The Zelevinsky classification of unramified representations of the metaplectic group

Igor Ciganović
Faculty of Science - University of Zagreb

joint work with Neven Grbac

Representation theory XVIII,
Dubrovnik, IUC

June 2023

F nonarchimedean field, char $F = 0$,
$\mathcal{O}$ ring of integers
$\mathfrak{p} = (\pi)$ maximal ideal
$q = p^f = |\mathcal{O}/\mathfrak{p}|$
$p \neq 2$ residual characteristic
Hilbert simbol of F is the function:

$$(\ , \ ) : F^\times \times F^\times \longrightarrow \{\pm 1\},$$

$$(a, b)_F = \begin{cases} 1, & \text{if } z^2 = ax^2 + by^2 \text{ has a non trivial solution in } F^3, \\ -1, & \text{else.} \end{cases}$$

$p \neq 2 \implies (\mathcal{O}^\times, \mathcal{O}^\times)_F = 1.$
$| \ |_F$ normalized absolute value on F, $|\pi|_F = \frac{1}{q}$
ν character of $GL(n, F)$:

$$\nu(g) = |\det g|_F, \ g \in GL(n, F).$$

W_n $2n$ dimensional vector space over F.
$\langle\ ,\ \rangle$ nondegenerate bilinear antisymmetric form on W_n,
$(e) = \{e_1, ..., e_n, e'_1, ..., e'_n\}$ basis of W_n such that:

$$\langle e_i, e_j\rangle = 0,\ \langle e'_i, e'_j\rangle = 0,\ \langle e_i, e'_j\rangle = \delta_{i,j},\ \ i, j = 1, ..., n.$$

$Sp(W_n)$, acts on W_n, on right, and we have a matrix representation

$$Sp(n, F) = \left\{\ h \in\ GL(2n, F) \mid h \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} h^t = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\ \right\},$$

where I_n is unit $n \times n$ matrix.

$$Sp(n, \mathcal{O}) = \{(a_{i,j}) \in Sp(n, F) \mid a_{i,j} \in \mathcal{O}\}$$

is a maximal compact subgroup. It is open, and it is the stabilizer of the $\mathcal{O}$ lattice of the base (e).
The stabilizer of the $\mathcal{O}$ lattice of $\{\pi^{-1}e_1, ..., \pi^{-1}e_n, e'_1, ..., e'_n\}$ gives the other conjugacy class of maximal compact subgroups.

The metaplectic group,

$$\widetilde{Sp(n,F)} = Sp(n,F) \times \mu_2, \text{ as a set}$$

where $\mu_2 = \{\pm 1\}$, is the unique non-trivial two-fold central extension of the $Sp(n,F)$:

$$1 \longrightarrow \{-1,1\} \xrightarrow{i} \widetilde{Sp(n,F)} \xrightarrow{p} Sp(n,F) \longrightarrow 1$$

The multiplication is given by

$$[h_1,\epsilon_1][h_2,\epsilon_2] = [h_1h_2, \epsilon_1\epsilon_2 \; c_{Rao}(h_1,h_2)], \; h_i \in Sp(n,F), \; \epsilon_i \in \mu_2,$$

where c_{Rao} is the Rao cocycle.

$Sp(n,\mathcal{O})$ splits into its cover, making $\widetilde{Sp(n,F)}$ an l-group.

Similarly,

$$GL(n,F) \to Sp(n,F), \quad g \mapsto \begin{bmatrix} g & 0 \\ 0 & g^{-t} \end{bmatrix},\ g \in GL(n,F),$$

where $(\ )^{-t}$ is an inverse of the transposition.

$$\widetilde{GL(n,F)} = GL(n,F) \times \mu_2 \qquad \text{as a set, and}$$

$$[g_1, \epsilon_1][g_2, \epsilon_2] = [g_1 g_2, \epsilon_1 \epsilon_2 \, (\det\ h_1, \det\ h_2)_F],\ g_i \in GL(n,F),\ \epsilon_i \in \mu_2$$

We have

$$1 \longrightarrow \mu_2 \stackrel{i}{\longrightarrow} \widetilde{GL(n,F)} \stackrel{p}{\longrightarrow} GL(n,F) \longrightarrow 1$$

The mapping $g \mapsto [g,1]$ is splitting of $GL(n,\mathcal{O})$, the maximal compact subgroup of $GL(n,F)$, into $\widetilde{GL(n,\mathcal{O})}$.

Let G be either $GL(n, F)$, or $Sp(n, F)$, and $K = GL(n, \mathcal{O})$, or $Sp(n, \mathcal{O})$, the fixed maximal compact subgroup of G. Let $\overline{K}$ be the image of the splitting of K into its cover.

A smooth representation of $\widetilde{G}$ (G) is said to be unramified if it contains a nontrivial vector fixed by $\overline{K}$ (K).

Fix a non-trivial additive character ψ of F of even conductor. Define

$$\chi_\psi([g,\epsilon]) = \epsilon\gamma\left(\psi_{\frac{1}{2}}\right)\gamma\left(\psi_{\frac{\det g}{2}}\right)^{-1}, \quad g \in GL(n,F),\ \epsilon \in \mu_2,$$

where $\psi_a(x) = \psi(ax)$, and $\gamma(\psi)$ is the Weil index, an 8-th root of unity associated to ψ by Weil.

χ_ψ is a genuine unramified representation of $\widetilde{GL(n,F)}$, moreover

$$\pi \longmapsto \chi_\psi \pi$$

$$Irr_{unr}\, GL(n,F) \longleftrightarrow Irr_{unr}\, \widetilde{GL(n,F)}.$$

Fix B the Borel subgroup of $Sp(n, F)$ of upper triangular matrices. Standard parabolic subgroups of $Sp(n, F)$ are parameterized by ordered partitions of n

$$s = (n_1, \ldots, n_k; n_0) \longleftrightarrow P_s = M_s N_s,$$

where M_s is the Levi factor and N_s the unipotent radical. Let $\widetilde{P_s}$ and $\widetilde{M_s}$ be the preimages of P_s and M_s in $\widetilde{Sp(n, F)}$ and $N_s' = N_s \times \{1\}$. We have parabolic subgroups in $\widetilde{Sp(n, F)}$:

$$\widetilde{P_s} = \widetilde{M_s} N_s'$$

and an epimorphism ϕ with finite kernel

$$\widetilde{GL(n_1, F)} \times \cdots \times \widetilde{GL(n_k, F)} \times \widetilde{Sp(n_0, F)} \xrightarrow{\phi} \widetilde{M_s}.$$

We have functors of normalised parabolic induction and Jacquet module

$$Ind_{\widetilde{M}_s}^{\widetilde{G}} : Alg\,\widetilde{M}_s \to Alg\,\widetilde{G},$$
$$r_s = Jacq_s = Jacq_{\widetilde{M}_s}^{\widetilde{G}} : Alg\,\widetilde{G} \to Alg\,\widetilde{M}_s,$$

We have the Frobenius reciprocity: for σ in $Alg\,\widetilde{G}$ and ρ in $Alg\,\widetilde{M}_s$,

$$Hom_{\widetilde{G}}\left(\sigma, Ind_{\widetilde{M_s}}^{\widetilde{G}}(\rho)\right) \cong Hom_{\widetilde{M_s}}\left(Jacq_{\widetilde{M_s}}^{\widetilde{G}}(\sigma), \rho\right).$$

For π_1, π_2 smooth irreducible representations of GL

$$\chi_\psi(\pi_1 \times \pi_2) \cong \chi_\psi \pi_1 \times \chi_\psi \pi_2.$$

If π is an irreducible unramified representation of $GL(n, F)$, then there exist unramified characters $\chi_1, \ldots, \chi_n$ of $F^\times$ such that

$$\pi \leq \chi_1 \times \cdots \times \chi_n$$

is a unique irreducible unramified subquotient.

If π is an irreducible unramified (genuine) representation of $\widetilde{Sp(n, F)}$, then there exist unramified characters $\chi_1, \ldots, \chi_n$ of $F^\times$ such that

$$\pi \leq \chi_\psi \chi_1 \times \cdots \times \chi_\psi \chi_n \rtimes \omega_0$$

is a unique irreducible unramified subquotient. Here ω_0 is an irreducible representation of $\mu_2 = \{\pm 1\}$

Lemma. Let G be either $GL(n, F)$ or $Sp(n, F)$, and K the fixed maximal compact subgroup of G. Let $\overline{K}$ be the image of the splitting of K. In the notation as above, we have

1. $\widetilde{G} = \widetilde{P_s}\overline{K}$ (Iwasawa decomposition)
2. $\widetilde{P_s} \cap \overline{K} = (\widetilde{M_s} \cap \overline{K})(N'_s \cap \overline{K})$
3. if $G = GL(n, F)$ then,
   $\overline{GL(n_1, \mathcal{O})} \times \cdots \times \overline{GL(n_k, \mathcal{O})} \overset{\phi}{\cong} \widetilde{M_s} \cap \overline{K}$
4. if $G = Sp(n, F)$ then,
   $\overline{GL(n_1, \mathcal{O})} \times \cdots \times \overline{GL(n_k, \mathcal{O})} \times \overline{Sp(n_0, \mathcal{O})} \overset{\phi}{\cong} \widetilde{M_s} \cap \overline{K}$.

Lemma. Let G be either $GL(n,F)$ or $Sp(n,F)$, and K its fixed maximal compact subgroup.

1. Let σ be a smooth $\widetilde{M_s} \cap \overline{K}$-spherical representation of $\widetilde{M_s}$. Then $Ind_{\widetilde{M_s}}^{\widetilde{G}}(\sigma)$ is $\overline{K}$-spherical.
2. Let σ be a smooth representation of finite length of $\widetilde{M_s}$ such that $Ind_{\widetilde{M_s}}^{\widetilde{G}}(\sigma)$ contains a $\overline{K}$-spherical subquotient. Then σ is $\widetilde{M_s} \cap \overline{K}$-spherical.
3. Let $\pi_1, \ldots, \pi_k$ be smooth genuine representations of finite length of $\widetilde{GL(n_i,F)}$, $i = 1, \ldots, k$, and ρ a smooth genuine representation of finite length of $\widetilde{Sp(n_0,F)}$. Then $\pi_1 \times \cdots \times \pi_k$ (resp., $\pi_1 \times \cdots \times \pi_k \rtimes \rho$) is unramified if and only if π_i's (resp., π_i's and ρ) are unramified.

Let χ be a character of $F^\times$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta + 1 \in \mathbb{Z}_{>0}$.

We have a unique irreducible subrepresentation, called Zelevinsky segment representation of the induced representation

$$\chi_\psi(\chi \circ \det)\nu^{\frac{-\beta+\alpha}{2}} \cong \zeta(-\beta, \alpha, \chi_\psi\chi) \longrightarrow \chi_\psi\nu^{-\beta}\chi \times \cdots \times \chi_\psi\nu^{\alpha}\chi$$

Lemma. Let π be a genuine unramified irreducible representation of $\widetilde{GL(n,F)}$. Then there exist a sequence of Zelevinsky segment representations, unique up to permutation, such that:

$$\pi \cong \zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1^u) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi \chi_k^u),$$

where $\chi_1^u, \ldots, \chi_k^u$ are unramified unitary characters of $F^\times$.

Let σ be a genuine irreducible unramified representation of $\widetilde{Sp(n,F)}$. We call σ negative if for every embedding of form $\sigma \hookrightarrow \chi_1\chi_\psi \times \cdots \times \chi_n\chi_\psi \rtimes \omega_0$, where $\chi_1, \ldots, \chi_n$ are characters of $F^\times$, we have

$$\begin{aligned} e(\chi_1) &\le 0 \\ e(\chi_1) + e(\chi_2) &\le 0 \\ &\cdots \\ e(\chi_1) + \cdots + e(\chi_n) &\le 0. \end{aligned}$$

If above inequalities are strict, σ is said to be strongly negative.

Theorem 1. (Weak form) Let σ be a genuine irreducible unramified representation of $\widetilde{Sp(n,F)}$. Then, either σ is negative, or there exist $k \in \mathbb{Z}_{>0}$, $\alpha_i, \beta_i \in \mathbb{R}$ such that $\alpha_i - \beta_i,\ \alpha_i + \beta_i + 1 \in \mathbb{Z}_{>0}$, unitary unramified characters χ_i of $F^\times$, $i = 1, \ldots, k$, and a genuine unramified irreducible negative representation σ_{neg} of the metaplectic group such that

$$\sigma \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi \chi_k) \rtimes \sigma_{neg}$$

as unique irreducible subrepresentation, and $\zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi \chi_k)$ is irreducible. Data $\zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1)$,..., $\zeta(-\beta_k, \alpha_k, \chi_\psi \chi_k)$ are unique up to permutation, while σ_{neg} is unique up to an isomorphism.

Theorem 2. Let σ be a genuine irreducible unramified negative representation of $\widetilde{Sp(n,F)}$. Then, either σ is strongly negative, or there exist $k \in \mathbb{Z}_{>0}$, unramified unitary characters $\chi_1, \ldots, \chi_k$ of $F^\times$, $\beta_i \in \mathbb{R}$ such that $2\beta_i + 1 \in \mathbb{Z}_{>0}$, $i = 1, \ldots, k$, and a genuine irreducible unramified strongly negative representation σ_{sn} of the metaplectic group such that

$$\sigma \hookrightarrow \zeta(-\beta_1, \beta_1, \chi_\psi \chi_1) \times \cdots \times \zeta(-\beta_k, \beta_k, \chi_\psi \chi_n) \rtimes \sigma_{sn}.$$

Data $\zeta(-\beta_1, \beta_1, \chi_\psi \chi_1), \ldots, \zeta(-\beta_k, \beta_k, \chi_\psi \chi_k)$ are unique up to permutation and replacing χ_i with χ_i^{-1}, while σ_{sn} is unique up to isomorphism.

Theta correspondance (with respect to ψ)

$$\mathit{Irr}\ \mathit{Mp}(W) \longleftrightarrow \mathit{Irr}\ \mathit{SO}(V^+) \cup \mathit{Irr}\ \mathit{SO}(V^-)$$

Let $\chi_0 = \nu^{\pi\sqrt{-1}/\ln q}$ be the unique unramified character of order two, and 1 the trivial character of $F^\times$.
For $\chi \in \{1, \chi_0\}$

$$\sigma \hookrightarrow \chi_\psi \chi \nu^{-\frac{1}{2}} \rtimes \omega_0 \quad \text{(reduces)}$$

is unramified, and strongly negative.

$$\mathit{Jacq}_{(1,1)}(\sigma) \cong \chi_\psi \chi \nu^{-\frac{1}{2}} \otimes \omega_0$$

Let $\chi \in \{1, \chi_0\}$.

Jordan block is a pair $(m, \chi_\psi \chi)$, where m is a positive integer and $\chi \in \{1, \chi_0\}$. *Jord* is a set built of Jordan blocks. Given $\chi \in \{1, \chi_0\}$ we denote $Jord(\chi_\psi \chi) = \{m \mid (m, \chi_\psi \chi) \in Jord\}$.
Let $k, l \in \mathbb{Z}_{\geq 0}$ and

$$Jord(\chi_\psi) = \{2m_1 + 1 < 2m_2 + 1 < \cdots < 2m_l + 1\},\ m_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0},$$

$$Jord(\chi_\psi \chi_0) = \{2n_1 + 1 < 2n_2 + 1 < \cdots < 2n_k + 1\},\ n_j \in \frac{1}{2} + \mathbb{Z}_{\geq 0}.$$

We denote by $\sigma(Jord)$ the unique unramified irreducible subquotient of the induced representation

$$\zeta(-m_{l-1}, m_l, \chi_\psi) \times \zeta(-m_{l-3}, m_{l-2}, \chi_\psi) \times \cdots \times \zeta(-n_{k-1}, n_k, \chi_\psi\chi_0) \\ \times\zeta(-n_{k-3}, n_{k-2}, \chi_\psi\chi_0) \times \cdots \rtimes \sigma_0(Jord),$$

where $\sigma_0(Jord)$ is the unique unramified irreducible subquotient of

$$\begin{aligned} \zeta(\frac{1}{2}, m_1, \chi_\psi) \times \zeta(\frac{1}{2}, n_1, \chi_\psi\chi_0) \rtimes \omega_0 \qquad & \text{if } k, l \in 2\mathbb{Z}+1, \\ \zeta(\frac{1}{2}, m_1, \chi_\psi) \rtimes \omega_0 \qquad & \text{if } k \in 2\mathbb{Z},\ l \in 2\mathbb{Z}+1, \\ \zeta(\frac{1}{2}, n_1, \chi_\psi\chi_0) \rtimes \omega_0 \qquad & \text{if } k \in 2\mathbb{Z}+1,\ l \in 2\mathbb{Z}, \\ \omega_0 \qquad & \text{if } k, l \in 2\mathbb{Z}. \end{aligned}$$

When $k = l = 0$, we have $Jord = \emptyset$, and $\sigma(Jord) = \omega_0$, which is by definition strongly negative.

$R_{gen}(\widetilde{GL(n,F)})$ - the Grothendieck group of the category of smooth genuine representations of $\widetilde{GL(n,F)}$ of a finite length.

$$R^{gen} = \bigoplus_{n \geq 0} R_{gen}(\widetilde{GL(n,F)})$$

Similarly

$$R_1^{gen} = \bigoplus_{n \geq 0} R_{gen}(\widetilde{Sp(n,F)}),$$

We have a map $\mu^* : R_1^{gen} \to R^{gen} \otimes R_1^{gen}$,

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.} \left(Jacq_{(k,n-k)}(\sigma) \right), \sigma \in R_1^{gen},$$

$$\mu^*(\zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma) = \sum_{\zeta \otimes \sigma' \leq \mu^*(\sigma)} \sum_{i=0}^{\alpha+\beta+1} \sum_{j=0}^{i} \zeta(-\alpha, \beta - i, \chi_\psi \chi^{-1}) \times \zeta(-\beta, -\beta - 1 + j, \chi_\psi \chi) \times \zeta \otimes \zeta(-\beta + j, -\beta - 1 + i, \chi_\psi \chi) \rtimes \sigma'.$$

Lemma. Let $m, n \in -\frac{1}{2} + \mathbb{Z}_{\geq 0}$, and let $\sigma_{m,n}$ be the unramified irreducible subquotient of $\zeta(\frac{1}{2}, m, \chi_\psi) \times \zeta(\frac{1}{2}, n, \chi_\psi\chi_0) \rtimes \omega_0$, Then $\sigma_{m,n}$ is strongly negative, and

$$\mu^*(\sigma_{m,n}) = \sum_{i=0}^{m+\frac{1}{2}} \sum_{i'=0}^{n+\frac{1}{2}} \zeta(-m, -\frac{1}{2} - i, \chi_\psi) \times \zeta(-n, -\frac{1}{2} - i', \chi_\psi\chi_0) \otimes \sigma_{i-\frac{1}{2}, i'-\frac{1}{2}},$$

and

$$\sigma_{m,n} \hookrightarrow \zeta(-m, -\frac{1}{2}, \chi_\psi) \times \zeta(-n, -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0.$$

Lemma

• Let σ be a genuine irreducible unramified strongly negative representation of $\widetilde{Sp(n,F)}$. Then there exist
-an unramified unitary character χ of $F^\times$,
-$\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta \in \mathbb{Z}_{\geq 0}$,
-an irreducible unramified representation σ' of the metaplectic group such that

$$\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'$$

Also $\alpha - \beta < 0$ and $\zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'$ reduces. If α is the largest possible for such embedding, then σ' is strongly negative.

• Let $\beta > 0$ be the maximal, such that $\nu^{\pm\beta}\chi_\psi\chi$ appears in the cuspidal support of σ, where χ is a unitary character of $F^\times$. Then there exist $\alpha \in \mathbb{R}$ such that $\alpha + \beta \in \mathbb{Z}_{\geq 0}$, and $\sigma' \in Irr_{unr}(\widetilde{Sp(n',F)})$ such that

$$\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'.$$

Theorem 3. The representation $\sigma(\mathit{Jord})$, attached to a set *Jord* of Jordan blocks, is strongly negative, and we have

$$\sigma(\mathit{Jord}) \hookrightarrow \zeta(-m_l, m_{l-1}, \chi_\psi) \times \zeta(-m_{l-2}, m_{l-3}, \chi_\psi) \times \dots \\ \times \zeta(-n_k, n_{k-1}, \chi_\psi\chi_0) \times \zeta(-n_{k-2}, n_{k-3}, \chi_\psi\chi_0) \times \dots \\ \rtimes \sigma_0(\mathit{Jord}).$$

If $\chi \in \{1, \chi_0\}$ and $card(\mathit{Jord}(\chi_\psi\chi)) \geq 2$, let $2\beta + 1 > 2\alpha + 1$ be two largest elements in $\mathit{Jord}(\chi_\psi\chi)$. Put $\mathit{Jord}' = \mathit{Jord} \backslash \{(2\beta + 1, \chi_\psi\chi), (2\alpha + 1, \chi_\psi\chi)\}$ and $\sigma' = \sigma(\mathit{Jord}'), \sigma = \sigma(\mathit{Jord})$. Then:

$$\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'.$$

Every strongly negative representation corresponds to some $\sigma(\mathit{Jord})$.

Theorem [Zelevinsky Classification] Let σ be a genuine irreducible unramified representation of $\widetilde{Sp(n,F)}$. Then, either σ is negative, or there exist $k \in \mathbb{Z}_{>0}$, and a sequence $\chi_1, \ldots, \chi_k$ of unramified unitary characters of $F^\times$, and there exist real numbers α_i, β_i, such that $\alpha_i + \beta_i \in \mathbb{Z}_{\geq 0}$ and $-\beta_i + \alpha_i > 0$, for $i = 1, \ldots, k$ and there exists a genuine irreducible unramified negative representation σ_{neg} of the metaplectic group, such that $\sigma \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \sigma_{neg}$. Data $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1), \ldots, \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k)$ are unique up to permutation, while σ_{neg} is unique up to isomorphism. Moreover

$$\sigma \cong \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \sigma_{neg}.$$

thank you