloor$. *Then corresponding* $\varepsilon-$*set of digits is*

$$\mathcal{N}_\varepsilon = \{-k, ..., |p_0| - 1 - k\}.$$

i) *Let* $\varepsilon \in \left[0, \frac{1}{2}\right)$ *or let* $\varepsilon = \frac{1}{2}$ *if* $|p_0|$ *is odd. Then* P *is a* $\varepsilon-$*CNS polynomial if and only if*

$$-k - 1 \leq p_1 \leq p_0 - k, \quad p_0 \geq 2$$

or additionally

$$k + 2 - |p_0| \leq p_1 \leq k - 1, \quad p_0 \leq -3 \quad \text{if} \quad \varepsilon \in \left[\frac{1}{|p_0|}, \frac{1}{2}\right].$$

ii) *Let* $\varepsilon \in \left(\frac{1}{2}, 1\right)$ *or let* $\varepsilon = \frac{1}{2}$ *if* $|p_0|$ *is even. Then* P *is a* $\varepsilon-$*CNS polynomial if and only if*

$$-p_0 + k \leq p_1 \leq k + 1, \quad p_0 \geq 2,$$

or additionally

$$-k + 1 \leq p_1 \leq -k - 2 + |p_0|, \quad p_0 \leq -3 \quad \text{if} \quad \varepsilon \in \left[\frac{1}{2}, \frac{|p_0| - 1}{|p_0|}\right).$$

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- This characterization provides a unified view of the well-known characterizations of the classical quadratic CNS polynomials ($\varepsilon = 0$) and quadratic SCNS polynomials ($\varepsilon = 1/2$).
- We reprove following well-know result to explain main idea of the proof.

Theorem (2)

Let $P(x) = x^2 + p_1 x + p_0 \in \mathbb{Z}[x]$. *Then* P *is a* $0-$*CNS polynomial if and only if* $-1 \leq p_1 \leq p_0$ *and* $p_0 \geq 2$.

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$$\mathcal{D}_{2,0}^0 \cap R(0) = \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \frac{2}{3}, -x \leq y < x + 1\right\} =: B(0),$$

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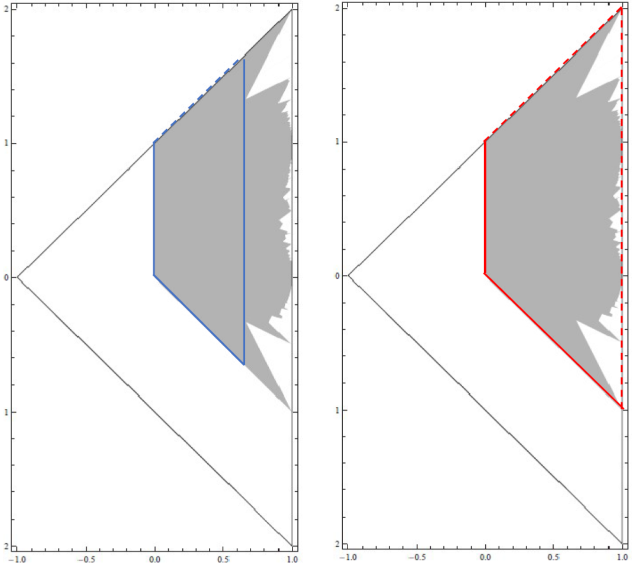
So we have:

$$B(0) \subset \mathcal{D}^0_{2,0} \subset D(0).$$



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$$P(x) = x^2 + p_1 x + p_0$$

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Proof:

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Proof:

$$\text{If } p_0 \ \ge \ 2, \ -1 \le p_1 \le p_0 \overset{\mathbf{p}_0 \ge \frac{3}{2}}{\Longrightarrow} 0 \le \frac{1}{p_0} \le \frac{2}{3}, \quad -\frac{1}{p_0} \le \frac{p_1}{p_0} \le 1 < \frac{1}{p_0} + 1$$

$$\Longrightarrow \left(\frac{1}{p_0}, \frac{p_1}{p_0} \right) \in B(0) \subset \mathcal{D}^0_{2,0} \overset{\textbf{Th. 1}}{\Longrightarrow} P \text{ is } 0-\text{CNS polynomial.}$$

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$$\text{If } P \text{ is } 0-\text{CNS polynomial} \overset{\mathbf{Th.\ 1}}{\Longrightarrow} \left(\frac{1}{p_0}, \frac{p_1}{p_0} \right) \in \mathcal{D}^0_{2,0} \subset D(0) \Longrightarrow$$

$$\Longrightarrow 0 \le \frac{1}{p_0} < 1 \text{ and } -\frac{1}{p_0} \le \frac{p_1}{p_0} < \frac{1}{p_0} + 1 \iff p_0 \geq 2,\ -1 \le p_1 \le p_0. \quad \square$$

- Note that the characterization of $0-$CNS polynomials

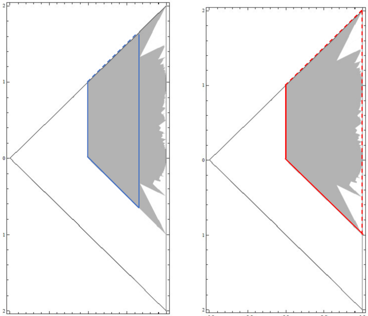
$$p_0 \geq 2, \ -1 \leq p_1 \leq p_0$$

is just the characterization of all $(p_0, p_1) \in \mathbb{Z}^2$, $|p_0| \geq 2$ such that $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in D(0)$.

- Also we see that if $P(x) = x^2 + p_1 x + p_0$ is $0-$CNS polynomial, then

$$\mathbf{r} = \left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in B(0) \subset \mathcal{D}^0_{2,0},$$

where $B(0)$ is the (best possible) "nice part" of the set $\mathcal{D}^0_{2,0}$ with the respect to the range of x.



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Let $\varepsilon \in [0, 1)$ and

$$D^*(\varepsilon) := \begin{cases} \{(x, y) \in \mathbb{R}^2 \colon -x - \varepsilon \le y < x + 1 - \varepsilon, \quad x < 1 - \varepsilon\} \text{ if } \varepsilon \in \left[0, \frac{1}{2}\right) \\ \{(x, y) \in \mathbb{R}^2 \colon -x - 1 + \varepsilon < y \le x + \varepsilon, \quad x \le \varepsilon\}, \text{ if } \varepsilon \in \left[\frac{1}{2}, 1\right) \end{cases}$$

and let half-plains $L(\varepsilon)$ and $R(\varepsilon)$ be given by

$$L(\varepsilon) := \begin{cases} \{(x, y) \in \mathbb{R}^2 : -\varepsilon \le x\}, \text{ if } \varepsilon \in \left[0, \frac{1}{2}\right) \\ \{(x, y) \in \mathbb{R}^2 : -(1 - \varepsilon) < x\}, \text{ if } \varepsilon \in \left[\frac{1}{2}, 1\right) \end{cases},$$

$$R(\varepsilon) := \begin{cases} \{(x, y) \in \mathbb{R}^2 : x \le 2/3 - \varepsilon\}, \text{ if } \varepsilon \in \left[0, \frac{1}{2}\right) \\ \{(x, y) \in \mathbb{R}^2 : x \le 2/3 - (1 - \varepsilon)\}, \text{ if } \varepsilon \in \left[\frac{1}{2}, 1\right) \end{cases}.$$

- We know that

$$\mathcal{D}^0_{2,\varepsilon} \subset D^*(\varepsilon) \tag{1}$$

(P. Surer for $\varepsilon \in (0,1) \setminus \left\{\frac{1}{2}\right\}$, for $\varepsilon = 0$ and $\varepsilon = \frac{1}{2}$, (1) easy follows previous mention results).

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- Cases $\varepsilon = 0$, $\varepsilon = \frac{1}{2}$, $\varepsilon = \frac{1}{5}$ and $\varepsilon = \frac{1}{10}$ indicate that for all $\varepsilon \in [0,1)$ sets $\mathcal{D}^0_{2,\varepsilon}$ and $D^*(\varepsilon)$ should coincide in the stripe $S(\varepsilon) := L(\varepsilon) \cap R(\varepsilon)$, i.e.

$$\mathcal{D}^0_{2,\varepsilon} \cap S(\varepsilon) = D^*(\varepsilon) \cap S(\varepsilon) \tag{2}$$

and

$$\mathcal{D}^0_{2,\varepsilon} \subset L(\varepsilon)\,. \tag{3}$$

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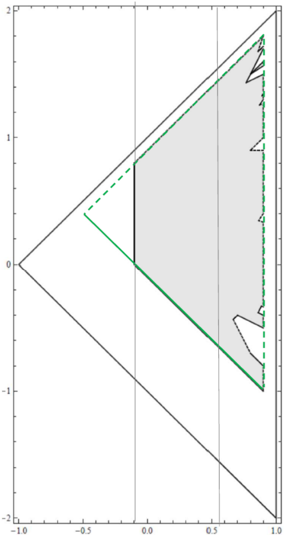
Let

$$D(\varepsilon) := D^*(\varepsilon) \cap L(\varepsilon) \quad \text{and} \quad B(\varepsilon) := D^*(\varepsilon) \cap S(\varepsilon).$$

Therefore, to prove our conjectures (2) and (3), it is enough to prove

$$B(\varepsilon) \subset \mathcal{D}^0_{2,\varepsilon} \subset D(\varepsilon),$$

since (1) holds.



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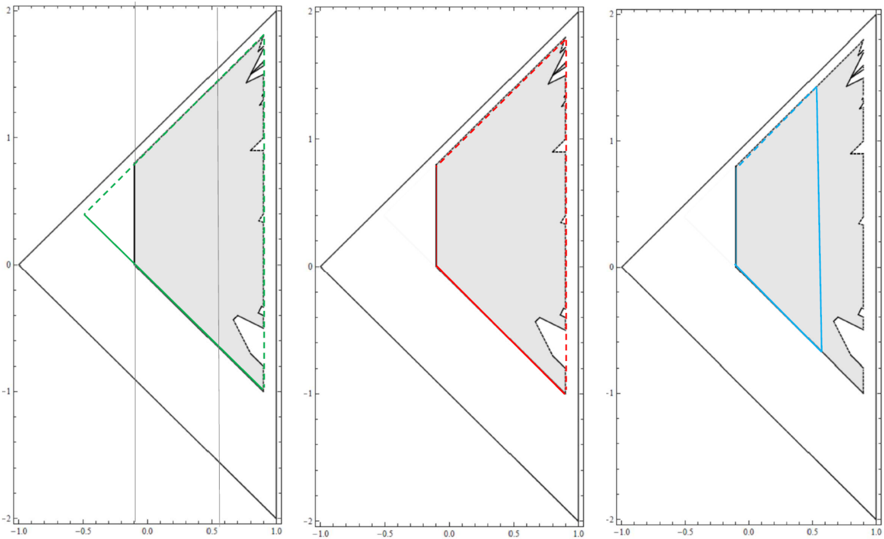

$\mathcal{D}^0_{2,\varepsilon} \subset D^*(\varepsilon)$, $\quad D(\varepsilon) := D^*(\varepsilon) \cap L(\varepsilon)$, $\quad B(\varepsilon) = D^*(\varepsilon) \cap S(\varepsilon)$

Since $D^*(\varepsilon)$, $L(\varepsilon)$ and $R(\varepsilon)$ are explicitly given sets, then $D(\varepsilon) = D^*(\varepsilon) \cap L(\varepsilon)$ and $B(\varepsilon) = D(\varepsilon) \cap R(\varepsilon)$ are also explicitly given sets:

$$D(\varepsilon) = \begin{cases} \left\{(x,y) \in \mathbb{R}^2 \colon -x-\varepsilon \leq y < x+1-\varepsilon, \ \ -\varepsilon \leq x < 1-\varepsilon\right\}, \text{ if } \varepsilon \in \left[0, \frac{1}{2}\right) \\ \left\{(x,y) \in \mathbb{R}^2 \colon -x-1+\varepsilon < y \leq x+\varepsilon, \ -(1-\varepsilon) < x \leq \varepsilon\right\}, \text{ if } \varepsilon \in \left[\frac{1}{2}, 1\right) \end{cases}$$

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- Note that the sets $B(\varepsilon)$ and $D(\varepsilon)$ are defined by the same inequalities except for the upper bound for x. Consequently,

$$\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in D(\varepsilon) \Longrightarrow \left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in B(\varepsilon)$$

except for finitely many possible positive small values of p_0.

Therefore, in order to prove our main result:

- we have to characterize all $(p_0, p_1) \in \mathbb{Z}^2$ such that $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in D(\varepsilon)$, for all $\varepsilon \in [0, 1)$;

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Then using $B(\varepsilon) \subset \mathcal{D}^0_{2,\varepsilon} \subset D(\varepsilon)$ and Theorem 1, similarly as in case $\varepsilon = 0$, we can easily obtain the characterization of $\varepsilon-$CNS polynomials expect for finitely many possible positive small values of p_0.

- Also note if $B(\varepsilon) \subset \mathcal{D}^0_{2,\varepsilon} \subset D(\varepsilon)$ holds, then

$$\mathcal{D}^0_{2,\varepsilon} \cap R(\varepsilon) = B(\varepsilon)$$

which means that we completely characterize sets $\mathcal{D}^0_{2,\varepsilon}$ for $x \leq \frac{2}{3} - \varepsilon$ if $\varepsilon \in \left[0, \frac{1}{2}\right)$ and for $x < \frac{2}{3} - (1-\varepsilon)$ if $\varepsilon \in \left[\frac{1}{2}, 1\right)$.

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- Note that in case $\varepsilon = 0$, the range of x mentioned above ($x \leq \frac{2}{3}$) is the best possible range of x where $\mathcal{D}^0_{2,\varepsilon}$ and $D(\varepsilon)$ coincide. Namely, for $\varepsilon = 0$ the range for x cannot go beyond $\frac{2}{3}$ since the points $(\frac{2}{3}, -\frac{1}{3})$ and $(\frac{2}{3}, \frac{4}{3})$ are on the boundary of a cutout polygons but not contained in them.

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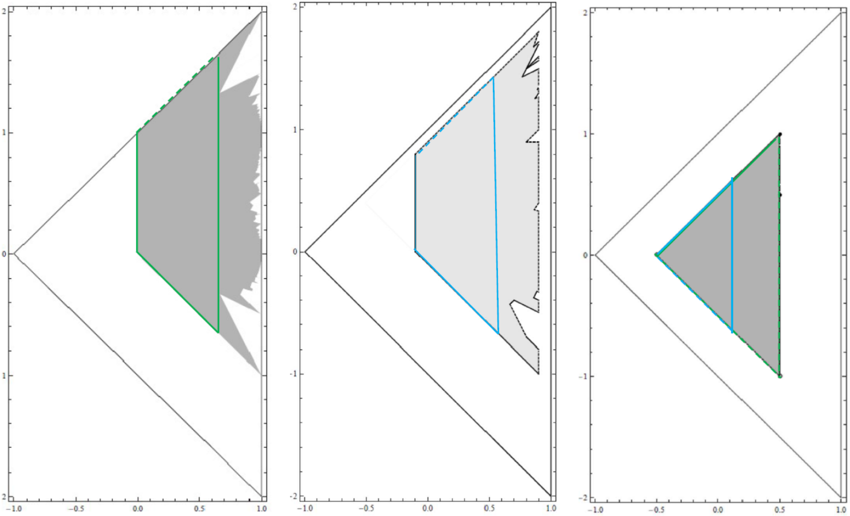
- For all others $\varepsilon \in (0, 1)$ it is probably not the case.
- For $\varepsilon = \frac{1}{2}$ the best possible range of x is $x < \frac{1}{2}$, and we have $x < \frac{1}{6} = \frac{2}{3} - (1-\varepsilon)$.
- For $\varepsilon = \frac{1}{5}$ the best possible range of x is $x < \frac{2}{3} - \frac{\varepsilon}{3} = \frac{3}{5}$.
- For $\varepsilon = \frac{1}{10}$ the best possible range of x is $x \leq \frac{2}{3}$.



The sets: $B\left(0\right)$, $B\left(\dfrac{1}{10}\right)$, $B\left(\dfrac{1}{2}\right)$

Characterization of all $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right)$ belonging to sets $D(\varepsilon)$ and $B(\varepsilon)$

Let $P(x) = x^d + p_{d-1}x^{d-1} + \ldots + p_1 x + p_0 \in \mathbb{Z}[x]$, with $|p_0| \geq 2$ and $\varepsilon \in [0, 1)$. Then

$$\mathcal{N}_\varepsilon = [-\varepsilon |p_0|, (1-\varepsilon)|p_0|) \cap \mathbb{Z}.$$

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Let us rewrite the interval $[0,1)$ as disjoint union of the subintervals as follows

$$[0,1) = \left[0, \frac{1}{|p_0|}\right) \cup \left[\frac{1}{|p_0|}, \frac{2}{|p_0|}\right) \cup ... \cup \left[\frac{k}{|p_0|}, \frac{k+1}{|p_0|}\right) \cup ... \cup \left[\frac{|p_0|-1}{|p_0|}, 1\right).$$

If $\varepsilon \in \left[\frac{k}{|p_0|}, \frac{k+1}{|p_0|}\right)$, $k = 0, ..., |p_0| - 1$, then

$$k = \lfloor \varepsilon |p_0| \rfloor \quad \text{and} \quad \mathcal{N}_\varepsilon = \{-k, ..., |p_0| - 1 - k\}.$$

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Therefore, $\mathcal{N}_\varepsilon = \mathcal{N}_{\varepsilon_k}$, where $\varepsilon_k = \frac{k}{|p_0|}$.

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Therefore, $\mathcal{N}_\varepsilon = \mathcal{N}_{\varepsilon_k}$, where $\varepsilon_k = \frac{k}{|p_0|}$.

Corollary

P is $\varepsilon-$CNS polynomial if and only if P is ε_k-CNS polynomial.

Proposition (1)

Let $(p_0, p_1) \in \mathbb{Z}^2$, $|p_0| \geq 2$, $\varepsilon \in [0,1)$, *and let* $k = \lfloor \varepsilon |p_0| \rfloor$.

i) *Let* $\varepsilon \in \left[0, \frac{1}{2}\right)$ *or let* $\varepsilon = \frac{1}{2}$ *if* $|p_0|$ *is odd. Then* $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in D(\varepsilon)$ *if and only if*

$$-k - 1 \leq p_1 \leq p_0 - k, \quad p_0 \geq 2$$

or additionally

$$k + 2 - |p_0| \leq p_1 \leq k - 1, \quad p_0 \leq -3 \ \ \text{if} \ \ \varepsilon \in \left[\frac{1}{|p_0|}, \frac{1}{2}\right].$$

ii) *Let* $\varepsilon \in \left(\frac{1}{2}, 1\right)$ *or let* $\varepsilon = \frac{1}{2}$ *if* $|p_0|$ *is even. Then* $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in D(\varepsilon)$ *if and only if*

$$-p_0 + k \leq p_1 \leq k + 1, \quad p_0 \geq 2,$$

or additionally

$$-k + 1 \leq p_1 \leq -k - 2 + |p_0|, \quad p_0 \leq -3 \ \ \text{if} \ \ \varepsilon \in \left[\frac{1}{2}, \frac{|p_0| - 1}{|p_0|}\right).$$

Consequently, $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in D(\varepsilon)$ *if and only if* $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in D(\varepsilon_k)$ *where* $\varepsilon_k = \frac{k}{|p_0|}$.

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Let $(p_0, p_1) \in \mathbb{Z}^2$, $|p_0| \geq 2$ *and* $\varepsilon \in [0,1)$. *Let* $k = \lfloor \varepsilon |p_0| \rfloor$, $\varepsilon_k = \frac{k}{|p_0|}$. *Then we have:*

i) *If* $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in D(\varepsilon)$ *and* $p_0 \neq 2, 3, 4, 5$, *than* $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in B(\varepsilon)$.

ii) *If* $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in D(\varepsilon)$, *than* $\left(\frac{1}{p_0}, \frac{p_1}{p_0}\right) \in B(\varepsilon_k)$, *except for* $p_0 = 2$ *or* 4 *and* $\varepsilon \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{|p_0|}\right)$.

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Proposition (3)

Let $\varepsilon \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{|p_0|}\right)$ *and* $p_0 = 2$ *or* $p_0 = 4$. *Then* $P(x) = x^2 + p_1 x + p_0$ *is a* $\varepsilon-$*CNS polynomial if and only if* $-p_0 + k \leq p_1 \leq k + 1$ *where* $k = \lfloor \varepsilon |p_0| \rfloor$.

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Remark

If $\varepsilon \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{|p_0|}\right)$ *and* $p_0 = 2$ *or* $p_0 = 4$, *then* $\varepsilon_k = \frac{1}{2}$.

Characterization of set $\mathcal{D}^0_{2,\varepsilon}$ in half-plain $R(\varepsilon)$

We have to prove

$$B(\varepsilon) \subset \mathcal{D}^0_{2,\varepsilon} \subset D(\varepsilon) \quad \text{for all } \varepsilon \in [0,1). \tag{1}$$

We note:

- We know that (1) holds for $\varepsilon = 0$ and $\varepsilon = \frac{1}{2}$ (Akiyama et al. ($\varepsilon = 0$), Akiyama and Scheicher ($\varepsilon = \frac{1}{2}$)).

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- We have
$$\mathcal{D}^0_{2,\varepsilon} \subset D^*(\varepsilon) \quad \text{and} \quad B(\varepsilon), D(\varepsilon) \subset D^*(\varepsilon).$$
- In general it is easier to examine if a certain region of $D^*(\varepsilon)$ does not belong to $\mathcal{D}^0_{d,\varepsilon}$ than opposite.

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- We know that (1) holds for $\varepsilon = 0$ and $\varepsilon = \frac{1}{2}$ (Akiyama et al. ($\varepsilon = 0$), Akiyama and Scheicher ($\varepsilon = \frac{1}{2}$)).
- It is enough to prove (1) for $\varepsilon \in \left(0, \frac{1}{2}\right)$ since the sets $\mathcal{D}^0_{d,\varepsilon}$ and $\mathcal{D}^0_{d,1-\varepsilon}$ are equal up to the boundary and the boundaries are reversed.
- We have
$$\mathcal{D}^0_{2,\varepsilon} \subset D^*(\varepsilon) \quad \text{and} \quad B(\varepsilon), D(\varepsilon) \subset D^*(\varepsilon).$$
- In general it is easier to examine if a certain region of $D^*(\varepsilon)$ does not belong to $\mathcal{D}^0_{d,\varepsilon}$ than opposite.
- We note that
$$D(\varepsilon) = D^*(\varepsilon) \diagdown T(\varepsilon)$$
where $T(\varepsilon)$ is triangle. Thus, to prove $\mathcal{D}^0_{2,\varepsilon} \subset D(\varepsilon)$, it suffices to prove that $T(\varepsilon)$ can be cut out from $D^*(\varepsilon)$, i.e. that $T(\varepsilon) \cap \mathcal{D}^0_{d,\varepsilon} = \emptyset$.

Characterization of set $\mathcal{D}^0_{2,\varepsilon}$ in half-plain $R(\varepsilon)$

We have to prove

$$B(\varepsilon) \subset \mathcal{D}^0_{2,\varepsilon} \subset D(\varepsilon) \quad \text{for all } \varepsilon \in [0,1). \tag{1}$$

We note:

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- In general it is easier to examine if a certain region of $D^*(\varepsilon)$ does not belong to $\mathcal{D}^0_{d,\varepsilon}$ than opposite.
- We note that
$$D(\varepsilon) = D^*(\varepsilon) \setminus T(\varepsilon)$$
where $T(\varepsilon)$ is triangle. Thus, to prove $\mathcal{D}^0_{2,\varepsilon} \subset D(\varepsilon)$, it suffices to prove that $T(\varepsilon)$ can be cut out from $D^*(\varepsilon)$, i.e. that $T(\varepsilon) \cap \mathcal{D}^0_{d,\varepsilon} = \emptyset$.
So, the set inclusion $\mathcal{D}^0_{2,\varepsilon} \subset D(\varepsilon)$ is easy to prove, but it turns out to be a hard problem to prove that some parts of the set $B(\varepsilon)$ belong to $\mathcal{D}^0_{2,\varepsilon}$.

Algorithm for ε-shift radix systems

- We start with $\mathcal{D}_{2,\varepsilon}$ and have to remove all points $\mathbf{r}$ where $\left(\tau^n_{\mathbf{r},\varepsilon}(\mathbf{z})\right)_{n\in\mathbb{N}}$ is periodic for some $\mathbf{z} \in\mathbb{Z}^2$, $\mathbf{z} \neq \mathbf{0}$. In particular, $\mathbf{r} \notin \mathcal{D}^0_{2,\varepsilon}$, when there exist nonzero points $\mathbf{z}_0,\dots,\mathbf{z}_{l-1} \in \mathbb{Z}^2$ with

$$\mathbf{z}_0 \xrightarrow{\tau_{\mathbf{r},\varepsilon}} \mathbf{z}_1 \xrightarrow{\tau_{\mathbf{r},\varepsilon}} \dots\dots \xrightarrow{\tau_{\mathbf{r},\varepsilon}} \mathbf{z}_{l-1} \xrightarrow{\tau_{\mathbf{r},\varepsilon}} \mathbf{z}_0$$

By definition of mapping $\tau_{\mathbf{r},\varepsilon}$ these points are of the form

$$\mathbf{z}_0 = (z_0, z_1)\,,\ \mathbf{z}_1 = (z_1, z_2)\,, \dots, \mathbf{z}_{l-1} = (z_{l-1}, z_0)\,.$$

We refer to such a sequence as a *cycle of $\tau_{\mathbf{r},\varepsilon}$ of period l* and we write it in the form $\pi = (z_0, z_1, \dots, z_{l-1})$.

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  We refer to such a sequence as a *cycle of $\tau_{\mathbf{r},\varepsilon}$ of period l* and we write it in the form $\pi = (z_0, z_1, ..., z_{l-1})$.
- By definition of mapping $\tau_{\mathbf{r},\varepsilon}$ for $\mathbf{r} = (r_1, r_2)$ we derive

$$\mathbf{z}_i = (z_i, z_{i+1}) \xrightarrow{\tau_{\mathbf{r},\varepsilon}} \mathbf{z}_{i+1} = (z_{i+1}, z_{i+2}), \quad \forall i = 0, ..., l-2$$

  if and only if

$$0 \leq r_1 z_i + r_2 z_{i+1} + z_{i+2} + \varepsilon < 1, \ \forall i = 0, ..., l-2. \qquad (2)$$

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- Hence, $\pi = (z_0, z_1, \ldots, z_{l-1})$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for those $\mathbf{r} = (r_1, r_2)$ that satisfy the system of inequalities (2).

- For a cycle $\pi = (z_0, z_1, ..., z_{l-1})$ of $\tau_{\mathbf{r},\varepsilon}$ we define

$$P_\varepsilon(\pi) = \left\{ \mathbf{r} = (r_1, r_2) \in \mathbb{R}^2 : (r_1, r_2) \text{ satisfies } (2) \right\}.$$

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- Since $\mathbf{r} \in \mathcal{D}^0_{2,\varepsilon}$ if and only if $\tau_{\mathbf{r},\varepsilon}$ has (0) as its only period we conclude that

$$\mathcal{D}^0_{2,\varepsilon} = \mathcal{D}_{2,\varepsilon} \diagdown \bigcup_{\pi \in \Pi} P_\varepsilon(\pi)$$

  where Π is set of all families of *cycles* $\pi \neq (0)$ of finite length. We call this family of (non-empty) polygons *the family of cutout polygons of* $\mathcal{D}^0_{2,\varepsilon}$.
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- The problem is that this representation is not very practicable, since Π is a infinite set.
- There is an algorithm, based on "convexity property" of $\tau_{\mathbf{r},\varepsilon}$, that allows us to check whether a given subset of $Int\mathcal{D}_{2,\varepsilon}$ is subset of set $\mathcal{D}^0_{2,\varepsilon}$ or not.

- Let $\varepsilon \in [0, 1)$ and let

$$Q \subset Int\mathcal{D}_{2,\varepsilon}$$

be closed and convex set, for example the convex hull H of $\mathbf{r}_1, ..., \mathbf{r}_k$. Then there exists an algorithm to create *a set of witnesses* $\mathcal{V} \subset \mathbb{Z}^2$ for Q and an algorithm to create a *finite directed graph* $G(\mathcal{V}, \varepsilon) = \mathcal{V} \times E$ with set of vertices $\mathcal{V} \subset \mathbb{Z}^2$ and set of edges $E \subset \mathcal{V} \times \mathcal{V}$.

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- Let $\mathcal{V}$ be a **finite** set of witnesses of the closed and convex set $Q \subset Int\mathcal{D}_{2,\varepsilon}$. Further let Λ be the set of graph-cycles π of $G(\mathcal{V}, \varepsilon)$ without the trivial one (0). Then
$$Q \cap \mathcal{D}^0_{2,\varepsilon} = Q \diagdown \bigcup_{\pi \in \Lambda} P_\varepsilon(\pi).$$

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- Consequently, if $G(\mathcal{V}, \varepsilon)$ has only trivial cycle (0) or if $\bigcup_{\pi \in \Lambda} P_\varepsilon(\pi) \cap Q = \emptyset$, then $Q \subset \mathcal{D}^0_{2,\varepsilon}$.
- If the algorithm to create a finite directed graph $G(\mathcal{V}, \varepsilon)$ does not converge, we have to subdivide Q into several parts and perform the algorithm for each of these parts.

- We know

$$\mathcal{D}_{2,\varepsilon}^{0} \subset \overline{D^{*}(\varepsilon)} \subset Int\mathcal{D}_{2,\varepsilon}$$

So we apply the algorithm for the closed convex set $\overline{D^{*}(\varepsilon)}$, precisely for the sets $Q = \overline{T(\varepsilon)}$ and $Q = \overline{B(\varepsilon)}$.

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- Denote $\Delta\left(A,B,C\right):=\overline{T\left(\varepsilon\right)}$. We obtain that corresponding graph $G(\mathcal{V},\varepsilon)$ has two nontrivial cycles: $\pi_1=\left(1,0\right),\pi_2=\left(-1,1\right)$ and

$$\Delta\left(A,B,C\right)\cap\mathcal{D}_{2,\varepsilon}^{0}=\Delta\left(A,B,C\right)\diagdown\left(P_{\varepsilon}\left(\pi_1\right)\cup P_{\varepsilon}\left(\pi_2\right)\right)=\overline{BC}\diagdown\left\{C\right\}$$

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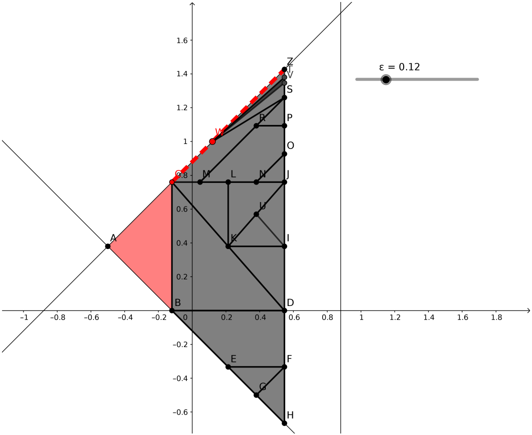
$$\Longrightarrow T(\varepsilon) \cap \mathcal{D}^0_{d,\varepsilon} = \emptyset \Longrightarrow \mathcal{D}^0_{2,\varepsilon} \subset D(\varepsilon).$$
- We have to subdivide the set $\overline{B(\varepsilon)}$ into several convex parts (for example 16 parts if $\varepsilon \in \left[\frac{1}{10}, \frac{1}{2}\right)$) since the algorithm that creates *a set of witnesses* $\mathcal{V}$ doesn't converge if the convex set isn't sufficiently small. The problem becomes harder and harder the closer ε is to 0 and the nearer we get to the line $y = x + 1 - \varepsilon$.



- For example for $\Delta(D, I, K) \subset \overline{B(\varepsilon)}$ we obtain that $G(\mathcal{V}, \varepsilon)$ has only one nontrivial cycles $\pi_3 = (1, 1, -1)$

$$\Delta(D, I, K) \cap P_\varepsilon(\pi_3) = \emptyset \Longrightarrow \Delta(D, I, K) \subset \mathcal{D}^0_{2,\varepsilon}.$$

- For for $\Delta(E, F, G) \subset \overline{B(\varepsilon)}$ we obtain that $G(\mathcal{V}, \varepsilon)$ has only trivial cycles $\pi = (0)$, then $\Delta(E, F, G) \subset \mathcal{D}^0_{2,\varepsilon}$,

$\varepsilon-$canonical number systems in number fields

Definition

Let L be a number field of degree n, and denote its ring of integers by $\mathbb{Z}_L$. Let $\alpha \in \mathbb{Z}_L$ and let $\mathcal{N} \subset \mathbb{Z}$ be a complete residue system modulo $\left|Norm_{L/\mathbb{Q}}(\alpha)\right|$ containing 0. The pair $(\alpha, \mathcal{N})$ is called a number system in $\mathbb{Z}_L$ (or in L) if each $\gamma \in \mathbb{Z}_L$ is represented uniquely in the form

$$\gamma = d_0 + d_1\alpha + \ldots + d_{l-1}\alpha^{l-1}, \quad \text{with } d_0, d_1, \ldots, d_{l-1} \in \mathcal{N},\ d_{l-1} \neq 0. \tag{1}$$

α is called base of number system $(\alpha, \mathcal{N})$ and $\mathcal{N}$ is called its set of digits.

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- In particular, number system $(\alpha, \mathcal{N})$ is called a $\varepsilon-$canonical number systems (or $\varepsilon-$CNS) in $\mathbb{Z}_L$ if

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where $\varepsilon \in [0, 1)$.

- Pethő and Thuswaldner (2019) gave more general definition of number system:

Definition (Generalized number system)

Let $\mathbb{K}$ be a number field of degree k and let $\mathcal{O}$ be an order in $\mathbb{K}$. Let $P \in \mathcal{O}[x]$ be monic polynomial and $\mathcal{N} \subset \mathcal{O}$ be a complete residue system modulo $P(0)$ containing 0. The pair $(P, \mathcal{N})$ is called a generalized number system over $\mathcal{O}$ (GNS for short) if each $a \in \mathcal{O}[x]$ admits a representation of the form a

$$a \equiv d_0 + d_1 x + \ldots + d_{l-1} x^{l-1} \pmod{P} \quad \text{with } d_0, d_1, \ldots, d_{l-1} \in \mathcal{N},\ d_{l-1} \neq 0.$$

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- If $\mathbb{K} = \mathbb{Q}$, $\mathcal{O} = \mathbb{Z}$, $P \in \mathbb{Z}[x]$ is monic **irreducible** polynomial then

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  with $d_0, ..., d_{l-1} \in \mathcal{N}$, $d_{l-1} \neq 0$. So, instead of $(P, \mathcal{N})$, can write $(\alpha, \mathcal{N})$.

Definition (Generalized number system)

Let $\mathbb{K}$ be a number field of degree k and let $\mathcal{O}$ be an order in $\mathbb{K}$. Let $P \in \mathcal{O}[x]$ be monic polynomial and $\mathcal{N} \subset \mathcal{O}$ be a complete residue system modulo $P(0)$ containing 0. The pair $(P, \mathcal{N})$ is called a generalized number system over $\mathcal{O}$ (GNS for short) if each $a \in \mathcal{O}[x]$ admits a representation of the form a

$$a \equiv d_0 + d_1 x + \ldots + d_{l-1} x^{l-1} \pmod P \quad \text{with } d_0, d_1, \ldots, d_{l-1} \in \mathcal{N},\ d_{l-1} \neq 0.$$

- If $\mathbb{K} = \mathbb{Q}$ and $\mathcal{O} = \mathbb{Z}$ - Pethő's definition of number system $(P, \mathcal{N})$
- If $\mathbb{K} = \mathbb{Q}$, $\mathcal{O} = \mathbb{Z}$, $P \in \mathbb{Z}[x]$ is monic **irreducible** polynomial then

  $$\mathbb{Z}[x]/P(x)\mathbb{Z}[x] \cong \mathbb{Z}[\alpha]$$

  for **any** root α of P. Note that $|P(0)| = |Norm_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha)|$.
- In this case the fact that $(P, \mathcal{N})$ is number system is equivalent to the fact that each $\gamma \in \mathbb{Z}[\alpha]$ admits a unique expansion of the form

  $$\gamma = d_0 + d_1\alpha + \ldots + d_{l-1}\alpha^{l-1}$$

  with $d_0, \ldots, d_{l-1} \in \mathcal{N}$, $d_{l-1} \neq 0$. So, instead of $(P, \mathcal{N})$, can write $(\alpha, \mathcal{N})$. Note

  $$(\alpha, \mathcal{N}) \text{ is number system in } \mathbb{Z}_{\mathbb{Q}(\alpha)} \text{ (in } L = \mathbb{Q}(\alpha)) \implies \mathbb{Z}_L = \mathbb{Z}[\alpha]$$

- Therefore, if there exist a number system $(\alpha, \mathcal{N})$ in the ring of integers $\mathbb{Z}_L$ of field L, then $\mathbb{Z}_L$ is monogenic and α generates power integral basis.

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- The real innovation of Pethő and Thuswaldner's paper (2019) is the definition of the digit sets of GNS by using fundamental domains $\mathcal{F}$ of the action of $\mathbb{Z}^k$ on $\mathbb{R}^k$ containing 0, where k denotes the degree of $\mathbb{K}$ over $\mathbb{Q}$. For $P \in \mathcal{O}[x]$ this enabled them to define the digit set,

$$\mathcal{N}_{\mathcal{F},P(0)} =: \mathcal{N}_{\mathcal{F}}$$

uniformly depending only on the chosen fundamental domain $\mathcal{F}$ and on $P(0)$.

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- If $\mathbb{K} = \mathbb{Q}$, $\mathcal{O} = \mathbb{Z}$,

$$\mathcal{F}_\varepsilon = [-\varepsilon, 1-\varepsilon),\ \varepsilon \in [0,1)$$

  and $P \in \mathbb{Z}[x]$ is a monic polynomial then GNS $(P, \mathcal{N}_{\mathcal{F}_\varepsilon})$ is $\varepsilon-$CNS $(P, \mathcal{N}_\varepsilon)$ since

$$\mathcal{N}_{\mathcal{F}_\varepsilon} = [-\varepsilon\,|P(0)|\,, (1-\varepsilon)\,|P(0)|) \cap \mathbb{Z} = \mathcal{N}_\varepsilon$$

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If P is additionally an **irreducible** polynomial, then GNS $(P, \mathcal{N}_{\mathcal{F}_\varepsilon})$ is $\varepsilon-$CNS $(\alpha, \mathcal{N}_\varepsilon(\alpha))$ in field $L = \mathbb{Q}(\alpha)$ where α is **any** root of P.

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- Kovács and Pethő (1992) proved that if α is a generator of power integral basis then, up to finitely many possible exceptions, $\alpha + m, m \in \mathbb{Z}$ is a basis of 0-CNS if and only if $m \leq M$, where M denotes a constant (for each sufficiently small m).

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- Pethő and Thuswaldner (2019): If $0 \in int(\mathcal{F})$ then all but finitely many generators of power integral bases of $\mathbb{Z}_L$ form a basis for number system in L. In particular, if $\varepsilon \in (0, 1)$ then all but finitely many generators of power integral bases of $\mathbb{Z}_L$ form a basis of $\varepsilon-$CNS in $\mathbb{Z}_L$.

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- This means that $0-$CNS are quite exceptional among other $\varepsilon-$CNS. They are kind of "boundary case" since $0 \in \partial\mathcal{F}$.
- Therefore, the problem of finding all $\varepsilon-$CNS $(\alpha, \mathcal{N}_\varepsilon(\alpha))$ in L, where $\varepsilon \in [0,1)$, is equivalent to find all $\varepsilon-$CNS $(P, \mathcal{N}_\varepsilon)$, where $P \in \mathbb{Z}[x]$ runs through the set of all minimal polynomials of all generators of power integral basis α of $\mathbb{Z}_L$.

- Let L be quadratic number filed and let $D \neq 0, 1$ be unique squarefree integer such that $L = \mathbb{Q}\left(\sqrt{D}\right)$. It is well known that the ring of integers $\mathcal{O}_{\mathbb{L}}$ of L is monogenic and that all generators of power integral basis of $\mathcal{O}_L$ are given by

$$\begin{aligned} \alpha &= \pm\sqrt{D} + m,\ m \in \mathbb{Z} \ \text{ if } \ D \equiv 2, 3 \pmod 4 \\ \alpha &= \frac{1 \pm \sqrt{D}}{2} + m,\ m \in \mathbb{Z} \ \text{ if } \ D \equiv 1 \pmod 4 \end{aligned}$$

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- The minimal polynomial of $\alpha = \pm\sqrt{D} + m$ is

$$P_m(x) = x^2 - 2mx + \left(m^2 - D\right),$$

while minimal polynomial of of $\alpha = \frac{1 \pm \sqrt{D}}{2} + m$ is

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- Using our characterization of quadratic $\varepsilon-$CNS polynomials we are able to find explicitly all $\varepsilon-$CNS in all quadratic number fields $L = \mathbb{Q}\left(\sqrt{D}\right)$. Namely, a generator of power integral basis α is $\varepsilon-$CNS basis in $\mathcal{O}_L$ if and only if corresponding minimal polynomial P_m is $\varepsilon-$CNS polynomial.

Case $D \equiv 2, 3 (\mathrm{mod}\, 4)$

Let $P_m(x) = x^2 - 2mx + \left(m^2 - D\right)$, where $m \in \mathbb{Z}$ and let $\varepsilon \in [0, 1)$. By our theorem on characterization of quadratic $\varepsilon-$CNS polynomials and using the properties of the floor and ceiling functions, we derive:

a) Let $\varepsilon \in \left[0, \frac{1}{2}\right)$ or let $\varepsilon = \frac{1}{2}$ if $\left|m^2 - D\right|$ is odd. Then P_m is a $\varepsilon-$CNS polynomial if and only if

$$-\varepsilon m^2 + \varepsilon D - 1 \leq -2m < (1 - \varepsilon)\, m^2 - (1 - \varepsilon)\, D + 1,\ \ m^2 - D \geq 2$$

or additionally

$$(1 - \varepsilon)\, m^2 - (1 - \varepsilon)\, D + 1 < -2m \leq -\varepsilon m^2 + \varepsilon D - 1, \ \ m^2 - D \leq -3 \text{ if } \ \varepsilon \in \left[\frac{1}{D - m^2}, \frac{1}{2}\right].$$

Case $D \equiv 2, 3 \pmod 4$

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b) Let $\varepsilon \in \left(\frac{1}{2}, 1\right)$ or let $\varepsilon = \frac{1}{2}$ if $\left|m^2 - D\right|$ is even. Then P_m is a $\varepsilon-$CNS polynomial if and only if

$$-(1 - \varepsilon) m^2 + (1 - \varepsilon) D - 1 < -2m \leq \varepsilon m^2 - \varepsilon D + 1, \;\; m^2 - D \geq 2,$$

or additionally

$$\varepsilon m^2 - D\varepsilon + 1 \leq -2m < -(1 - \varepsilon) m^2 + (1 - \varepsilon) D - 1, \;\; m^2 - D \leq -3 \text{ if } \; \varepsilon \in \left[\frac{1}{2}, 1 - \frac{1}{D - m^2}\right)$$

Theorem 1 *Let* $\varepsilon \in [0,1)$ *and let* $D \equiv 2,3 \pmod 4$ *be squarefree rational integer. Let* $\mathcal{O}_{\mathbb{L}}$ *be the ring of integers of the number field* $\mathbb{L} = \mathbb{Q}\left(\sqrt{D}\right)$.
Then $\alpha \in \mathcal{O}_{\mathbb{L}}$ *is a* $\varepsilon-$*CNS basis in* $\mathcal{O}_{\mathbb{L}}$ *if and only if* $\alpha = \pm\sqrt{D} + m$, $m \in \mathbb{Z}$ *where* $m \in \mathbb{Z}$ *satisfies the following conditions:*
i) Case $\varepsilon = 0$

$$\begin{array}{ll} m \leq 0, & \text{if } D \leq -2,\ \varepsilon = 0 \\ m \leq -1, & \text{if } D = -1,\ \varepsilon = 0 \\ m \leq -\left\lfloor \sqrt{D} \right\rfloor - 2, & \text{if } D > 0,\ \varepsilon = 0 \end{array}$$

ii) Case $D > 0$ *and* $\varepsilon \in (0,1)$

$$\begin{array}{ll} m \leq \left\lceil M_1^{(-)} \right\rceil - 1 \ \text{ or } \ m \geq \left\lceil M_2^{(+)} \right\rceil, & \text{if } D > 0, \varepsilon \in \left(0, \frac{1}{2}\right) \\ m \leq \left\lfloor -\sqrt{D+2} \right\rfloor - 2 \ \text{ or } \ m \geq \left\lceil \sqrt{D+2} \right\rceil + 3, & \text{if } D > 0,\ \sqrt{D+2} \in \mathbb{Z},\ \varepsilon = \frac{1}{2} \\ m \leq \left\lfloor -\sqrt{D+2} \right\rfloor - 2 \ \text{ or } \ m \geq \left\lceil \sqrt{D+2} \right\rceil + 2, & \text{if } D > 0,\ \sqrt{D+2} \notin \mathbb{Z},\ \varepsilon = \frac{1}{2}. \\ m \leq -\left\lceil M_2^{(+)} \right\rceil \ \text{ or } \ m \geq -\left\lceil M_1^{(-)} \right\rceil + 1, & \text{if } D > 0, \varepsilon \in \left(\frac{1}{2}, 1\right) \end{array}$$

and additionally

$$
\begin{array}{ll}
m = 0, & \textit{if } D = 3,\ \varepsilon \in \left[\frac{1}{3}, \frac{2}{3}\right) \\
\left\lceil M_2^{(-)} \right\rceil \le m \le \left\lfloor \sqrt{D - \frac{1}{\varepsilon}} \right\rfloor, & \textit{if } D > 3,\ \varepsilon \in \left[\frac{1}{D}, \frac{1}{2\sqrt{D-1}}\right) \\
\left\lceil M_2^{(-)} \right\rceil \le m \le \left\lceil M_1^{(+)} \right\rceil - 1, & \textit{if } D > 3,\ \varepsilon \in \left[\frac{1}{2\sqrt{D-1}}, \frac{1}{2}\right) \\
-\left\lceil \sqrt{D+2} \right\rceil + 3 \le m \le \left\lfloor \sqrt{D+2} \right\rfloor - 2, & \textit{if } D > 3,\ \varepsilon = \frac{1}{2} \\
-\left\lceil M_1^{(+)} \right\rceil + 1 \le m \le -\left\lceil M_2^{(-)} \right\rceil, & \textit{if } D > 3,\ \varepsilon \in \left(\frac{1}{2}, 1 - \frac{1}{2\sqrt{D-1}}\right] \\
-\left\lceil M_1^{(+)} \right\rceil + 1 \le m \le \left\lfloor \sqrt{D - \frac{1}{1-\varepsilon}} \right\rfloor, & \textit{if } D > 3,\ \varepsilon \in \left(1 - \frac{1}{2\sqrt{D-1}}, 1 - \frac{1}{D}\right)
\end{array}
$$

iii) Case $D < 0$ and $\varepsilon \in (0,1)$

$$m \geq \left\lceil M_2^{(+)} \right\rceil \;\text{ or } m \leq -1, \qquad \text{if } D=-1,\ \varepsilon \in \left(0, \tfrac{3-\sqrt{5}}{2}\right)$$

$$m \geq \left\lceil M_2^{(+)} \right\rceil \;\text{ or } m \leq \left\lceil M_1^{(-)} \right\rceil -1 \;\text{ or }\; m=-1, \qquad \text{if } D=-1,\ \varepsilon \in \left[\tfrac{3-\sqrt{5}}{2}, \tfrac{1}{2}\right)$$

$$m \geq 4 \;\text{ or }\; m \leq -3 \text{ or } m=-1, \qquad \text{if } D=-1,\ \varepsilon = \tfrac{1}{2}$$

$$m \geq -\left\lceil M_1^{(-)} \right\rceil +1 \text{ or } m \leq -\left\lceil M_2^{(+)} \right\rceil \;\text{ or } m=-1, \qquad \text{if } D=-1,\ \varepsilon \in \left(\tfrac{1}{2}, \tfrac{-1+\sqrt{5}}{2}\right]$$

$$m \geq -\left\lceil M_1^{(-)} \right\rceil +1 \text{ or } m \leq -1 \qquad \text{if } D=-1,\ \varepsilon \in \left(\tfrac{-1+\sqrt{5}}{2}, 1\right)$$

$$m \geq \left\lceil M_2^{(+)} \right\rceil \;\text{ or } m \leq \left\lfloor M_2^{(-)} \right\rfloor, \qquad \text{if } D=-2, \varepsilon \in \left(0, \tfrac{1}{2}\right)$$

$$|m| \geq 3, \qquad \text{if } D=-2,\ \varepsilon = \tfrac{1}{2}$$

$$m \geq -\left\lfloor M_1^{(-)} \right\rfloor +1 \text{ or } m \leq -\left\lceil M_1^{(+)} \right\rceil -1, \qquad \text{if } D=-2, \varepsilon \in \left(\tfrac{1}{2}, 1\right)$$

$$m \geq \left\lceil M_2^{(+)} \right\rceil \;\text{ or } m \leq \left\lfloor M_2^{(-)} \right\rfloor, \qquad \text{if } D \leq -3,\ \varepsilon \in \left(0, \tfrac{1}{2}\right),\ D \geq -\tfrac{1-\varepsilon}{\varepsilon^2}$$

$$m \geq -\left\lfloor M_1^{(-)} \right\rfloor +1 \text{ or } m \leq -\left\lceil M_1^{(+)} \right\rceil -1, \qquad \text{if } D \leq -3,\ \varepsilon \in \left(\tfrac{1}{2}, 1\right),\ D \geq -\tfrac{\varepsilon}{(1-\varepsilon)^2}$$

$$m \in \mathbb{Z}, \qquad \text{if} \begin{cases} D \leq -3,\ \varepsilon = \frac{1}{2} \\ \quad or \\ D \leq -3,\ \varepsilon \in \left(0, \frac{1}{2}\right),\ D < -\frac{1-\varepsilon}{\varepsilon^2} \\ \quad or \\ D \leq -3,\ \varepsilon \in \left(\frac{1}{2}, 1\right),\ D < -\frac{\varepsilon}{(1-\varepsilon)^2} \end{cases},$$

where $M_1^{(\pm)} = M_1^{(\pm)}(\varepsilon, D)$ *and* $M_2^{(\pm)} = M_2^{(\pm)}(\varepsilon, D)$ *are given by*

$$M_2^{(-)}(\varepsilon, D) = -\frac{1}{\varepsilon}\left(\sqrt{(1-\varepsilon)+\varepsilon^2 D} - 1\right),\ M_2^{(+)}(\varepsilon, D) = \frac{1}{\varepsilon}\left(\sqrt{(1-\varepsilon)+\varepsilon^2 D} + 1\right)$$

$$M_1^{(-)}(\varepsilon, D) = -\frac{1}{1-\varepsilon}\left(\sqrt{\varepsilon + D(1-\varepsilon)^2} + 1\right),\ M_1^{(+)}(\varepsilon, D) = \frac{1}{1-\varepsilon}\left(\sqrt{\varepsilon + D(1-\varepsilon)^2} - 1\right).$$