Combinatorial relations among relations for level 2 standard $C_n^{(1)}$-modules

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Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

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PROVEDBA VRHUNSKIH ISTRAŽIVANJA U SKLOPU
ZNANSTVENOG CENTRA IZVRSNOSTI
ZA KVANTNE I KOMPLEKSNE SUSTAVE
TE REPREZENTACIJE LIEJEVIH ALGEBRI

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Affine Lie algebras

- Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $\langle\ ,\ \rangle$ a symmetric invariant bilinear form on $\mathfrak{g}$ and we assume that $\langle\theta,\theta\rangle = 2$ for the maximal root θ
- Denote by $\Delta(= \Delta_+ \cup \Delta_-)$ roots (positive and negative roots)
- Triangular decomposition $\mathfrak{g} = \mathfrak{N}_+ + \mathfrak{h} + \mathfrak{N}_-$
- Fix root vectors X_α
- $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c \quad , \quad \tilde{\mathfrak{g}} = \hat{\mathfrak{g}} + \mathbb{C}d$ is the associated untwisted affine Kac-Moody Lie algebra
- $x(m) = x \otimes t^m$ for $x \in \mathfrak{g}$ and $i \in \mathbb{Z}$, c is the canonical central element, and $[d, x(m)] = mx(m)$
- $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{<0} + (\mathfrak{g} + \mathbb{C}c) + \hat{\mathfrak{g}}_{>0} \quad , \quad \hat{\mathfrak{g}}_{<0} = \sum_{m<0} \mathfrak{g}(m)$

Highest weight modules

- Λ highest weight, v_Λ highest weight vector
- Verma modul $M(\Lambda)$, $L(\Lambda)$ irr. modul
- level of representation $k = \Lambda(c)$ (for us $k = 1, 2, \cdots$)
- we can form the induced $\tilde{\mathfrak{g}}$-module (a generalized Verma modul)
$$N(k\Lambda_0) = \mathcal{U}(\tilde{\mathfrak{g}}) \otimes_{\mathcal{U}(\tilde{\mathfrak{g}})_{\geq 0}} \mathbb{C}v_{k\Lambda_0}$$
- $N(k\Lambda_0) \cong \mathcal{U}(\tilde{\mathfrak{g}})_{<0}$ (as vector space)

Vertex operator algebras

- $(N(k\Lambda_0), Y, \mathbb{1}, \omega)$ is VOA with generating fields $x(z) = \sum_{m\in\mathbb{Z}} x_m z^{-m-1}$, $x \in \mathfrak{g}$
- max. $\tilde{\mathfrak{g}}-$submodul $N^1(k\Lambda_0) \subset N(k\Lambda_0)$ generated by singular vector $x_\theta(-1)^{k+1}\mathbb{1}$
- we define the irr $\mathfrak{g}$-module $R = \mathcal{U}(\mathfrak{g}) \cdot X_\theta(-1)^{k+1}\mathbb{1} \subset N(k\Lambda_0)$ and the corresponding loop $\tilde{\mathfrak{g}}$-module $\bar{R} = \langle r_i \mid r \in R, i \in \mathbb{Z}\rangle_{\mathbb{C}}$
- M is a standard module $\Leftrightarrow$ $\bar{R}$ annihilates M
- $L(\Lambda) = M(\Lambda)/M^1(\Lambda) = M(\Lambda)/(\bar{R}M(\Lambda))$
- we shall call elements r_i relations and $Y(v, z)$, $v \in N^1(k\Lambda_0)$ annihilating fields (of standard modules)

Annihilating fields

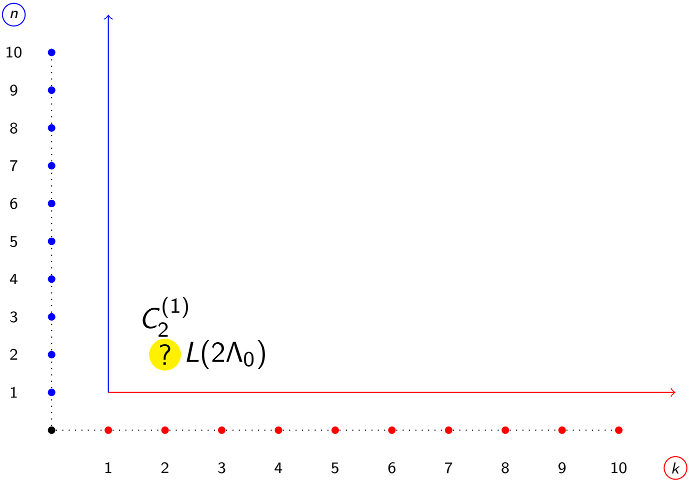
- Field $Y(x_\theta(-1)^{k+1}, z) = x_\theta(z)^{k+1}$ generates all annihilating fields of $L(k\Lambda_0)$
- $x_\theta(z)^{k+1} = \sum_{m\in\mathbb{Z}} r_{(k+1)\theta}(m) z^{-m-k-1}$
- $L(k\Lambda_0) = N(k\Lambda_0)/N^1(k\Lambda_0)$
- $L(k\Lambda_0) = N(k\Lambda_0)/(\bar{R}N(k\Lambda_0))$
- $N^1(k\Lambda_0) = \bar{R}N(k\Lambda_0) = \mathcal{U}(\tilde{\mathfrak{g}})\bar{R}v_\Lambda \rightsquigarrow \bar{R}$ *Relations*

Combinatorial and Gröbner bases

Problem:
Find a combinatorial basis of $L(k\Lambda_0)$ $\Leftrightarrow$ Find a "Gröbner basis" of $\bar{R}N(k\Lambda_0)$

- solved for all $\tilde{\mathfrak{sl}}_2$-modules $L(\Lambda)$
  [Meurman - Primc: *Annihilating Fields of Standard Modules of $\tilde{\mathfrak{sl}}_2$ and Combinatorial Identities*; Memoirs of AMS 1999]
- solved for basic modules $L(\Lambda_0)$ for all affine symplectic Lie algebras $C_n^{(1)}$
  [Primc-Š: *Combinatorial bases of basic modules for affine Lie algebras $C_n^{(1)}$*;J. Math. Phys. 2016]
- conjectured for standard modules $L(k\Lambda_0)$ for affine symplectic Lie algebras $C_n^{(1)}$
  [Primc-Š: *Leading terms of relations for standard modules of affine Lie algebras $C_n^{(1)}$*; Ramanujan J. 2019]

Starting point of this talk



New frontiers

- Case $C_n^{(1)}$ for $k = 2$
- Case $C_2^{(1)}$ for $k \geq 2$
- Case $C_n^{(1)}$ for $n \geq 2$ and $k \geq 2$

Colored partitions

- let B be the ordered basis of $\mathfrak{g}$
- We fix the basis $\bar{B}$ of $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$,

$$\bar{B} = \bigcup_{j \in \mathbb{Z}} B \otimes t^j,$$

- Let $\preceq$ be a linear order on $\bar{B}$ such that

$$i < j \quad \text{implies} \quad b(i) \prec b'(j).$$

- degree $|b(i)| = i$

Colored partitions

$$\pi = \prod_{i=1}^{\ell} b_i(j_i), \quad b_i(j_i) \in \bar{B},$$

- π is a colored partition of degree $|\pi| = \sum_{i=1}^{\ell} j_i \in \mathbb{Z}$ and length $\ell(\pi) = \ell$, with parts $b_i(j_i)$ of degree j_i and color b_i
- we shall usually assume that parts of π are indexed so that

$$b_1(j_1) \preceq b_2(j_2) \preceq \cdots \preceq b_\ell(j_\ell).$$

- we associate with a colored partition π its shape $\mathrm{sh}\,\pi$,

$$j_1 \leq j_2 \leq \cdots \leq j_\ell \quad (\text{"plain" partition}).$$

- the set of all colored partitions with parts $b_i(j_i)$ of degree $j_i (j_i < 0)$ is denoted as $\mathcal{P}(\mathcal{P}_{<0})$

Colored partitions

- $N(k\Lambda_0) \cong \mathcal{U}(\hat{\mathfrak{g}}_{<0}) \cong \mathcal{S}(\mathfrak{g}_{<0})$
  (Thx to PBW Thm ; like vec.space)
- $$(\prod_{b \in \bar{B}} b^{mult(b)}) \cdot v_{k\Lambda_0} \cong \prod_{b \in \bar{B}} b^{mult(b)} \quad \text{ordered monomials as in } \mathcal{P}_{<0}$$

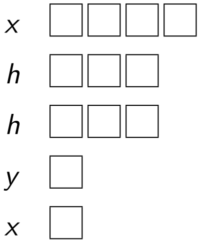
Colored partitions - example

Case: $\hat{\mathfrak{s}l}_2$; $B = \{x, h, y\}$; $y \prec h \prec x$

ordered monomial $u(\pi) = x(-4)h(-3)^2y(-1)x(-1)v_{k\Lambda_0}$

colored partitions

$u(\pi) = x(-4)h(-3)^2y(-1)x(-1)v_{k\Lambda_0} \quad \rightsquigarrow$



$$\ell(\pi) = \sum_{b \in \bar{B}} mult(b) = 5 \quad |\pi| = \sum_{i=1}^{\ell} j_i = 12$$

Relations on $L(\Lambda)$

On level k standard module $L(\Lambda)$ we have vertex operator relations

$$x_\theta(z)^{k+1} = \sum_{m\in\mathbb{Z}} r_{(k+1)\theta}(m) z^{-m-k-1} = 0$$

i.e. the coefficient (relations) of above annihilating fields are

$$r_{(k+1)\theta}(m) = \sum_{j_1+\cdots+j_{k+1}=m} x_\theta(j_1)\cdots x_\theta(j_{k+1}) \ .$$

The smallest summand in this sum is proportional to

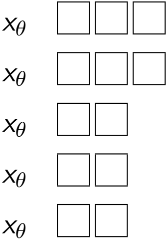
$$x_\theta(-j-1)^b x_\theta(-j)^a$$

for $a+b=k+1$ and $(-j-1)b+(-j)a=m$. Moreover, the shape of every other term Φ which appears in the sum is greater than the shape $(-j-1)^b(-j)^a$, so we can write

$$r_{(k+1)\theta}(m) = c\, x_\theta(-j-1)^b x_\theta(-j)^a + \sum_{\text{sh}\,\Phi \succ (-j-1)^b(-j)^a} c_\Phi X(\Phi)$$

Leading terms of relation - example

$$\ell t\left(r_{5\theta}(-12)\right) = x_\theta(-3)^2 x_\theta(-2)^3 \quad \rightsquigarrow$$



Remark:
For $a+b=k+1$ and $(-j-1)b+(-j)a=m$ we have only one possible shape. $b=|m|-(k+1)j$ $i.e.$ $b \equiv |m|(k+1)$.

$$k=4\ ,\ m=-12 \Rightarrow b=2 \Rightarrow a=3 \Rightarrow j=-2$$

Leading terms of relation $r(m)$

The adjoint action of $U(\mathfrak{g})$ on $r_{(k+1)\theta}(m)$, $m \in \mathbb{Z}$, gives all other relations in $\bar{R}$. For $u \in U(\mathfrak{g})$ the relation $r(m) = u \cdot r_{(k+1)\theta}(m)$ can be written as

$$r(m) = \sum_{\operatorname{sh}\Psi=(-j-1)^b(-j)^a} c_\Psi X(\Psi) + \sum_{\operatorname{sh}\Psi \succ (-j-1)^b(-j)^a} c_\Psi X(\Psi) + \sum_{\ell(\Psi)<k+1} .$$

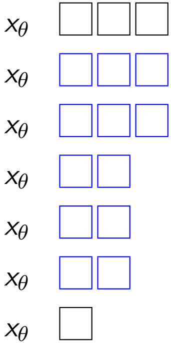
The actions of $u \in U(\mathfrak{g})$ in $\mathfrak{g}$-modules $\mathcal{U}$ and $\mathcal{S}$ are different, but we have $u\Big(c\, x_\theta(-j-1)^b x_\theta(-j)^a\Big) = \sum_{\operatorname{sh}\Psi=(-j-1)^b(-j)^a} c_\Psi \Psi$ with the same coefficients c_Ψ as in the first summand in above equation. The smallest $\Psi \in \mathcal{P}^{k+1}(m)$ which appears in the first sum we call **the leading term of relation** $r(m)$ and we denote it as $\ell t\, r(m)$. Hence we can rewrite above equation as

$$r(m) = c_\Phi X(\Phi) + \sum_{\Psi \succ \Phi} c_\Psi X(\Psi), \qquad \Phi = \ell t\, r(m).$$

Embeddings of leading terms

- ...$\ell t\bar{R} = \{\ell t\, r(m)\}$ parametrize a basis $\{r(\rho) \mid \rho \in \ell t\bar{R}\}$ of $\bar{R}$
- for $\kappa \in \mathcal{P}$, $\rho \in \ell t\bar{R}$ and $\pi = \kappa\rho$ we say that ρ is embedded in π (we write $\rho \subset \pi$)
- $u(\rho \subset \pi) = u(\kappa)r(\rho)$
- $\ell t(u(\rho \subset \pi)) = \pi$

$$x_\theta(-3)\ r_{5\theta}(-12)\ x_\theta(-1) \quad \rightsquigarrow$$



Let's summarize

- $L(k\Lambda_0) = N(k\Lambda_0)/N^1(k\Lambda_0) = N(k\Lambda_0)/(\bar{R}N(k\Lambda_0))$
- $\bar{R}N(k\Lambda_0) = \mathcal{U}(\tilde{\mathfrak{g}})\overline{R}v_\Lambda \rightsquigarrow \overline{R}$ *Relations*
- $r_{(k+1)\theta} = x_\theta(-1)^{k+1}\mathbb{1}; \ell t\left(r_{(k+1)\theta}(n)\right) = x_\theta(-j-1)^a x_\theta(-j)^b$
- all other elements $r(n)$ for $r \in R$ by the adjoint action of $\mathfrak{g}$, which does not change the length and degree, and sh $\ell t\left(r(n)\right) = (-j-1)^a(-j)^b$
- $\mathcal{D} = \ell t(\bar{R}) \cap \mathcal{P}_{<0}\,, \qquad \mathcal{RR} = P_{<0} \backslash \left(\mathcal{D} \cdot P_{<0}\right)$
- $u(\pi)\mathbb{1},\ \pi \in \mathcal{RR}$ will be a basis of the standard module $L(k\Lambda_0)$ with certain additional conditions (for $C_n^{(1)}$)

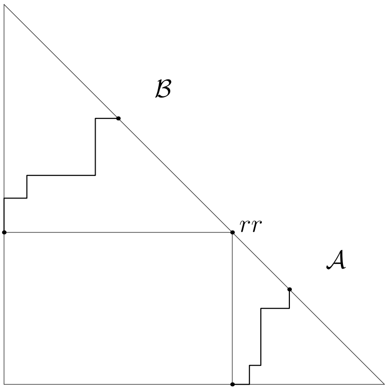
digression: **Simple Lie algebra of type C_n ($\mathfrak{sp}_{2n}$):**

These vectors form a basis B of $\mathfrak{g}$ which we shall write in a triangular scheme, e.g. for $n = 3$ the basis B is

$$\begin{array}{llllll}
11 & & & & & \\
12 & 22 & & & & \\
13 & 23 & 33 & & & \\
1\underline{3} & 2\underline{3} & 3\underline{3} & \underline{33} & & \\
1\underline{2} & 2\underline{2} & 3\underline{2} & \underline{32} & \underline{22} & \\
1\underline{1} & 2\underline{1} & 3\underline{1} & \underline{31} & \underline{21} & \underline{11}
\end{array}$$

digression: **Case** $C_n^{(1)}$

For general rank we may visualize admissible pair of cascades as figure below



digression: **Case** $C_n^{(1)}$

Theorem

Let $(-j-1)^b(-j)^a, \quad j \in \mathbb{Z}, \quad a+b=k+1, \quad b \geq 0,$ *be a fixed shape and let* $\mathcal{B}$ *and* $\mathcal{A}$ *be two cascades in degree* $-j-1$ *and* $-j$*, with multiplicities* $(m_{\beta,j+1}, \beta \in \mathcal{B})$ *and* $(m_{\alpha,j}, \alpha \in \mathcal{A})$*, such that* $\sum_{\beta \in \mathcal{B}} m_{\beta,j+1} = b, \quad \sum_{\alpha \in \mathcal{A}} m_{\alpha,j} = a$*. Let* $r \in \{1, \cdots, n, \underline{n}, \cdots, \underline{1}\}$*. If the points of cascade* $\mathcal{B}$ *lie in the upper triangle* $\triangle_r$ *and the points of cascade* $\mathcal{A}$ *lie in the lower triangle* ${}^r\triangle$*, then*

$$\prod_{\beta \in \mathcal{B}} X_\beta(-j-1)^{m_{\beta,j+1}} \prod_{\alpha \in \mathcal{A}} X_\alpha(-j)^{m_{\beta,j}}$$

is the leading term of a relation for level k *standard module for affine Lie algebra of the type* $C_n^{(1)}$*.*

digression: **Conjecture**

Let $n \geq 2$ and $k \geq 2$. We consider the standard module $L(k\Lambda_0)$ for the *aff* $\mathcal{LA}$ of type $C_n^{(1)}$ ($\{X_{ab}(j) \mid ab \in B, j \in \mathbb{Z}\} \cup \{c, d\}$ base).
We conjecture that the set of monomial vectors

$$\prod_{ab \in B,\, j>0} X_{ab}(-j)^{m_{ab;j}}\, v_0,$$

satisfying difference conditions $\sum_{ab\in\mathcal{B}} m_{ab;j+1} + \sum_{ab\in\mathcal{A}} m_{ab;j} \leq k$ *for any admissible pair of cascades* $(\mathcal{B}, \mathcal{A})$*, is a basis of* $L(k\Lambda_0)$*.*

The conjecture is true for

- $n = 1$ and all $k \geq 1$ [Meurman-Primc]
- $k = 1$ for all $n \geq 2$ [Primc-Š]

still digression: **Proposition**

If for each $\ell \in \{k+2, \ldots, 2k+1\}$ there exists a finite-dimensional subspace $Q_\ell \subset \ker\left(\Phi \,|\, (\bar{R}\mathbf{1} \otimes V)_\ell\right)$ such that $\ell(\pi) = \ell$ for all $\pi \in \ell t\,(Q_\ell(n))$ and

$$\sum_{\pi \in \mathcal{P}^\ell(n)} N(\pi) = \dim Q_\ell(n),$$

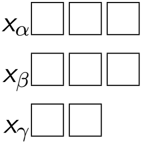
for all $n \leq -k-2$, then the set of vectors

$$u(\pi)\mathbf{1}\,, \qquad \pi \in \mathcal{RR},$$

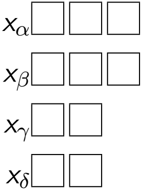
is a basis of the standard module $L(k\Lambda_0)$.

end of digression: **Why is $\ell \in \{k+2, \ldots, 2k+1\}$ important?**

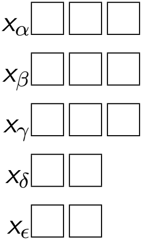
$$k = 1 \Rightarrow k + 2 = 2k + 1 = 3 \quad \rightsquigarrow$$



$$k = 2 \Rightarrow k + 2 = 4 \text{ and } 2k + 1 = 5$$



and



Embeddings of leading terms

Thm: Vectors of the form

$$u(\rho \subset \pi)\mathbb{1} \ , \ \rho \in (\ell t\bar{R}) \cap \mathcal{P}_{<0}$$

are the spanning set of $N^1(k\Lambda_0)$.

- Conjecture B ($C_n^{(1)}$ case): For each $\pi \in \mathcal{P}_{<0}$ which alows at least one embedding take (only) one vector of the form

  $$u(\rho \subset \pi)\mathbb{1}$$

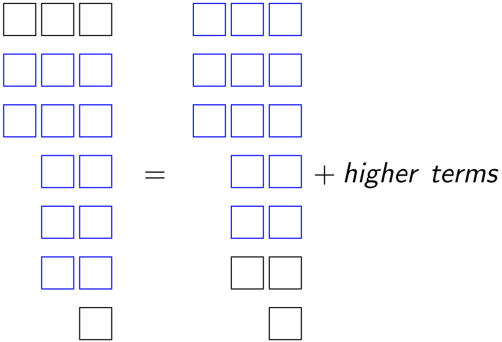
  these vectors are the basis of $N^1(k\Lambda_0)$
- Conjecture A ($C_n^{(1)}$ case): For any π with two embeddings $\rho_1 \subset \pi$ and $\rho_2 \subset \pi$ we have a relation among relations

  $$u(\rho_2 \subset \pi)\mathbb{1} = u(\rho_1 \subset \pi)\mathbb{1} + \text{higher terms}$$

Relation among relations

Conjecture B: For any π with two embeddings $\rho_1 \subset \pi$ and $\rho_2 \subset \pi$ we have a relation among relations

$$u(\rho_2 \subset \pi)\mathbb{1} = u(\rho_1 \subset \pi)\mathbb{1} + \text{higher terms}$$



Relation among relations - $C_n^{(1)}$ case

By using representation theory and the relation

$$x_\theta(z)\frac{d}{dz}(x_\theta(z)^{k+1}) = (k+1)x_\theta(z)^{k+1}\frac{d}{dz}x_\theta(z)$$

for element $u\mathbb{1}$ in $N^1(k\Lambda_0)$ with the leading terms $\ell t(u)$ of $\ell(\pi) = k+2$ and $\mid \pi \mid = m$ we get linearly independent relation

$$dimL((k+1)\theta) + dimL((k+2)\theta) + dimL((k+2)\theta - \alpha^*) \ .$$

For level 2 $C_n^{(1)}$-standard modul above relation is equal to

$$2n\binom{2n+2k+2}{2k+3} \ .$$

Relation among relations - $C_n^{(1)}$ case

Moreover, the following inequality holds

$$(\star) \qquad 2n\binom{2n+2k+2}{2k+3} \leq \sum_{|\pi|=m;\ell(\pi)=k+2} N(\pi) \quad \text{where}$$

$N(\pi) = max\{card(\varepsilon(\pi)) - 1, 0\}$, $\varepsilon(\pi) = \{\rho \in \ell t(\bar{R}) \mid \rho \subset \pi\}$.

If in $(\star)$ equality holds for all m than Conjecture A is true for all π of lenght $\ell(\pi) = k+2$.

A much more appropriate notation

For general rank we may visualize admissible pair of cascades as figure below

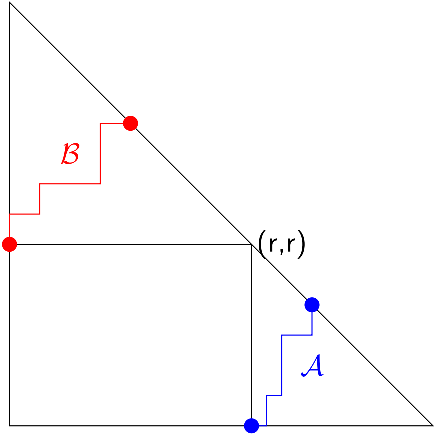


Figure 1

A much more appropriate notation

We will reinterpret the term of cascade. It is interesting to observe the following figure

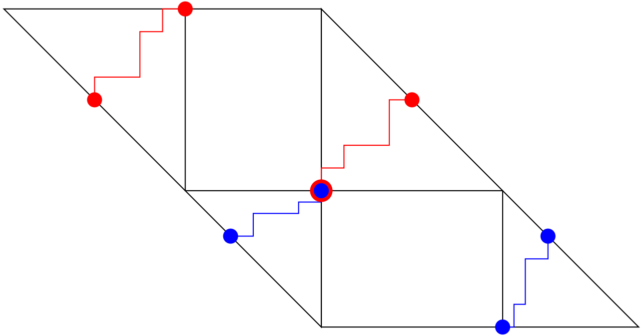


Figure 2

which consists of two identical triangles from Figure 1, but one is rotated and mirrored.

A much more appropriate notation

If we rotate the Figure 2 we have

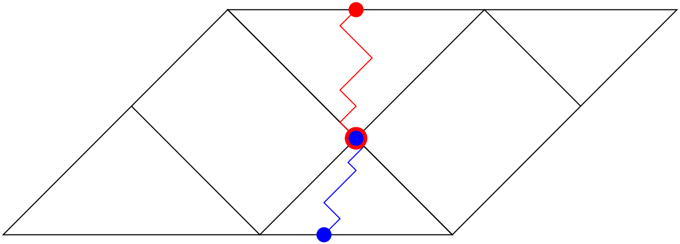


Figure 3

From Figure 3, it is already obvious that the pair of admissible cascades has become a zig-zag line.

A much more appropriate notation

In order to simplify the counting of embeddings of leading terms we introduce a slightly different indexation of a triangular scheme for a basis B. For instance for C_3 we have

					61					
				15		52				
			41		42		43			
		31		32		33		34		
	21		22		23		24		25	
11		12		13		14		15		16

Figure 4

A much more appropriate notation

and for the arbitrary C_n triangular decomposition looks like this

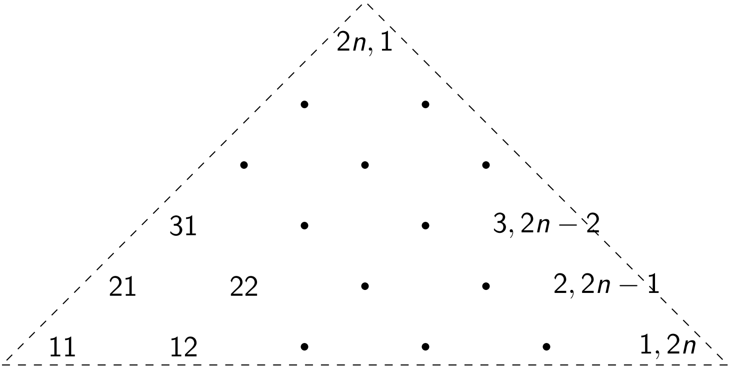


Figure 5

A much more appropriate notation

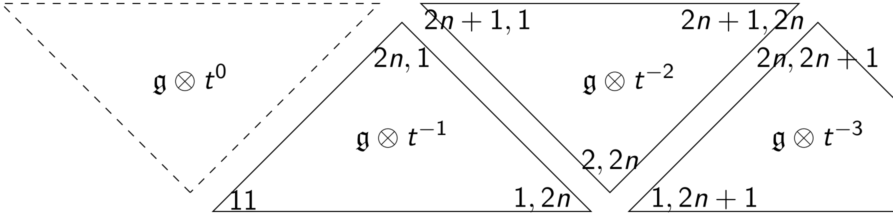


Figure 6

A much more appropriate notation

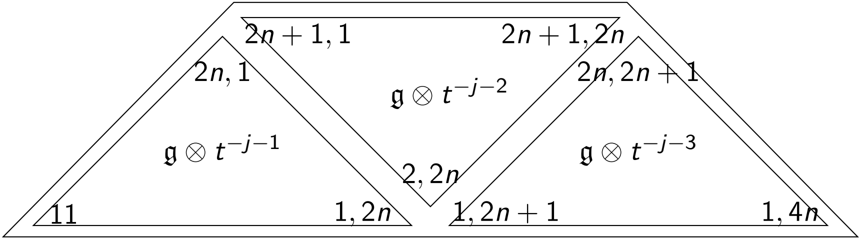


Figure 7

A much more appropriate notation

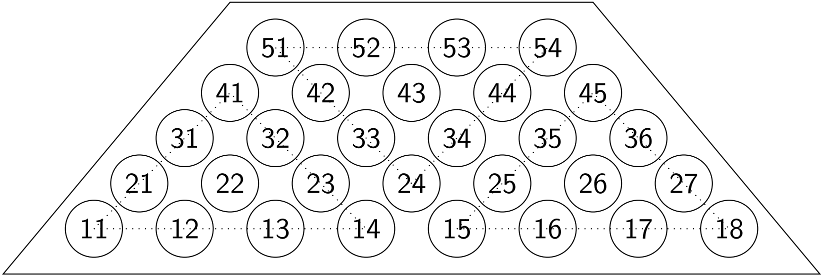


Figure 8

A much more appropriate notation

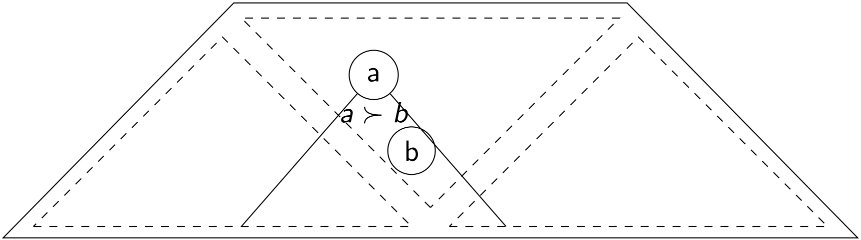


Figure 9

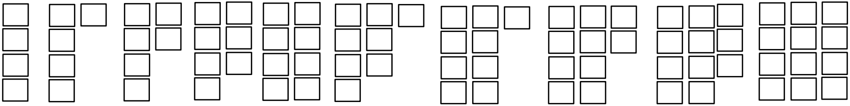
Case $C_2^{(1)}$ for $k = 2$

$$(\star) \qquad 2n\binom{2n+2k+2}{2k+3} =?= \sum_{|\pi|=m;\ell(\pi)=k+2} N(\pi) \quad \text{where}$$

$$2n\binom{2n+2k+2}{2k+3} = 4\binom{10}{7} = 480$$

Case $C_2^{(1)}$ for $k = 2$ and $m = 4, \cdots, 12$

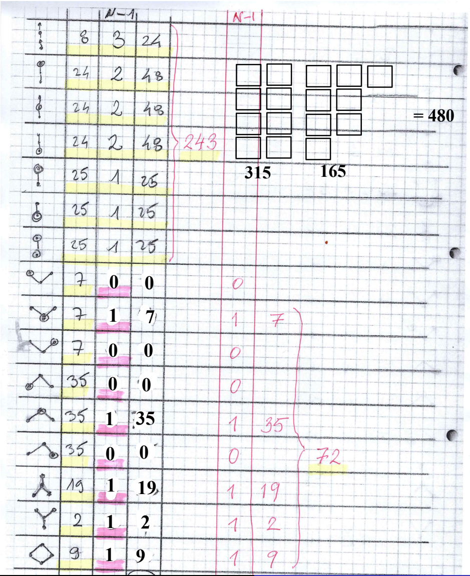
Young Diagrams



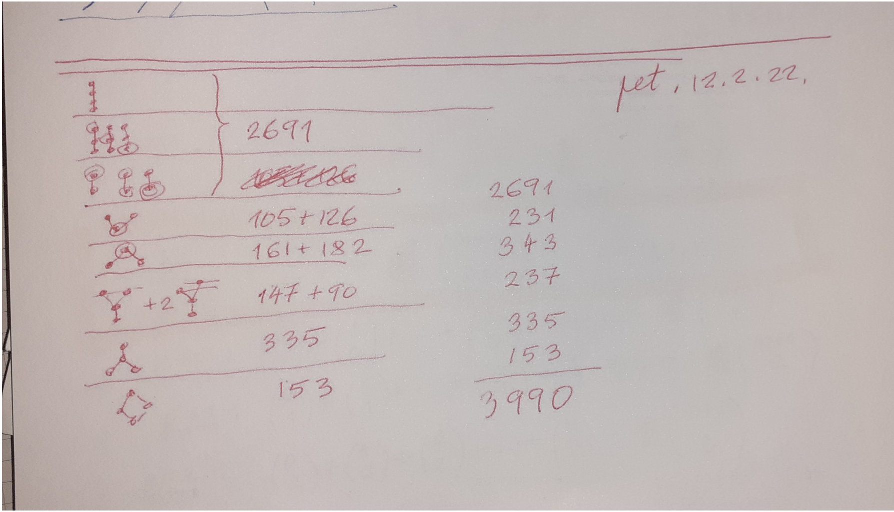
315 480 480 480 315 165 480 480 480 315

= 7 x 480 + 2 x 315 = 3990

Case $C_2^{(1)}$ for $k = 2$ and $m = 8$



Case $C_2^{(1)}$ for $k = 2$



THE END

THANK YOU!