The classification of simple (semi)relaxed admissible affine $\mathfrak{sl}_3$ modules and their modular properties at level $-\frac{3}{2}$

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Overview

1. Finite $\mathfrak{sl}_3$

2. Affine $\mathfrak{sl}_3$

3. Modular properties at $k = -\frac{3}{2}$

Before we start

I have an open 2 year postdoc postion starting October 2023.

`sites.google.com/view/imvoa/job-advertisement`

Application portal opens this week.

Finite $\mathfrak{sl}_3$

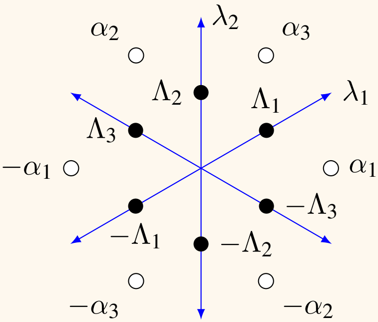
Def $\mathfrak{sl}_3$

The Lie algebra $\mathfrak{sl}_3$ can be realised using elementary 3×3 matrices:

$$
\begin{aligned}
e^1 =& E_{1,2}, \quad e^2 = E_{2,3}, \quad e^3 = E_{1,3} \\
h^1 =& E_{1,1} - E_{2,2}, \quad h^2 = E_{2,2} - E_{3,3} \\
f^1 =& E_{2,1}, \quad f^2 = E_{3,2}, \quad e^3 = E_{3,1}
\end{aligned}
$$

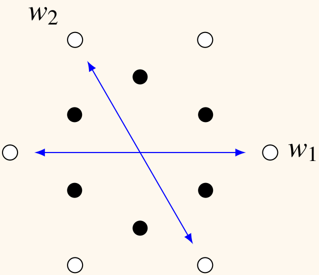
Finite $\mathfrak{sl}_3$

Roots and weights



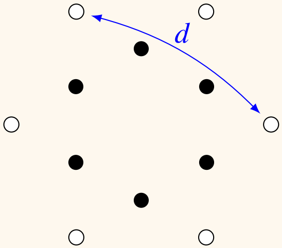
Finite $\mathfrak{sl}_3$

Weyl group, $W \cong S_3$



Finite $\mathfrak{sl}_3$

Root system automorphisms, $W \times$ outer Automorphisms $\cong D_6$



Weight modules

Def: Weight module

For $\mathfrak{g}$ finite complex reductive with Cartan $\mathfrak{h}$, a module M is weight, if

1. $\mathfrak{h}$-weight space decomposition: $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$,
2. each weight space is finite dimensional.

M is dense if its weight support fills out an entire root lattice coset.

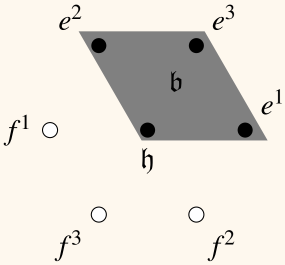
Thm: Fernando

Let $\mathfrak{g}$ a simple finite complex Lie algebra.

- $\mathfrak{g}$ admits simple dense modules if and only if $\mathfrak{g}$ is of type A or C.
- The following contructs a complete set of simples:
  1. Choose a parabolic $\mathfrak{p}$ and an irreducible dense module D over the Levi facor $\mathfrak{l}$ of $\mathfrak{p}$.
  2. Extend D to a $\mathfrak{p}$-module by letting the nilradical of $\mathfrak{p}$ act trivially.
  3. Take the simple quotient of the induced module $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} D$.

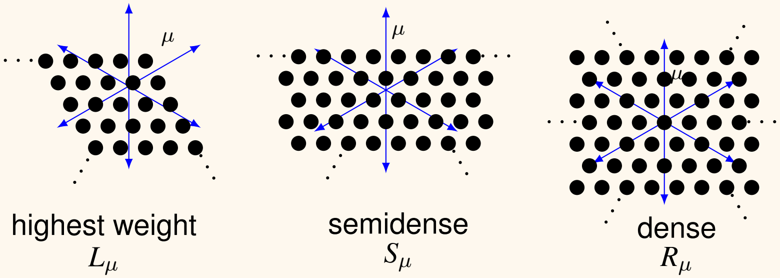
Borels and parabolics

A parabolic subalgebra is one which contains a Borel.



Parabolic	Levli	Nilradical
$\mathfrak{b}$	$\mathfrak{h}$	$\mathfrak{n}_+$
$\mathfrak{b} \oplus \mathbb{C}f^1$	$\mathrm{span}\{e^1, h^1, h^2, f^1\}$	$\mathrm{span}\{e^2, e^3\}$
$\mathfrak{b} \oplus \mathbb{C}f^2$	$\mathrm{span}\{e^2, h^1, h^2, f^2\}$	$\mathrm{span}\{e^1, e^3\}$
$\mathfrak{sl}_3$	$\mathfrak{sl}_3$	0

Modules constructed from Fernando's algorithm



Classifying simple dense modules

Thm: Mathieu

Let $\mathfrak{g}$ finite complex reductive with Cartan $\mathfrak{h}$. Every simple dense module can be uniquely realised as a direct summand of a (semisimple) *coherent family*

$$\mathcal{C} = \bigoplus_{[\mu]\in\mathfrak{h}^*/Q} \mathcal{C}_{[\mu]},$$

where coherent families satisfy

- All weight spaces have the same dimension.
- The trace of the action of the centraliser of $\mathfrak{h}$ in $U(\mathfrak{g})$ on weight spaces is polynomial in the weight.

A coherent family $\mathcal{C}$ has only finitely many highest weight direct summands and these weights form and orbit under the shifted Weyl group action. This orbit characterises the coherent family uniquely. We can therefore label $\mathcal{C}$, by a representative λ of the orbit: $\mathcal{C}^\lambda$

Highest weight affine $\mathfrak{sl}_3$ modules

Def: Admissible levels and weights, Kac-Wakimoto

An $\mathfrak{sl}_3$ level k is admissible, if it is of the form

$$k = \frac{u}{v} - 3, \quad u \in \mathbb{Z}_{\geq 3}, v \in \mathbb{Z}_{\geq 1}, \gcd\{u, v\} = 1.$$

Denote the simple quotient affine VOA at this level by $\mathsf{A}_2(u, v)$. For these levels, an affine weight is admissible if it is of the form

$$\hat{\lambda} = w \cdot \left(\hat{\lambda}^I - \frac{u}{v} \hat{\lambda}^{F,w} \right),$$

$w \in \{\mathrm{id}, w_1\} \subset W$, $\hat{\lambda}^I \in \hat{P}_+^{u-3}$, $\hat{\lambda}^{F,w} \in \hat{P}_+^{v-1}$, $\lambda_1^{F,w_1} \neq 0$

Thm: Arakawa

The simple highest weight modules over $\mathsf{A}_2(u, v)$ are precisely those with admissible highest weights. Every highest weight $\mathsf{A}_2(u, v)$ module is simple.

Highest weight affine $\mathfrak{sl}_3$ modules

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Thm: Ridout Kawasetsu

The inductions of coherent families are modules over $\mathsf{A}_2(u, v)$ precisely when they are labelled by admissible weights.

Partition the admissible weights as $A_{u,v} = A^{\mathbf{1}}_{u,v} \coprod A^{w_1}_{u,v}$

Prop

An admissible highest weight module $\hat{L}_{\hat{\lambda}}$ has finite dimensional top space if and only if $\hat{\lambda} \in A^{\mathbf{1}}_{u,v}$ and $\lambda_1^{F,\mathbf{1}} = \lambda_2^{F,w_1} = 0$. There are $\frac{1}{2}(u-1)(u-2)$ such weights.

Semirelaxed highest weight modules

Consider the admissible weights

$$B_{u,v} = \{\hat{\lambda} \in A^{\mathbf{1}}_{u,v} : \lambda_1^{F,\mathbf{1}} \neq 0\},$$

Prop

The semirelaxed simple modules $\hat{S}^{\lambda}_{[\mu]}$ for the parabolic $\mathfrak{p} = \mathrm{span}\{e^1, e^2, e^3, h^1, h^2, f^1\}$ (which distinguishes α_1) are precisely those for which $\hat{\lambda} \in B_{u,v}$ and where $[\mu] \in (\lambda + \mathbb{C}\alpha_1)/\mathbb{Z}\alpha_1$ satisfies $[\mu] \neq [\lambda]$ and $[\mu] \neq [w_1 \cdot \lambda]$.
There are $\frac{1}{2}(u-1)(u-2)v(v-1)$ such families of modules.
The semirelaxed simple modules for the parabolic $d(\mathfrak{p})$ are precisely the d twists of these modules.

Relaxed highest weight modules

Consider the admissible weights

$$C_{u,v} = \{\hat{\lambda} \in A^{\mathbf{1}}_{u,v} : \lambda_1^{F,\mathbf{1}} \neq 0, \lambda_2^{F,\mathbf{1}} = 0\}$$

Prop

The relaxed simple modules $R^{\lambda}_{[\mu]}$ are precisely those for which $\hat{\lambda} \in C_{u,v}$ and $[\mu] \in (\mathfrak{h}^*/Q) \setminus \text{sing}(\lambda)$, where

$$\text{sing}(\lambda) = [\lambda + \mathbb{C}\alpha_1] \cup [w_1 \cdot \lambda + \mathbb{C}\alpha_2] \cup [\lambda + \mathbb{C}\alpha_3].$$

There are $\frac{1}{4}(u-1)(u-2)(v-1)$ such families of modules.

Classification of simple positive energy weight modules

Thm: Arakawa-Futorny-Ramirez, Kawasetsu-Ridout-SW

Every simple $\mathsf{A}_2\,(u,v)$ positive energy weight module is isomorphic to a W-twist of one from the following list of mutually inequivalent modules

- Highest weight modules $\hat{L}_\lambda$ with $\hat{\lambda} \in A_{u,v}$.
- Semirelaxed highest weight modules $\hat{S}^{\lambda}_{[\mu]}$, with $B_{u,v}$ and $[\mu] \in (\lambda + \mathbb{C}\alpha_1)/\mathbb{Z}\alpha_1$ satisfying $[\mu] \neq [\lambda]$ and $[\mu] \neq [w_1 \cdot \lambda]$.
- d-twists of the semirelaxed highest weight modules above.
- Relaxed highest weight mdoules $\hat{R}^{\lambda}_{[\mu]}$ with $\hat{\lambda} \in C_{u,v}$ and $[\mu] \in (\mathfrak{h}^*/Q) \setminus \mathrm{sing}(\lambda)$.

Relation to nilpotent orbits observation

- If $\hat{\lambda}$ labels relaxed highest weight modules ($\hat{\lambda} \in C_{u,v}$) then λ lies in the minimal nilpotent orbit. This a restatement of a result of Mathieu.
- If $\hat{\lambda}$ labels semirelaxed highest weight modules but not relaxed ones, then λ lies in the principal nilpotent orbit.
- If $\hat{\lambda}$ only labels a highest weight module, then λ belongs to the zero nilpotent orbit.

Recall the Verlinde formula for rational theories

Let V be a rational VOA. By Huang's theorem V admits a category of modules which is a modular tensor category (so abelian, linear, finite, semisimple, monoidal, rigid, non-degenerate braiding) and hence satisfies the Verlinde formula: Let $V = M_0, \ldots, M_n$ be a complete set of simples with fusion tensor product

$$M_i \otimes M_j \cong \bigoplus_k N_{i,j}^k M_k$$

and let $S_{i,j}$ be the S-matrix, then

$$N_{i,j}^k = \sum_l \frac{S_{i,l} S_{j,l} \overline{S_{k,l}}}{S_{0,l}}.$$

For non-integral admissible levels (levels for which the affine VOAs are not rational), the Verlinde formula gives negative structure coefficients $N_{i,j}^k$. In examples the *standard module formalism* gives non-negative coefficients.

Recall the standard module formalism

Conjectured generalisation of the Verlinde formula.

Procedure:

1. Identify a suitable category of modules such that it is closed under contragredients, closed under tensor products, modules have characters and the span of these carries an action of the modular group.
2. Identify *standard* modules.
3. Standards parametrised by measurable space (M, μ).
4. Non-simple standards measure 0 in (M, μ).
5. The characters of standards form (topological) basis for all characters.
6. The characters of standards span representation of the modular group. In standard basis, the S-transformation is symmetric, unitary and squares to conjugation.
7. Plugging S-transformation into Verlinde formula gives non-negative integer fusion rules on Grothendieck group.

Standard module Verlinde formula

Let $R_\mu, \mu \in M$ be the standard modules, then the Verlinde product on the Grothendieck group is defined to be

$$[W_1] \times [W_2] = \int_M N_{1,2}^{\mu} [R_\mu] d\mu,$$

where

$$N_{1,2}^{\mu} = \int_M \frac{S_{W_1,\nu} S_{W_2,\nu} S^*_{\mu,\nu}}{S_{\mathsf{vac},\nu}} d\nu,$$

and where $S_{\mathsf{vac},\nu}$ is the modular S-transformation of the character of the VOA (expanded in standard characters).

What does this mean for $\mathfrak{sl}_3$?

1 The category of positive energy weight modules is closed under contragredients, but the other criteria fail. In analogy to other examples allow spectral flow twists and all weight finite length extensions.

2 The natural candidates for standard modules are the real weight $[\mu]$ relaxed highest weight modules and the semirelaxed highest weight modules, which do not appear in relaxed ones, and their spectral flows.

3 Spectral flow is labelled by the coweight lattice $P^\vee$, so standard modules are labelled by $P^\vee$ and cosets by discrete subgroups in $\mathfrak{h}^*_\mathbb{R}$. This inherits a measure from the translation invariant Lebesgue measure on $\mathfrak{h}^*_\mathbb{R}$.

4 Relaxed and semirelaxed modules degenerate on sets of measure 0.

What does this mean for $\mathfrak{sl}_3$?

5 Characters are linearly dependent outside of $u = 3, v = 2$, $k = -\frac{3}{2}$ because some (semi)relaxed modules have the same values of the quadratic Casimir and require the cubic Casimir to be disambiguated. For $v = 2$ there are no semirelaxed standard modules.
Non-standard simple modules can be resolved through non-simple standard modules.

6 We shall see shortly that at $k = -\frac{3}{2}$, the characters of spectral flows of relaxed modules carry a suitable action of the modular group.

Simple modules at $k = -\frac{3}{2}$

- Four highest weight modules $\hat{L}_0, \hat{L}_{-3\Lambda_1/2}, \hat{L}_{-3\Lambda_2/2}, \hat{L}_{-\rho/2}$ (previously classified by Perše) and their twists.
- One family of semirelaxed highest weight modules $\hat{S}_{[\mu]}^{-3\Lambda_1/2}$ for all $[\mu] \in (-\frac{3}{2}\Lambda_1 + \mathbb{C}\alpha_1)/\mathbb{Z}\alpha_1 \setminus \{[-\frac{3}{2}\Lambda_1], [-\frac{1}{2}\rho]\}$ and its twists.
- One family of relaxed highest weight modules $\hat{R}_{[\mu]}^{-3\Lambda_1/2}$ for all $[\mu] \in (\mathfrak{h}^*/Q) \setminus \text{sing}(-\frac{3}{2}\Lambda_1)$, where

$$\text{sing}(-\frac{3}{2}\Lambda_1) = [-\frac{3}{2}\Lambda_1 + \mathbb{C}\alpha_1] \cup [-frac12\rho + \mathbb{C}\alpha_2] \cup [-\frac{3}{2}\Lambda_1 + \mathbb{C}\alpha_3].$$

  These were first constructed by Adamovic and shown to be complete by Kawasetsu-Ridout.

Characters

Prop

For $\xi \in P^\vee, [\mu] \in \mathfrak{h}^*/Q$, $\theta \in \mathbb{R}, \zeta \in \mathfrak{h}, \tau \in \mathbb{H}$ the character of the spectrally flowed relaxed highest weight module $\sigma^\xi(\hat{R}_{[\mu]})$ is

$$\text{ch}\left[\sigma^\xi(\hat{R}_{[\mu]})\right](\theta|\zeta|\tau) = \text{tr}_{\sigma^\xi(\hat{R}_{[\mu]})} e^{2\pi i\theta k} e^{2\pi i\zeta} e^{2\pi i\tau(L_0 - \frac{c}{24})}$$
$$= \frac{e^{-3\pi i\theta} e^{-3\pi i\langle\xi,\zeta+\tau\frac{\xi}{2}\rangle} e^{2\pi i\langle\mu,\zeta+\tau\xi\rangle}}{\eta(\tau)^4} \sum_{\xi'\in P^\vee} \delta(\zeta+\tau\xi-\xi').$$

Modular transformations

Def: modular action

Define the following action of $SL(2,\mathbb{Z})$ on $\mathbb{C}\times\mathfrak{h}\times\mathbb{H}$ via the generators

$$S:(\theta|\zeta|\tau)\mapsto\left(\theta-\frac{\langle\zeta,\zeta\rangle}{2\tau}-\frac{2\arg(\tau)-\arg(-1)}{3\pi}\left|\frac{\zeta}{\tau}\right|-\frac{1}{\tau}\right),$$
$$T:(\theta|\zeta|\tau)\mapsto\left(\theta-\frac{\arg(-1)}{9\pi}\left|\zeta\right|\tau+1\right).$$

Thm

$$T\left\{\text{ch}\left[\sigma^{\xi}(\hat{R}_{[\mu]})\right]\right\}=e^{2\pi i(\langle-3\xi/4+\mu,\xi\rangle-\frac{1}{6})}\text{ch}\left[\sigma^{\xi}(\hat{R}_{[\mu]})\right]$$
$$S\left\{\text{ch}\left[\sigma^{\xi}(\hat{R}_{[\mu]})\right]\right\}=\sum_{\xi'\in P^\vee}\int_{\mathfrak{h}_{\mathbb{R}}/Q}S^{\xi,\xi'}_{[\mu][\mu]}\text{ch}\left[\sigma^{\xi}(\hat{R}_{[\mu']})\right]d[\mu'],$$

where $S^{\xi,\xi'}_{[\mu][\mu]}=e^{2\pi i(3\langle\xi,\xi'\rangle/2-\langle\mu,\xi'\rangle-\langle\mu',\xi\rangle)}$.

Verlinde formula product

The standard module Verlinde formula yields.

$$\left[(\hat{R}_{[\mu]})\right] \boxtimes \left[(\hat{R}_{[\mu']})\right] = \begin{array}{ccc} & \left[\sigma^{\xi_2}(\hat{R}_{[\mu+\mu']})\right] & \\ \left[\sigma^{\xi_3}(\hat{R}_{[\mu+\mu']})\right] & & \left[\sigma^{\xi_1}(\hat{R}_{[\mu+\mu']})\right] \\ & 2\left[\sigma^{\xi}(\hat{R}_{[\mu+\mu']})\right] & \\ \left[\sigma^{-\xi_1}(\hat{R}_{[\mu+\mu']})\right] & & \left[\sigma^{-\xi_3}(\hat{R}_{[\mu+\mu']})\right] \\ & \left[\sigma^{-\xi_2}(\hat{R}_{[\mu+\mu']})\right] & \end{array}$$

Verlinde products of other simples can be computed from this.

The End

Thank you for coming!