On the representation theory of the affine $sl(4)$ at level $-5/2$ and application to the higher ranks

Joint work with D. Adamović, O. Perše and T. Creutzig

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Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

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Reference:

- D. Adamović, O. Perše, I. Vukorepa, *On the representation theory of the vertex algebra* $L_{-5/2}(sl(4))$, Communications in Contemporary Mathematics (2021), arXiv:2103.02985

Content

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $(\cdot,\cdot)$ be a non-degenerate, symmetric bilinear form on $\mathfrak{g}$.
The **affine Lie algebra** $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K,$$

where K is central element and Lie algebra structure is given by

$$[x(m), y(n)] = [x, y](m+n) + m\delta_{m,-n}(x, y)K,$$

where $x(m)$ denotes $x \otimes t^m \in \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

We say that M is a $\hat{\mathfrak{g}}$–module of level $k \in \mathbb{C}$ if the central element K acts on M as a multiplication with k.

Affine vertex algebra

- $V^k(\mathfrak{g})$ universal affine vertex algebra of level k, $k \neq -h^\vee$
- As $\hat{\mathfrak{g}}$–module, we have

$$V^k(\mathfrak{g}) = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{g}\otimes\mathbb{C}[t]+\mathbb{C}K)} \mathbb{C}1.$$

- $L_k(\mathfrak{g})$ simple quotient of $V^k(\mathfrak{g})$.
- Let $u^{(i)}, u_{(j)}$, $i,j = 1,\dots,\dim\mathfrak{g}$ be dual bases of $\mathfrak{g}$ with respect to $(\cdot,\cdot)$. Then

$$\omega = \frac{1}{k+h^\vee}\sum_{i=1}^{\dim\mathfrak{g}} u^{(i)}(-1)u_{(i)}(-1)\mathbf{1}$$

is Sugawara Virasoro vector in $L_k(\mathfrak{g})$ of central charge $c = \frac{k\cdot\dim\mathfrak{g}}{k+h^\vee}$.

- Non-negative integers levels: Frenkel-Zhu, Li
  The category of $\mathbb{Z}_{\geq 0}$–graded $L_k(\mathfrak{g})$–modules is semi-simple.
- Admissible levels: Adamović-Milas, Dong-Li-Mason, Arakawa, Perše
  The category of $L_k(\mathfrak{g})$–modules which are in the category $\mathcal{O}$ as $\hat{\mathfrak{g}}$–modules is semi-simple.

Negative integer levels which appear in:

- free-field realizations of certain simple affine vertex algebras (Adamović-Perše),
- in the context of affine vertex algebras associated to the Deligne exceptional series (Arakawa-Moreau),
- in the context of collapsing levels for minimal affine $\mathcal{W}$–algebras (Adamović-Kac-Moseneder Frajria-Papi-Perše).

Admissible and almost admissible levels

- For $\mathfrak{g} = sl(n)$ level k is admissible if

$$k + n = \frac{p}{q}, \quad p, q \in \mathbb{N}, \ (p, q) = 1, \ p \geq n.$$

- We are interested in levels which are almost admissible, i.e.

$$k = -n + \frac{n-1}{q}, \quad q \in \mathbb{N}, \quad (n-1, q) = 1.$$

- First such example is $V^{-1}(sl(n))$, $n \geq 3$.
- For $q = 2$, we have $k = -\frac{n+1}{2}$.

On the vertex algebra $L_{-1}(sl(4))$

- Adamović and Perše determined an explicit formula for the singular vector in $V^{-1}(sl(4))$ and classified irreducible $L_{-1}(sl(4))$–modules in the category $\mathcal{O}$.
- Category $\mathcal{O}$ for $L_{-1}(sl(n))$ is not semi-simple unlike the admissible case.
- Description of the maximal ideal in $V^{-1}(sl(4))$ was obtained using minimal QHR functor H_{f_θ} (Arakawa-Moreau).
- Level $k = -1$ is collapsing for $W^k(sl(4), f_\theta)$ and $W_{-1}(sl(4), f_\theta) = M(1)$.
- Category KL_{-1} is semi-simple (AKMPP).
- Category KL_{-1} is a rigid braided tensor category (Creutzig-Yang).

Vertex algebra $L_{-5/2}(sl(4))$

- New example of non-admissible, half-integer level.
- It appears in conformal embedding $L_{-5/2}(sl(4)) \otimes M(1) \hookrightarrow L_{-5/2}(sl(5))$ (AKMPP), where $M(1)$ denotes the Heisenberg vertex algebra associated to abelian Lie algebra of rank one.
- The level $k = -5/2$ is admissible for $\widehat{sl(5)}$.

Theorem

Let $\mathfrak{g} = sl(4)$. The following vector v is a singular vector of weight $-\frac{5}{2}\Lambda_0 - 4\delta + 2\omega_2$ in $V^{-5/2}(\mathfrak{g})$:

$$\begin{aligned} v = {} & e_{\varepsilon_1-\varepsilon_3}(-1)e_{\varepsilon_2-\varepsilon_4}(-3)\mathbf{1} + e_{\varepsilon_1-\varepsilon_3}(-3)e_{\varepsilon_2-\varepsilon_4}(-1)\mathbf{1} + \frac{1}{2}e_{\varepsilon_1-\varepsilon_3}(-2)e_{\varepsilon_2-\varepsilon_4}(-2)\mathbf{1} \\ & - e_{\varepsilon_1-\varepsilon_4}(-1)e_{\varepsilon_2-\varepsilon_3}(-3)\mathbf{1} - e_{\varepsilon_1-\varepsilon_4}(-3)e_{\varepsilon_2-\varepsilon_3}(-1)\mathbf{1} - \frac{1}{2}e_{\varepsilon_1-\varepsilon_4}(-2)e_{\varepsilon_2-\varepsilon_3}(-2)\mathbf{1} \\ & + e_{\varepsilon_2-\varepsilon_4}(-1)e_{\varepsilon_2-\varepsilon_3}(-2)e_{\varepsilon_1-\varepsilon_2}(-1)\mathbf{1} - e_{\varepsilon_2-\varepsilon_4}(-2)e_{\varepsilon_2-\varepsilon_3}(-1)e_{\varepsilon_1-\varepsilon_2}(-1)\mathbf{1} \\ & - e_{\varepsilon_1-\varepsilon_3}(-1)e_{\varepsilon_2-\varepsilon_3}(-2)e_{\varepsilon_3-\varepsilon_4}(-1)\mathbf{1} - 3e_{\varepsilon_1-\varepsilon_3}(-2)e_{\varepsilon_2-\varepsilon_3}(-1)e_{\varepsilon_3-\varepsilon_4}(-1)\mathbf{1} \\ & + 2e_{\varepsilon_1-\varepsilon_2}(-1)e_{\varepsilon_2-\varepsilon_3}(-1)^2e_{\varepsilon_3-\varepsilon_4}(-1)\mathbf{1} - \frac{2}{3}e_{\varepsilon_1-\varepsilon_3}(-1)e_{\varepsilon_2-\varepsilon_4}(-1)h_2(-2)\mathbf{1} + \cdots \end{aligned}$$

The remaining terms can be found in the referenced paper.

Let us denote

$$\widetilde{L}_{-5/2}(\mathfrak{g}) = V^{-5/2}(\mathfrak{g})/\langle v \rangle.$$

Vertex algebra $V^{-5/2}(\mathfrak{g})$ has an order two automorphism σ which is lifted from the automorphism of the Dynkin diagram of $\mathfrak{g}$, defined by:

$$\sigma(\alpha_1) = \alpha_3,\ \sigma(\alpha_2) = \alpha_2,\ \sigma(\alpha_3) = \alpha_1.$$

- One easily checks that $\sigma(v) = v$ for the singular vector v.
- This implies that σ induces an automorphism of $\widetilde{L}_{-5/2}(\mathfrak{g})$.

Theorem

The complete list of irreducible $\widetilde{L}_{-5/2}(sl(4))$–modules in the category $\mathcal{O}$ is given by

$$\left\{L_{-5/2}(\mu_i(t)) \mid i = 1, \ldots, 16,\ t \in \mathbb{C}\right\},$$

where:

$\mu_1(t) = t\omega_1,$

$\mu_2(t) = t\omega_3,$

$\mu_3(t) = t\omega_1 + (-t - \frac{5}{2})\omega_2,$

$\mu_4(t) = t\omega_2 + (-t - \frac{5}{2})\omega_3,$

$\mu_5(t) = t\omega_1 - \frac{3}{2}\omega_2,$

$\mu_6(t) = -\frac{3}{2}\omega_2 + t\omega_3,$

$\mu_7(t) = t\omega_1 + (-t - 1)\omega_2,$

$\mu_8(t) = t\omega_2 + (-t - 1)\omega_3,$

$\mu_9(t) = -\frac{3}{2}\omega_1 + t\omega_3,$

$\mu_{10}(t) = t\omega_1 - \frac{3}{2}\omega_3,$

$\mu_{11}(t) = -\frac{3}{2}\omega_1 + t\omega_2 + (-t - 1)\,\omega_3,$

$\mu_{12}(t) = (-t - 1)\,\omega_1 + t\omega_2 - \frac{3}{2}\omega_3,$

$\mu_{13}(t) = -\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2 + t\omega_3,$

$\mu_{14}(t) = -\frac{1}{2}\omega_1 + t\omega_2 + (-t - \frac{3}{2})\omega_3,$

$\mu_{15}(t) = t\omega_1 - \frac{1}{2}\omega_2 - \frac{1}{2}\omega_3,$

$\mu_{16}(t) = (-t - \frac{3}{2})\omega_1 + t\omega_2 - \frac{1}{2}\omega_3.$

Corollary

The complete list of irreducible $\widetilde{L}_{-5/2}(sl(4))$–modules in the category $KL_{-5/2}$ is given by

$$\left\{L_{-5/2}(t\omega_1) \mid t \in \mathbb{Z}_{\geq 0}\right\} \cup \left\{L_{-5/2}(t\omega_3) \mid t \in \mathbb{Z}_{\geq 0}\right\}.$$

Next goal: Prove the simplicity of $\widetilde{L}_{-5/2}(sl(4))$.

- It turns out that in this case we can not use $\mathcal{W}$–algebra $W^k(sl(4), f_\theta)$ as in the case $k = -1$.

$\mathcal{W}$–algebra $W^{-5/2}(sl(4), f_{subreg})$

- We use subregular nilpotent element

$$f = f_{subreg} = f_{\varepsilon_2-\varepsilon_3} + f_{\varepsilon_3-\varepsilon_4}.$$

- Let

$$x = \omega_2 + \omega_3$$

  be a semisimple element of $sl(4)$ which defines a good grading with respect to f.
- Vertex algebra $W^{-5/2}(sl(4), f)$ is strongly generated by five elements; $J, \bar{L} = L + \partial J, W, G^+, G^-$ having conformal weights 1,2,3,1,3, respectively.
- The OPE formulas are presented by T. Creutzig and A. Linshaw.

Let us denote $\mathfrak{g} = sl(4)$.

Theorem

Level $k = -5/2$ is a collapsing level for $W^k(\mathfrak{g}, f_{subreg})$ and

$$W_{-5/2}(\mathfrak{g}, f_{subreg}) \cong M_J(1),$$

where $M_J(1)$ is the Heisenberg vertex algebra generated by J.

Lemma

The image of singular vector v in $W^{-5/2}(\mathfrak{g}, f_{subreg})$ coincides (up to a non-zero scalar) with the vector G^+.

Proposition

We have:

(1) $H_{f_{subreg}}(\widetilde{L}_{-5/2}(\mathfrak{g})) \cong M_J(1)$.

(2) $H_{f_{subreg}}(L_{-5/2}(\mathfrak{g})) \cong M_J(1)$.

Problem: The properties of the QHR functor $H_{f_{subreg}}(\cdot)$ are not presented so explicitly as in the case of the minimal reduction.

- In the case $k=-1$ we have $H_{f_\theta}(L_{-1}(n\omega_i)) \neq \{0\}$, $i=1,3$.

Theorem

For any $n \in \mathbb{Z}_{>0}$ we have:
(P) $H_{f_{subreg}}(L_{-5/2}(n\omega_3)) \neq \{0\}$ and $H_{f_{subreg}}(M) = \{0\}$ for any highest weight $\widetilde{L}_{-5/2}(\mathfrak{g})$–module M in $KL_{-5/2}$ of $\mathfrak{g}$–weight $n\omega_1$.

- The proof is based on a construction of singular vectors in generalized Verma modules $V^{-5/2}(n\omega_i)$, $i = 1, 3$, and the description of their submodules $\langle v \rangle \cdot V^{-5/2}(n\omega_i)$.
- As a consequence, we obtain a description of the universal $\widetilde{L}_{-5/2}(\mathfrak{g})$–modules

$$\overline{M}(n\omega_i) = \frac{V^{-5/2}(n\omega_i)}{\langle v \rangle \cdot V^{-5/2}(n\omega_i)}$$

for which we prove vanishing and non-vanishing of $H_{f_{subreg}}(\overline{M}(n\omega_i))$.

The main idea in the case $k = -5/2$ is to use property (P) and the automorphism σ which interchanges the weights $n\omega_1$ and $n\omega_3$.

Theorem

We have:

(i) $\langle v \rangle$ is the maximal ideal in $V^{-5/2}(sl(4))$, i.e. $L_{-5/2}(sl(4)) \cong V^{-5/2}(sl(4))/\langle v \rangle$.

(ii) The category $KL_{-5/2}$ is semi-simple.

Sketch of proof: (i)

- We have $H_{f_{subreg}}(\widetilde{L}_{-5/2}(\mathfrak{g})) = W_{-5/2}(\mathfrak{g}, f_{subreg}) = M_J(1)$.
- If $\widetilde{L}_{-5/2}(\mathfrak{g})$ is not simple, it must contain singular vector w_μ of $\mathfrak{g}$–weight $\mu = n\omega_1$ or $\mu = n\omega_3$, for $n \in \mathbb{Z}_{>0}$.

- Using an automorphism σ we conclude that $\widetilde{L}_{-5/2}(\mathfrak{g})$ contains a singular vector of $\mathfrak{g}$–weight $\mu = n\omega_3$.
- Property (P) implies that ideal generated by this vector is mapped by QHR functor $H_{f_{subreg}}$ to a non-trivial ideal in $W_{-5/2}(\mathfrak{g}, f_{subreg})$, which is simple.
- Using exactness of $H_{f_{subreg}}$ in the category $KL_{-5/2}$, we get $H_{f_{subreg}}(L_{-5/2}(\mathfrak{g})) = \{0\}$, which is a contradiction.

(ii) The similar arguments as in (i) prove that any highest weight module in $KL_{-5/2}$ is irreducible. Then the result of AKMPP implies that $KL_{-5/2}$ is semi-simple.

Proposition (AKMPP (2016))

There is a conformal embedding

$$L_k(sl(2n)) \otimes M(1) \hookrightarrow L_k(sl(2n+1)), \quad k = -\frac{2n+1}{2}, \ n \geq 2,$$

and we have the following decomposition of $L_k(sl(2n+1))$ as an $L_k(sl(2n)) \otimes M(1)$–module:

$$L_k(sl(2n+1)) = \bigoplus_{i=0}^{\infty} L_k(i\omega_1) \otimes M(1,i) \ \oplus \ \bigoplus_{i=1}^{\infty} L_k(i\omega_{2n-1}) \otimes M(1,-i).$$

We introduce the following notation for the irreducible $L_k(sl(2n))$–modules in the category $KL_k(sl(2n))$:

$$U_i^{(n)} = L_k(i\omega_1), \ U_{-i}^{(n)} = L_k(i\omega_{2n-1}), \ i \in \mathbb{Z}_{\geq 0}.$$

Fusion rules between irreducible modules in $KL_{-5/2}$

Proposition

Let $i, j \in \mathbb{Z}$. We have the following fusion rule:

$$U_i^{(2)} \times U_j^{(2)} = U_{i+j}^{(2)}.$$

This means that for $i, j, k \in \mathbb{Z}$:

$$\dim I \begin{pmatrix} & U_k^{(2)} & \\ U_i^{(2)} & & U_j^{(2)} \end{pmatrix} = \delta_{i+j,k}.$$

Corollary

$KL_{-5/2}$ is a semi-simple rigid braided tensor category with the fusion rules

$$U_i^{(2)} \boxtimes U_j^{(2)} = U_{i+j}^{(2)} \quad (i, j \in \mathbb{Z}).$$

Tensor categories and conformal embeddings

- Creutzig, McRae, Yang and collaborators use tensor category approach for studying VOAs.
- Let us consider conformal embeddings $L_k(\mathfrak{g}_0) \hookrightarrow L_k(\mathfrak{g})$ from AKMPP.
- Based on their results, one expects that the category KL_k of ordinary $L_k(\mathfrak{g}_0)$–modules will have the structure of a rigid braided tensor category.
- Together with the decompositions of $L_k(\mathfrak{g})$ as $L_k(\mathfrak{g}_0)$–modules from AKMPP, their results should imply also the fusion rules in KL_k.

Singlet $\mathcal{M}(2)$

<u>Goal:</u> Using tensor category approach, extend results on $KL_{-5/2}(sl(4))$ to $KL_k(sl(2n))$, $k = -\frac{2n+1}{2}$.

Results on singlet $\mathcal{M}(2)$ [A'03] [AM'17], [CMY'21]:

- The category of all C_1–cofinite $\mathcal{M}(2)$–modules $\mathcal{O}_{\mathcal{M}(2)}$ is a rigid braided tensor category.
- Let $\mathcal{M}_i$, $i \in \mathbb{Z}$ be all atypical $\mathcal{M}(2)$–modules.

  Modules $\mathcal{M}_i$, $i \in \mathbb{Z}$ are simple currents in $\mathcal{O}_{\mathcal{M}(2)}$ with the following fusion rules

$$\mathcal{M}_i \times \mathcal{M}_j = \mathcal{M}_{i+j}, \quad i, j \in \mathbb{Z}.$$

Structure of $W_{-7/2}(sl(6), f_\theta)$

Using that $KL_{-5/2}(sl(4))$ is a braided tensor category and above results on singlet, we obtain:

- $W_{-7/2}(sl(6), f_\theta)$ is a simple current extension of $L_{-5/2}(sl(4)) \otimes \mathcal{H} \otimes \mathcal{M}(2)$, where $\mathcal{H}$ denotes the rank one Heisenberg vertex algebra generated by h previously denoted by $M(1)$.
- We have the following decomposition

$$W_{-7/2}(sl(6), f_\theta) = \bigoplus_{i \in \mathbb{Z}} U_i^{(2)} \otimes \mathcal{F}_i \otimes \mathcal{M}_i,$$

  where $\mathcal{F}_i$ denotes Fock $\mathcal{H}$–module generated by highest weight vector v_i such that

$$h(n)v_i = \delta_{n,0} i v_i \ (n \geq 0),$$

  previously denoted by $M(1, i)$.

Theorem

(1) Set $\{U_i^{(3)} | \ i \in \mathbb{Z}\}$ provides all irreducible modules in $KL_{-7/2}(sl(6))$ and we have the following fusion rules:

$$U_i^{(3)} \times U_j^{(3)} = U_{i+j}^{(3)}, \quad i, j \in \mathbb{Z}.$$

(2) $KL_{-7/2}(sl(6))$ is a semi-simple rigid braided tensor category.

By mathematical induction for $m = 2n$ we proved that above is true for any $m \geq 4$ even and $k = -\frac{m+1}{2}$.

Theorem

(1) Set $\{U_i^{(n)} | \ i \in \mathbb{Z}\}$ provides all irreducible modules in $KL_k(sl(2n))$ and we have the following fusion rules:

$$U_i^{(n)} \times U_j^{(n)} = U_{i+j}^{(n)}, \quad i, j \in \mathbb{Z}.$$

(2) $KL_k(sl(2n))$ is a semi-simple rigid braided tensor category.

(3) For $n \geq 3$, $W_k(sl(2n), f_\theta)$ is a simple current extension of $L_{k+1}(sl(2n-2)) \otimes \mathcal{H} \otimes \mathcal{M}(2)$, and we have the following decomposition

$$W_k(sl(2n), f_\theta) = \bigoplus_{i \in \mathbb{Z}} U_i^{(n-1)} \otimes \mathcal{F}_i \otimes \mathcal{M}_i.$$

Thank you!