Duality of hook-type W-superalgebras via convolution operations

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Triality of principal $\mathcal{W}$-algebras

The $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{sl}_n)$ enjoy the *triality*

$\mathcal{W}^k(\mathfrak{sl}_n)$ —— FF duality —— $\mathcal{W}^{\check{k}}(\mathfrak{sl}_n)$

GKO

$\mathrm{Com}(V^\ell(\mathfrak{sl}_n), V^{\ell-1}(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n))$

Relation of levels

$$(k+n)(\check{k}+n) = 1$$

$$\frac{1}{k+n} + \frac{1}{\ell+n} = 1$$

Ex $\mathcal{W}^k(\mathfrak{sl}_3) = \langle L(z), W_3(z) \rangle$ with OPEs

$$L(z)L(w) \sim \frac{\frac{1}{2}c_3(k)}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}, \quad c_3(k) = 25 - 12\left(k+3+\frac{1}{k+3}\right)$$

$$L(z)W_3(w) \sim \frac{3W_3(w)}{(z-w)^2} + \frac{\partial_w W_3(w)}{(z-w)}$$

$$W_3(z)W_3(w) \sim \frac{-4(3k+4)(5k+12)}{(z-w)^6} + \frac{-12(3k+4)(5k+12)L(w)}{(z-w)^4} + \frac{-6(3k+4)(5k+12)\partial_w L(w)}{(z-w)^3} + \cdots$$

More trialities?

Feigin and Semikhatov proposed the following triality:

$\mathrm{Com}(\pi, \mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}}))$ —— $\mathrm{Com}(\pi, \mathcal{W}^{\check{k}}(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}}))$

$\mathrm{Com}(V^\ell(\mathfrak{gl}_n), V^\ell(\mathfrak{sl}_{n|1}))$

Relation of levels

$(k+n)(\check{k}+n-1) = 1$

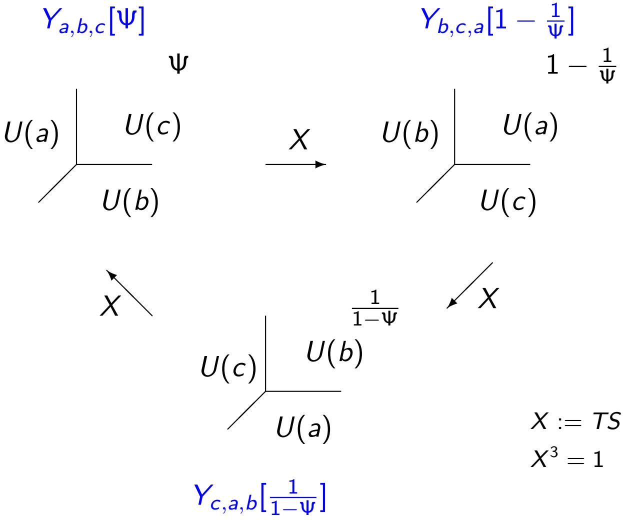
$\frac{1}{k+n} + \frac{1}{\ell+n} = 1$

Ex $\mathrm{Com}(\pi, V^k(\mathfrak{sl}_2)) \simeq \mathrm{Com}(\pi, \mathcal{W}^{\check{k}}(\mathfrak{sl}_{2|1}))$

$$\Leftarrow \ \mathcal{W}^{\check{k}}(\mathfrak{sl}_{2|1}) \simeq \mathrm{Com}(\pi^\Delta, V^k(\mathfrak{sl}_2) \otimes V_{\mathbb{Z}})$$

Gaiotto and Rapcak explained this triality as symmetry of boundary conditions of 4D gauge theory and generalized to $\mathcal{W}$-superalgebras of certain type

Gaiotto–Rapcak's triality



$$X := TS = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$
$$X^3 = 1$$

Hook-type $\mathcal{W}$-superalgebras

For type A, the affine coset of the following $\mathcal{W}$-superalgebras appear

$$\mathcal{W}^k_{A^+}(n,m) = \mathcal{W}^k(\mathfrak{sl}_{n+m}, f_{n,1^m}),$$
$$\mathcal{W}^k_{A^-}(n,m) = \mathcal{W}^k(\mathfrak{sl}_{n+m|m}, f_{n+m,|1^m}).$$

In this talk, we focus on the Feigin–Frenkel type duality:

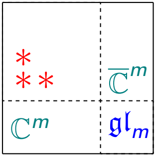
Fact 1 (Creutzig–Linshaw '20)

For $k,\ell \in \mathbb{C}\backslash\mathbb{Q}$ *with* $(k+n+m)(\ell+n)=1$,

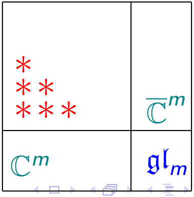
$$\mathrm{Com}(V^{k_\sharp}(\mathfrak{gl}_m), \mathcal{W}^k_{A^+}(n,m)) \simeq \mathrm{Com}(V^{\ell_\sharp}(\mathfrak{gl}_m), \mathcal{W}^\ell_{A^-}(n,m)).$$

$\mathcal{W}(1^{m^2}, 2, \cdots, n, (\frac{n+1}{2})^{2m})$, $\mathcal{W}(1^{m^2}, 2, \cdots, n+m, (\frac{n+m+1}{2})^{2m})$

$\mathfrak{sl}_{n+m}$



$\mathfrak{sl}_{n+m|m}$



Orthosymplectic cases

$$\mathfrak{osp}_{n|2m} \simeq (\mathfrak{so}_n \oplus \mathfrak{sp}_{2m}) \oplus \mathbb{C}^n \otimes \mathbb{C}^{2m}.$$

	$\mathcal{W}^k_{B+}(n,m)$	$\mathcal{W}^k_{C+}(n,m)$	$\mathcal{W}^k_{D+}(n,m)$	$\mathcal{W}^k_{O+}(n,m)$
$\mathfrak{g}$	$\mathfrak{so}_{2(n+m+1)}$	$\mathfrak{sp}_{2(n+m)}$	$\mathfrak{so}_{2(n+m)+1}$	$\mathfrak{osp}_{1\|2(n+m)}$
f	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}_{2n}$	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}_{2n}$
$\mathfrak{g}_\sharp$	$\mathfrak{so}_{2m+1}$	$\mathfrak{sp}_{2m}$	$\mathfrak{so}_{2m}$	$\mathfrak{osp}_{1\|2m}$

	$\mathcal{W}^k_{B-}(n,m)$	$\mathcal{W}^k_{C-}(n,m)$	$\mathcal{W}^k_{D-}(n,m)$	$\mathcal{W}^k_{O-}(n,m)$
$\mathfrak{g}$	$\mathfrak{osp}_{2m+1\|2(n+m)}$	$\mathfrak{osp}_{2(n+m)+1\|2m}$	$\mathfrak{osp}_{2m\|2(n+m)}$	$\mathfrak{osp}_{2(n+m)+2\|2m}$
f	$\mathfrak{sp}_{2(n+m)}$	$\mathfrak{so}_{2(n+m)+1}$	$\mathfrak{sp}_{2(n+m)}$	$\mathfrak{so}_{2(n+m)+1}$
$\mathfrak{g}_\sharp$	$\mathfrak{so}_{2m+1}$	$\mathfrak{sp}_{2m}$	$\mathfrak{so}_{2m}$	$\mathfrak{osp}_{1\|2m}$

Fact 2 (Creutzig–Linshaw '21)

For $k, \ell \in \mathbb{C}\backslash\mathbb{Q}$ with $r_{X,Y}(k + h^\vee_{X+})(\ell + h^\vee_{Y-}) = 1$,

$$\mathrm{Com}(V^{k_\sharp}(\mathfrak{g}_\sharp), \mathcal{W}^k_{X+}(n,m))^{\mathbb{Z}_2} \simeq \mathrm{Com}(V^{\ell_\sharp}(\mathfrak{g}_\sharp), \mathcal{W}^\ell_{Y-}(n,m))^{\mathbb{Z}_2}.$$

Kazama–Suzuki coset construction

Let's ask whether we can upgrade this duality to that of $\mathcal{W}$-superalgebras. Kazama–Suzuki coset construction is a prototype in this direction:

$$\begin{array}{rccc} \textbf{KS}: & \mathcal{W}^{\ell}(\mathfrak{sl}_{2|1}) & \xrightarrow{\simeq} & \mathrm{Com}\left(\pi^{H^{\Delta}_{+}}, V^{k}(\mathfrak{sl}_2)\otimes V_{\mathbb{Z}}\right) \\ & G^{+} & \mapsto & \sqrt{\frac{2}{k+2}}e\otimes|1\rangle \\ & G^{-} & \mapsto & \sqrt{\frac{2}{k+2}}f\otimes|\text{-}1\rangle \end{array}$$

We have its inverse construction due to Feigin–Semikhatov–Tipunin

$$V^{k}(\mathfrak{sl}_2)\simeq \mathrm{Com}\left(\pi^{H^{\Delta}_{-}}, \mathcal{W}^{\ell}(\mathfrak{sl}_{2|1})\otimes V_{\sqrt{\text{-}1}\,\mathbb{Z}}\right).$$

These two constructions yield categorical equivalence

$$\Omega^{+}_{-\lambda}\colon V^{k}(\mathfrak{sl}_2)\text{-}\mathrm{mod}^{[\lambda]}_{\mathrm{wt}} \rightleftarrows \mathcal{W}^{\ell}(\mathfrak{sl}_{2|1})\text{-}\mathrm{mod}^{[\epsilon\lambda]}_{\mathrm{wt}}\colon \Omega^{-}_{\epsilon\lambda},\quad (\epsilon=\tfrac{2}{k+2}).$$

Rephrasing the coset functors

$$V^k(\mathfrak{sl}_2) \simeq \bigoplus_{n\in\mathbb{Z}} \mathscr{C}_n^{k,+} \otimes \pi_n^{\frac{1}{2}h}, \quad \mathcal{W}^\ell(\mathfrak{sl}_{2|1}) \simeq \bigoplus_{n\in\mathbb{Z}} \mathscr{C}_n^{\ell,-} \otimes \pi_n^{J}.$$

The coset functors work well for Heisenberg vertex algebras but don't for affine vertex algebras in general:

$$\mathrm{Com}(\pi^\Delta, \pi\otimes\pi) \simeq \pi^\perp, \quad \mathrm{Com}(V^\ell(\mathfrak{gl}_n), V^{\ell-1}(\mathfrak{gl}_n)\otimes L_1(\mathfrak{gl}_n)) \simeq \mathcal{W}^k(\mathfrak{gl}_n).$$

We have another nice tool, *relative semi-infinite cohomology functor*:

Fact 3 (I.Frenkel–Garland–Zuckerman)

For $\lambda, \mu \in P_+$,

$$H_{\mathrm{rel}}^{\frac{\infty}{2}+n}\left(\mathfrak{g}, \mathbb{V}_\lambda^{\kappa_1}\otimes\mathbb{V}_\mu^{\kappa_2}\right) \simeq \delta_{n,0}\delta_{\lambda,\mu^\dagger}\mathbb{C}[\mathrm{tr}_\mu]$$

holds for (κ_1,κ_2) *such that* $\kappa_1+\kappa_2 = -\kappa_{\mathfrak{g}}$.

Gluing objects: first example

$$V^k(\mathfrak{sl}_2) \simeq \bigoplus_{n\in\mathbb{Z}} \mathscr{C}_n^{k,+} \otimes \pi_n^{\frac{1}{2}h}, \quad \mathcal{W}^\ell(\mathfrak{sl}_{2|1}) \simeq \bigoplus_{n\in\mathbb{Z}} \mathscr{C}_n^{\ell,-} \otimes \pi_n^{J}.$$

$$\Rightarrow \quad K_{+\to-} \quad := \bigoplus_{n\in\mathbb{Z}} \pi_{-n}^{\sqrt{-1}\frac{1}{2}h} \otimes \pi_n^{J} \qquad K_{-\to+} \quad := \bigoplus_{n\in\mathbb{Z}} \pi_{-n}^{\sqrt{-1}J} \otimes \pi_n^{\frac{1}{2}h}$$

$$\simeq V_{\mathbb{Z}} \otimes \pi \qquad\qquad \simeq V_{\sqrt{-1}\mathbb{Z}} \otimes \pi.$$

Proposition 2.1 (CGNS '21)

For $k, \ell \in \mathbb{C}$ with $(k+2)(\ell+1) = 1$, there exist isomorphisms of vertex superalgebras

$$H^0_{\mathrm{rel}}\left(\mathfrak{gl}_1; V^k(\mathfrak{sl}_2) \otimes K_{+\to-}\right) \simeq \mathcal{W}^\ell(\mathfrak{sl}_{2|1}),$$
$$H^0_{\mathrm{rel}}\left(\mathfrak{gl}_1; \mathcal{W}^\ell(\mathfrak{sl}_{2|1}) \otimes K_{-\to+}\right) \simeq V^k(\mathfrak{sl}_2).$$

Gluing objects: general case

Let's continue the same game to see what's these gluing objects are:

$$\mathcal{W}^k_{A^+}(n,m) = \mathcal{W}^k(\mathfrak{sl}_{n+m}, f_{n,1^m}) \simeq \bigoplus_{\lambda\in P_+} \mathscr{C}^{k,+}_\lambda \otimes \mathbb{V}^{k^*_\sharp}_\lambda(\mathfrak{gl}_m)$$

$$\mathcal{W}^\ell_{A^-}(n,m) = \mathcal{W}^\ell(\mathfrak{sl}_{n+m|m}, f_{n+m|1^m}) \simeq \bigoplus_{\lambda\in P_+} \mathscr{C}^{\ell,-}_\lambda \otimes \mathbb{V}^{\ell^*_\sharp}_\lambda(\mathfrak{gl}_m)$$

$$\Rightarrow \quad K_{+\to -} := \bigoplus_{\lambda\in P_+} \mathbb{V}^{-k^*_\sharp-2h^\vee}_{\lambda^*}(\mathfrak{gl}_m) \otimes \mathbb{V}^{\ell^*_\sharp}_\lambda(\mathfrak{gl}_m)$$

$$K_{-\to +} := \bigoplus_{\lambda\in P_+} \mathbb{V}^{-\ell^*_\sharp-2h^\vee}_{\lambda^*}(\mathfrak{gl}_m) \otimes \mathbb{V}^{k^*_\sharp}_\lambda(\mathfrak{gl}_m)$$

They are specializations of the families

$$K^\pm(\mathfrak{gl}_m,\alpha) = \bigoplus_{\lambda\in P_+} \mathbb{V}^\alpha_\lambda(\mathfrak{gl}_m) \otimes \mathbb{V}^\beta_{\lambda^*}(\mathfrak{gl}_m), \quad \tfrac{1}{\alpha+h^\vee} + \tfrac{1}{\beta+h^\vee} = \pm 1$$

$\Rightarrow$ Something happens when $\frac{1}{\alpha+h^\vee} + \frac{1}{\beta+h^\vee} \in \mathbb{Z}$??
$\Rightarrow$ LHS$=0$ is realized as the algebra of chiral differential operators:

$$K^0(\mathfrak{gl}_m,\alpha) \simeq \mathscr{D}^{\mathrm{ch}}_{\mathrm{GL}_m,\alpha}$$

Easy example of CDO: $\mathscr{D}^{\mathrm{ch}}_{GL_1,k}$ $(k \neq 0)$

$$\mathscr{D}^{\mathrm{ch}}_{GL_1,k} = \mathcal{U}(\widehat{\mathfrak{gl}}_{1,k}) \underset{\mathcal{U}(\mathfrak{gl}_1[\![t]\!])}{\otimes} \mathbb{C}[J_\infty GL_1] \simeq \mathbb{C}[h_{-n}, x^{\pm}, \partial^n x \mid n \geq 1].$$

The OPE is given by

$$h(z)h(w) \sim \tfrac{k}{(z-w)^2}, \;\; h(z)x^{\pm}(w) \sim \tfrac{x^{\pm}}{(z-w)}, \;\; x^{\pm}(z)x^{\pm}(w) \sim 0$$

$$\begin{array}{ccccc} \pi^L := \pi^k & \hookrightarrow & \mathscr{D}^{\mathrm{ch}}_{GL_1,k} & \hookleftarrow & \pi^{-k} =: \pi^R \\ H_k(z) & \rightharpoonup & h(z) & & \\ & & -h(z) + kx^{-1}(z)\partial_z x(z) & \leftharpoondown & H_{-k}(z). \end{array}$$

$$\mathbb{C}[GL_1] \simeq \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_n \otimes \mathbb{C}_{-n} \;\Rightarrow\; \bigoplus_{n\in\mathbb{Z}} \pi^L_n \otimes \pi^R_{-n} \simeq \mathscr{D}^{\mathrm{ch}}_{GL_1,k}$$

"Shifted" chiral differential operators $\mathscr{D}^{\mathrm{ch}}_{G,k}[n]$

We consider the case $X = A, C, D$ and set $G = GL_m, \mathrm{Sp}_{2m}, \mathrm{SO}_{2m}$

$$\begin{array}{ccc}
\mathbf{KL}^{\kappa}_G \boxtimes \mathbf{KL}^{\kappa^*}_G & \xrightarrow[\text{Kazhdan-Lusztig}]{\simeq} & \mathcal{U}_q(\mathfrak{g})\text{-mod} \boxtimes \mathcal{U}_{q^{-1}}(\mathfrak{g})\text{-mod} \\
\Downarrow & & \Downarrow \\
\mathscr{D}^{\mathrm{ch}}_{G,\kappa} & \longmapsto & \mathbb{C}_q[G] \quad \simeq_{\mathrm{BTC}} \text{ [KW,TW]} \\
\mathbf{KL}^{\kappa}_G \boxtimes \mathbf{KL}^{\kappa'}_G & \xrightarrow{\simeq} & \mathcal{U}_q(\mathfrak{g})\text{-mod} \boxtimes \mathcal{U}_{\pm q^{-1}}(\mathfrak{g})\text{-mod}_\omega \\
\Downarrow & & \Downarrow \\
\mathscr{D}^{\mathrm{ch}}_{G,\kappa}[n] & \longleftarrow & \mathbb{C}^{[n]}_q[G]
\end{array}$$

$\left(\frac{1}{\kappa+h^\vee} + \frac{1}{\kappa'+h^\vee} = rn\right)$ $\qquad$ $\left(q = \exp\left(\frac{\pi\sqrt{-1}}{r(k+h^\vee)}\right)\right)$

Fact 4 (Moriwaki '21)

For $G = \mathrm{GL}_m, \mathrm{Sp}_{2m}, \mathrm{SO}_{2m}$ and $\kappa \notin \mathbb{Q}$, there exist vertex superalgebras

$$\mathscr{D}^{\mathrm{ch}}_{G,\kappa}[n] = \bigoplus_{\lambda \in P_+(G)} \mathbb{V}^{\kappa}_{\lambda}(\mathfrak{g}) \otimes \mathbb{V}^{\kappa'}_{\lambda^*}(\mathfrak{g}), \quad \frac{1}{\kappa+h^\vee} + \frac{1}{\kappa'+h^\vee} = rn.$$

Reconstruction via convolution operation

Theorem 2.2 (CLNS'22)

For irrational levels in duality relation and $X = A, C, D$, there exist isomorphisms of vertex superalgebras

$$\mathcal{W}^{\ell}_{X^-}(n,m) \simeq H^0_{\mathrm{rel}}\left(\mathfrak{g}_\sharp; \mathcal{W}^{k}_{X^+}(n,m) \otimes \mathscr{D}^{\mathrm{ch}}_{G_\sharp, k^*_\sharp}[1]\right),$$

$$\mathcal{W}^{k}_{X^+}(n,m) \simeq H^0_{\mathrm{rel}}\left(\mathfrak{g}_\sharp; \mathcal{W}^{\ell}_{X^-}(n,m) \otimes \mathscr{D}^{\mathrm{ch}}_{G_\sharp, \ell^*_\sharp}[-1]\right).$$

Proof:

$$\mathcal{W}^{k}_{A^+}(n,m) = \mathcal{W}(1^{m^2}, 2, \cdots, n, (\tfrac{n+1}{2})^{2m}),$$

$$\mathcal{W}^{\ell}_{A^-}(n,m) = \mathcal{W}(1^{m^2}, 2, \cdots, n+m, (\tfrac{n+m+1}{2})^{2m}).$$

$$\mathcal{W}^{k}_{A^+}(n,m) = \mathscr{C}^{k,+}_0 \otimes V^{k_\sharp}(\mathfrak{gl}_m) \oplus \mathscr{C}^{k,+}_{\varpi_1} \otimes \mathbb{V}^{k_\sharp}(\mathbb{C}^m) \oplus \mathscr{C}^{k,+}_{\varpi_1^\dagger} \otimes \mathbb{V}^{k_\sharp}(\overline{\mathbb{C}}^m) \oplus \cdots$$

$$\mathcal{W}^{\ell}_{A^-}(n,m) = \mathscr{C}^{\ell,-}_0 \otimes V^{\ell_\sharp}(\mathfrak{gl}_m) \oplus \mathscr{C}^{\ell,-}_{\varpi_1} \otimes \mathbb{V}^{\ell_\sharp}(\mathbb{C}^m) \oplus \mathscr{C}^{\ell,-}_{\varpi_1^\dagger} \otimes \mathbb{V}^{\ell_\sharp}(\overline{\mathbb{C}}^m) \oplus \cdots$$

Reconstruction via convolution operation

Theorem 2.3 (CLNS'22)

For irrational levels in duality relation and $X = A, C, D$, there exist isomorphisms of vertex superalgebras

$$\mathcal{W}^{\ell}_{X^-}(n,m) \simeq H^0_{\mathrm{rel}}\left(\mathfrak{g}_\sharp; \mathcal{W}^{k}_{X^+}(n,m) \otimes \mathscr{D}^{\mathrm{ch}}_{G_\sharp, k_\sharp^*}[1]\right),$$

$$\mathcal{W}^{k}_{X^+}(n,m) \simeq H^0_{\mathrm{rel}}\left(\mathfrak{g}_\sharp; \mathcal{W}^{\ell}_{X^-}(n,m) \otimes \mathscr{D}^{\mathrm{ch}}_{G_\sharp, \ell_\sharp^*}[-1]\right).$$

Proof:

$$\mathcal{W}^{k}_{A^+}(n,m) = \mathcal{W}(1^{m^2}, 2, \cdots, n, (\tfrac{n+1}{2})^{2m}),$$

$$\mathcal{W}^{\ell}_{A^-}(n,m) = \mathcal{W}(1^{m^2}, 2, \cdots, n+m, (\tfrac{n+m+1}{2})^{2m}).$$

$$\mathcal{W}^{k}_{A^+}(n,m) = \mathscr{C}^{k,+}_{0} \otimes V^{k_\sharp}(\mathfrak{gl}_m) \oplus \mathscr{C}^{k,+}_{\varpi_1} \otimes \mathbb{V}^{k_\sharp}(\mathbb{C}^m) \oplus \mathscr{C}^{k,+}_{\varpi_1^\dagger} \otimes \mathbb{V}^{k_\sharp}(\overline{\mathbb{C}}^m) \oplus \cdots$$

$$\mathscr{D}^{\mathrm{ch}}_{G_\sharp, k_\sharp^*}[1] = V^{k_\sharp^*}(\mathfrak{gl}_m) \otimes V^{\ell_\sharp^*}(\mathfrak{gl}_m) \oplus \mathbb{V}^{k_\sharp^*}(\overline{\mathbb{C}^m}) \otimes \mathbb{V}^{\ell_\sharp^*}(\mathbb{C}^m) \oplus \mathbb{V}^{k_\sharp^*}(\mathbb{C}^m) \otimes \mathbb{V}^{\ell_\sharp^*}(\overline{\mathbb{C}}^m) \oplus$$

$$\mathcal{W}^{\ell}_{A^-}(n,m) = \mathscr{C}^{\ell,-}_{0} \otimes V^{\ell_\sharp}(\mathfrak{gl}_m) \oplus \mathscr{C}^{\ell,-}_{\varpi_1} \otimes \mathbb{V}^{\ell_\sharp}(\mathbb{C}^m) \oplus \mathscr{C}^{\ell,-}_{\varpi_1^\dagger} \otimes \mathbb{V}^{\ell_\sharp}(\overline{\mathbb{C}}^m) \oplus \cdots$$

Kazhdan–Lusztig category and intertwining operators

We continue the case $X = A, C, D$

$$\mathcal{W}^k_{X^+}(n,m) \leftrightsquigarrow \mathcal{W}^\ell_{X^-}(n,m)$$

We introduce the following module categories

$$\mathbf{KL}^k_{X^\pm}(n,m) = \{M \mid G_\sharp[\![t]\!] \curvearrowright M\} \subseteq \mathcal{W}^k_{X^\pm}(n,m)\text{-mod}$$

Then we have functors

$$\mathrm{H}_+ \colon \mathbf{KL}^k_{X^+}(n,m) \rightleftarrows \mathbf{KL}^\ell_{X^-}(n,m) \colon \mathrm{H}_-$$

together with functorial homomorphism of intertwining operators

$$\mathrm{H}^{\mathrm{int}}_\pm \colon I^{log}_{\mathcal{W}_{X^\pm}(n,m)}\begin{pmatrix} M_3 \\ M_1 \;\; M_2 \end{pmatrix} \to I^{log}_{\mathcal{W}_{X^\mp}(n,m)}\begin{pmatrix} \mathrm{H}_\pm(M_3) \\ \mathrm{H}_\pm(M_1) \;\; \mathrm{H}_\pm(M_2) \end{pmatrix}.$$

Conjecture 2.4 (CLN, work in progress)

(i)The functors $\mathrm{H}_\pm$ give an equivalence of categories and indeed quasi-inverse to each other.
(ii) $\mathrm{H}^{\mathrm{int}}_\pm$ are functorial isomorphisms which are quasi-inverse to each other.

To summarize...

$$H^0_{\mathrm{rel}}(\mathfrak{g}_\sharp; \bullet \otimes \mathscr{D}^{\mathrm{ch}}_{G_\sharp, k_\sharp^*}[1])$$

Feigin-Frenkel duality

$$\mathcal{W}^k_{X^+}(n,m) \supset \mathcal{W}^k_{X^+}(n,m)^{G_\sharp[\![t]\!]} \simeq \mathcal{W}^\ell_{X^-}(n,m)^{G_\sharp[\![t]\!]} \subset \mathcal{W}^\ell_{X^-}(n,m)$$

$$H^0_{\mathrm{rel}}(\mathfrak{g}_\sharp; \bullet \otimes \mathscr{D}^{\mathrm{ch}}_{G_\sharp, \ell_\sharp^*}[-1])$$

$$\mathbf{KL}^k_{X^+}(n,m) \longleftrightarrow \mathbf{KL}^\ell_{X^-}(n,m)$$

exact functor

$$[\bullet \otimes Y_{\mathscr{D}^{\mathrm{ch}}_{G_\sharp, k_\sharp^*}[1]}(\cdot, z)]$$

$$I^{log}_{\mathcal{W}^k_{X^+}(n,m)}\binom{M_3}{M_1\ M_2} \rightleftarrows I^{log}_{\mathcal{W}^\ell_{X^-}(n,m)}\binom{M_3}{M_1\ M_2}$$

ex. func hom

$$[\bullet \otimes Y_{\mathscr{D}^{\mathrm{ch}}_{G_\sharp, \ell_\sharp^*}[-1]}(\cdot, z)]$$

Application I – Special functions in characters

$$\mathbf{KL}^k_{A^+}(n,m) \xrightarrow[\text{(exact)}]{H^0_{\mathrm{rel}}(\mathfrak{g}_\sharp;\bullet\otimes\mathscr{D}^{\mathrm{ch}}_{G_\sharp,k_\sharp^*}[1])} \mathbf{KL}^\ell_{A^-}(n,m)$$

Type: $\mathcal{W}(1^{m^2}, 2, \cdots, n, (\frac{n+1}{2})^{2m})$

Type: $\mathcal{W}(1^{m^2}, 2, \cdots, n+m, (\frac{n+m+1}{2})^{2m})$

$$\text{ch:}\quad \underbrace{\frac{s_\lambda(z_1,\cdots,z_m)}{R_+(z_1,\cdots,z_m,q)}}_{\mathrm{ch}\mathbb{W}^k_{A^+}(\lambda)(z,q)} \xrightarrow{\text{trigometric } \beta\text{-integral}} \underbrace{\frac{s_{\mathbb{D}(\lambda)}(q,\cdots,q^m)}{R_-(z_1,\cdots,z_m,q)}}_{\mathrm{ch}H^0_{\mathrm{rel}}(\mathbb{W}^k_{A^+}(\lambda))(z,q)}$$

binomial identity

q-binomial identity

$$\underbrace{\sum_{a=0}^m (-1)^a \begin{bmatrix} m \\ n \end{bmatrix} z^a = (1-z)^m}_{\text{comes from Wakimoto resol.}} \longrightarrow \sum_{a=0}^m (-1)^a q^{\frac{1}{2}a(a+1)} \begin{bmatrix} m \\ n \end{bmatrix}_q z^a = (z;q)_m$$

Application II – Free field modules for F.S. case

Let us compare some basic classes of modules for the pair

$$\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}}) = \mathcal{W}^k_{A^+}(n-1,1) \leftrightsquigarrow \mathcal{W}^\ell_{A^-}(n-1,1) = \mathcal{W}^\ell(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}}).$$

Theorem 3.1 (CGN '20)

$$\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}}) \simeq \mathrm{Com}(\pi^{H^\Delta_-}, \mathcal{W}^\ell(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}}) \otimes V_{\sqrt{-1}\mathbb{Z}}),$$
$$\mathcal{W}^\ell(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}}) \simeq \mathrm{Com}(\pi^{H^\Delta_+}, \mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}).$$

The goal is to explain the following dictionary [N,CLN] at generic levels:

$\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}})$-side	$\mathcal{W}^\ell(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}})$-side
Wakimoto modules	"thin" Wakimoto modules
"thick" Wakimoto modules	Wakimoto modules
relaxed highest weight modules	"Verma modules"

$V^k(\mathfrak{sl}_2)$ v.s. $V^c(\mathfrak{ns}_2)$ (Feigin–Semikhatov–Tipunin)

$V^k(\mathfrak{sl}_2)$-side	$V^c(\mathfrak{ns}_2)$-side
affine Verma modules	topological Verma modules
relaxed highest weight modules	massive Verma modules

We have an exact sequence

$$V^k(\mathfrak{sl}_2)\colon 0 \to \mathbb{M}^k_{-(n+2)\varpi_1} \to \mathbb{M}^k_{n\varpi_1} \to \mathbb{V}^k_{n\varpi_1} \to 0.$$

By setting $\epsilon = \frac{2}{k+2}$, the counterpart is

$$V^c(\mathfrak{ns}_2)\colon 0 \to \underbrace{\Omega^+_{-n/2}(\mathbb{M}^k_{-(n+2)\varpi_1})}_{S_\bullet \mathbf{M}^c\left(\frac{-\epsilon(n+2)}{4}, \frac{-\epsilon(n+2)}{2}\right)} \to \underbrace{\Omega^+_{-n/2}(\mathbb{M}^k_{n\varpi_1})}_{\mathbf{M}^c\left(\frac{\epsilon n}{4}, \frac{\epsilon n}{2}\right)} \to L_c\left(\tfrac{\epsilon n}{4}, \tfrac{\epsilon n}{2}\right) \to 0$$

$$\mathbf{M}^c\left(\tfrac{\epsilon n}{4}, \tfrac{\epsilon n}{2}\right) = \underbrace{\mathbb{M}^c\left(\tfrac{\epsilon n}{4}, \tfrac{\epsilon n}{2}\right)}_{\text{massive}} / (G^+_{-1/2}|\tfrac{1}{4}\epsilon n, \tfrac{1}{2}\epsilon n\rangle)\colon \text{topological}$$

Relaxed highest weight modules

We thicken $\mathbb{M}^k_{n\varpi_1}$ to obtain the massive Verma module $\mathbb{M}^c(\frac{\epsilon n}{4}, \frac{\epsilon n}{2})$:

1. Weight module $R_{a,b} = \mathcal{U}(\mathfrak{sl}_2) \underset{\mathbb{C}[\Omega,h]}{\otimes} \mathbb{C}_{a,b}$

$$\mathbb{M}^c(\tfrac{\epsilon n}{4}, \tfrac{\epsilon n}{2}) \simeq \Omega^+_{-n/2}(R^k_{n(n+2)/2,n})$$

2. Adamovic's embedding

$$V^k(\mathfrak{sl}_2) \hookrightarrow \mathcal{W}^k(\mathfrak{sl}_2) \otimes \Pi, \quad \Pi := \bigoplus_{p \in \mathbb{Z}} \pi_{p(1+\sqrt{-1})} \subset V_{\mathbb{Z}} \otimes V_{\sqrt{-1}\mathbb{Z}}$$

$$\curvearrowright \mathrm{M}^k_{r,s}[\lambda] := \mathrm{M}^{c(k)}_{r,s} \otimes \Pi[\lambda] \underset{\lambda:\ \mathrm{gen}}{\simeq} R^k_{a,b}$$

$$(a = 2(k+2)(h_{r,s} + \tfrac{k}{4}),\ b = 2(\lambda - \tfrac{k}{2}))$$

For λ generic, the Feigin–Fuch's resolution gives [N]:

$$0 \to \mathrm{M}^k_{-r,s}[\lambda] \to \mathrm{M}^k_{r,s}[\lambda] \to \mathrm{L}^k_{r,s}[\lambda] \to 0.$$

$$\Downarrow \Omega^+_{-b/2}$$

$$0 \to \mathbb{M}^c(\alpha_-, \beta_\lambda) \to \mathbb{M}^c(\alpha_+, \beta_\lambda) \to L_c(\alpha_+, \beta_\lambda) \to 0.$$

Generalization to Feigin–Semikhatov duality [N,CLN]

(A) affine Verma modules $\mathbb{M}^k_{n\varpi_1} \rightsquigarrow$ Wakimoto modules $\mathbb{W}^{k,+}_\lambda$

$$\mathcal{W}^k(\mathfrak{sl}_n, f_{\mathrm{sub}}) \hookrightarrow \pi^{k+n}_{\mathfrak{h}} \otimes \beta\gamma \curvearrowright \pi^{k+n}_{\mathfrak{h},\lambda} \otimes \beta\gamma =: \mathbb{W}^{k,+}_\lambda$$

$$0 \to H^0_{\mathrm{DS}}(\mathbb{V}^k_\lambda) \to \mathbb{W}^{k,+}_\lambda \to \bigoplus_{\ell(w)=1} \mathbb{W}^{k,+}_{w^{-1}*\lambda} \to \cdots \to 0.$$

For the super-side, we get

$$0 \to \mathrm{H}_+(H^0_{\mathrm{DS}}(\mathbb{V}^k_\lambda)) \to W^{\ell,-}_{\lambda_\diamond} \to \bigoplus_{\ell(w)=1} S_\bullet W^{\ell,-}_{(w^{-1}*\lambda)_\diamond} \to \cdots \to 0.$$

$$\mathrm{ch}\ W^{\ell,-}_\mu \sim \frac{(-zq^{\frac{n+1}{2}}, -z^{-1}q^{\frac{-n+3}{2}}; q)_\infty}{(q;q)^n_\infty}$$

(B) massive Verma $\mathbb{M}^c(\frac{\epsilon n}{4}, \frac{\epsilon n}{2}) \rightsquigarrow$ Wakimoto modules $\mathbb{W}^{\ell,-}_\mu$

$$\mathcal{W}^\ell(\mathfrak{sl}_{n|1}, f_{\mathrm{prin}}) \hookrightarrow \pi^{\ell+h^\vee}_{\mathfrak{h}} \otimes V_{\mathbb{Z}} \curvearrowright \pi^{\ell+h^\vee}_{\mathfrak{h},\mu} \otimes V_{\mathbb{Z}} =: \mathbb{W}^{\ell,-}_\mu.$$

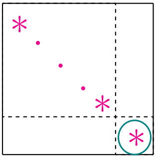
$$\pi^{k+h^\vee}_{\mathfrak{h},\lambda} \otimes \beta\gamma = \mathbb{W}^{k,+}_{\lambda} \xrightarrow{H_{+,\bullet}} W^{\ell,-}_{\lambda_\diamond}$$

$$\cap \qquad\qquad \cap$$

$$\pi^{k+h^\vee}_{\mathfrak{h},\lambda} \otimes \Pi = \widehat{\mathbb{W}}^{k,+}_{\lambda} \xrightarrow{H_{+,\bullet}} \mathbb{W}^{\ell,-}_{\lambda_\diamond} = \pi^{\ell+h^\vee}_{\mathfrak{h},\lambda_\diamond} \otimes V_{\mathbb{Z}}.$$

$$\lambda_\diamond = -(\ell + h^\vee)(\lambda + (\lambda, \varpi_{n-1})\varpi_n)$$

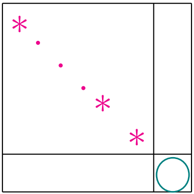
$\mathfrak{sl}_n$



$$\chi_\lambda \leftrightarrow \chi_{-\tilde{\ell}\lambda}$$

$$u \leftrightarrow -\tilde{\ell} u$$

$\mathfrak{sl}_{n|1}$



(C) relaxed highest weight modules [Fehily] v.s. "Verma modules"

$$R^k_+(\chi_\lambda, u) := \mathbb{M}^k_{\chi_\lambda} \otimes \Pi_\bullet[u], \quad R^\ell_-(\chi_\mu) = \mathbb{M}^\ell_{\chi_\mu} \otimes V_{\mathbb{Z}}$$

$$\rightsquigarrow \mathrm{H}_+(R^k_+(\chi_\lambda, u)) \simeq S_\bullet R^\ell_-(\chi_\mu), \quad \mu = -(\ell + h^\vee)(\lambda + u\varpi_n).$$