Reflective forms on orthogonal groups and their expansions at 1-dimensional cusps

T. Driscoll-Spittler, N. Scheithauer, J. Wilhelm
Work in progress

October 4, 2022

1. Automorphic forms on orthogonal groups
2. Bounds for reflective forms
3. Classification of reflective forms of singular weight
4. Expansions at 1-dimensional cusps
5. Relation to vertex operator algebras
6. Summary

Automorphic forms on orthogonal groups

Let L be a rational lattice and $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. A primitive element $\alpha \in L$ of positive norm is called a root of L if $\sigma_\alpha : V \to V$, $x \mapsto x - 2(x,\alpha)\alpha/\alpha^2$ is in $\mathrm{O}(L)$.

The level of an even lattice L is the smallest positive integer N such that $N\alpha^2/2 \in \mathbb{Z}$ for all $\alpha \in L'$.

An even lattice L of level N is called regular if L'/L contains an isotropic element of order N.

Automorphic forms on orthogonal groups

Let L be an even lattice of signature $(n, 2)$, $n > 2$ and $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Define $\tilde{\mathcal{H}} = \{Z \in V(\mathbb{C}) \,|\, (Z, Z) = 0, \, (Z, \overline{Z}) < 0\}^+$.

Let $\Gamma \subset \mathrm{O}(L)^+$ and $\chi : \Gamma \to \mathbb{C}^*$ a character. A meromorphic function $\Psi : \tilde{\mathcal{H}} \to \mathbb{C}$ is called an automorphic form of weight k for Γ with character χ if

$$\Psi(MZ) = \chi(M)\Psi(Z)$$
$$\Psi(tZ) = t^{-k}\Psi(Z)$$

for all $M \in \Gamma$ and $t \in \mathbb{C}^*$.

If Ψ is holomorphic then $k = 0$ or $k \geq (n-2)/2$. Ψ has singular weight if $k = (n-2)/2$.

Automorphic forms on orthogonal groups

For $v \in V$ we define the rational quadratic divisor $v^\perp = \{Z \in \tilde{\mathcal{H}} \,|\, (Z, v) = 0\}$. Ψ is called reflective if the divisor of Ψ is supported on $\bigcup \alpha^\perp$ where α ranges over the roots of L.

Let L be an even lattice of signature $(n, 2)$, $n > 2$ and F a modular form for the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$. Then Borcherds' singular theta correspondence maps F to an automorphic form Ψ for $\mathrm{O}(L, F)^+$. The weight of Ψ is explicitly known and its zeros and poles lie on rational quadratic divisors $\lambda^\perp$ with $\lambda \in L$, $\lambda^2 > 0$.

Bruinier's converse theorem states that if L splits $II_{1,1} \oplus II_{1,1}$ and Ψ is an automorphic form for the discriminant kernel of $\mathrm{O}(L)^+$ whose divisor is supported on a union of rational quadratic divisors then Ψ is a theta lift.

Bounds for reflective forms

Theorem

Let L be a regular lattice of signature $(n, 2)$, $n \geq 4$ and n even splitting $II_{1,1} \oplus II_{1,1}$. Suppose L carries a reflective automorphic form Ψ vanishing or having poles at all divisors $\lambda^{\perp}$ with $\lambda \in L$, $\lambda^2 = 2$. Then $n \leq 26$ and the level N of L divides M where M is given in the following table

n	M
4	$2^7 3^4 5^2 7^2 11^2 13^1 17^1 19^1 23^1$
6	$2^5 3^3 5^2 7^1 11^1$
8	$2^4 3^2 5^1 7^1$
10	$2^3 3^2 5^1$

n	M
12	$2^2 3^1$
14	$2^2 3^1$
16	2^1
18	2^1

n	M
20	1
22	1
24	1
26	1

Bounds for reflective forms

Proof: After symmetrization we can assume that Ψ is the theta lift of a vector valued modular form $F = \sum F_\gamma e^\gamma$ for L. A suitable linear combination f of the components F_γ is a scalar valued modular form for $\Gamma_0(N)$ with small pole orders at the cusps. Then the Riemann-Roch theorem implies bounds on N.

Classification of reflective forms of singular weight

Theorem

Let L be a regular lattice of signature $(n, 2)$, $n \geq 4$ and n even splitting $II_{1,1} \oplus II_{1,1}$. Suppose L carries a reflective holomorphic automorphic form Ψ of singular weight for the discriminant kernel of $O(L)^+$ vanishing at all divisors $\lambda^\perp$ with $\lambda \in L$, $\lambda^2 = 2$. Then L is one of the following lattices

n	Lattice
6	$II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$
8	$II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$, $II_{8,2}(2_1^{+1}4_1^{+1}8_{II}^{+4})$, $II_{8,2}(7^{-5})$
10	$II_{10,2}(2_{II}^{+6}3^{-6})$, $II_{10,2}(5^{+6})$
12	$II_{12,2}(2_2^{+2}4_{II}^{+6})$
14	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$, $II_{14,2}(3^{-8})$
18	$II_{18,2}(2_{II}^{+10})$
26	$II_{26,2}$

Proof: Ψ is the theta lift of a vector valued modular form $F = \sum F_\gamma e^\gamma$ for L. Pairing F with lifts $F_{f,0}$ of Eisenstein series and cusp forms for $\Gamma_0(N)$ imposes restrictions on L.

Theorem (Uniqueness)

Let L be one of the above lattices. If L carries a reflective automorphic form Ψ of singular weight vanishing at all divisors $\lambda^\perp$ with $\lambda \in L$, $\lambda^2 = 2$, then Ψ is unique up to $\mathrm{O}(L)$.

Proof: Here we pair F with lifts $F_{f,\gamma}$ of cusp forms for $\Gamma(N)$.

Classification of reflective forms of singular weight

<u>Theorem</u> (Existence)

Let L be one of the above lattices. Then L carries a reflective automorphic form Ψ of singular weight vanishing at all divisors $\lambda^\perp$ with $\lambda \in L$, $\lambda^2 = 2$.

If L has squarefree level N then Ψ can be constructed as follows. Define $f(\tau) = \prod_{d|N} \eta(d\tau)^{-24/\sigma_1(N)}$. Then the liftings

$$f \mapsto F_{f,0} \mapsto \Psi$$

map f to the desired automorphic form for $\mathrm{O}(L)$. Here the second map is Borcherds' singular theta correspondence.

The other cases are similar but more complicated.

Expansions at 1-dimensional cusps

Theorem (Classification of 1-dimensional cusps)

Let Ψ be one of the above 11 reflective automorphic forms of singular weight. Then the number of inequivalent 1-dimensional cusps of type $L = K \oplus II_{1,1} \oplus II_{1,1}$ of $\mathrm{O}(L,F)^+ \backslash \mathcal{H}$ is given by

L	#	L	#	L	#
$II_{26,2}$	24	$II_{12,2}(2_2^{+2}4_{II}^{+6})$	5	$II_{8,2}(2_1^{+1}4_1^{+1}8_{II}^{+4})$	1
$II_{18,2}(2_{II}^{+10})$	17	$II_{10,2}(5^{+6})$	2	$II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$	2
$II_{14,2}(3^{-8})$	6	$II_{10,2}(2_{II}^{+6}3^{-6})$	2	$II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$	1
$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$	9	$II_{8,2}(7^{-5})$	1		

Expansions at 1-dimensional cusps

Proof: This amounts to determining the cardinalities of the double cosets

$$\overline{\mathrm{O}(L,F)}\backslash \mathrm{O}(D)/\overline{\mathrm{O}(K)}$$

where $D = L'/L$.

Expansions at 1-dimensional cusps

<u>Theorem</u> (First non-vanishing coefficient)

Let Ψ be one of the above 11 reflective automorphic forms of singular weight corresponding to the lattice L. Then the first non-vanishing Fourier-Jacobi coefficient of Ψ at a 1-dimensional cusp of type $K \oplus II_{1,1} \oplus II_{1,1}$ is up to a constant of absolute value 1 of the form

$$e^{2\pi i \omega c}\, \eta(\tau)^{\mathrm{rk}(K)} \prod_{\alpha \in R^+} \frac{\vartheta((z,\alpha),\tau)}{\eta(\tau)}$$

where R is a root system in K' and $c = (\dim(\mathfrak{g}) - 24)/24$.

The root systems that we obtain in this way are exactly the 70 root systems from Schellekens' list together with their scalings.

Relation to vertex operator algebras

Let V be a holomorphic vertex operator algebra of central charge 24 with $\mathfrak{g} = V_1 \neq 0$ semisimple. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For $z \in \mathfrak{h}$ define

$$\chi_V(\tau, z) = \mathrm{tr}_V\, e^{2\pi i z_0} q^{L_0 - 1}\,.$$

Then χ_V is a Jacobi form of weight 0 for M where M is a sum of scaled coroot lattices. It has a theta decomposition

$$\chi_V(\tau, z) = \sum_{\lambda \in M'/M} g_\lambda(\tau) \theta_\lambda^M(\tau, z)\,.$$

Dividing by the cominimal simple currents of $\langle V_1 \rangle$ we can write

$$\chi_V(\tau, z) = \sum_{\lambda \in K'/K} f_\lambda(\tau) \theta_\lambda^K(\tau, z)$$

for some lattice $K \supset M$. Define $L = K \oplus II_{1,1} \oplus II_{1,1}$ and $F = \sum_{\lambda \in K'/K} f_\lambda e^\lambda$.

Theorem

The theta lift Ψ of F is a reflective automorphic form. It has singular weight and vanishes at all divisors $\lambda^\perp$ with $\lambda \in L$, $\lambda^2 = 2$.

Proof: There is a generalized Kac-Moody algebra $\mathfrak{g}(V)$ associated with V. The expansion of Ψ at the 0-dimensional cusp corresponding to $M \oplus II_{1,1}$ with $M = K \oplus II_{1,1}$ is the denominator function of $\mathfrak{g}(V)$ and antisymmetric under the Weyl group W of $\mathfrak{g}(V)$.

Hence Ψ is locally reflective and since it is the lift of a modular form also globally.

Theorem

The first non-vanishing coefficient of the expansion of Ψ at the cusp corresponding to $K \oplus II_{1,1} \oplus II_{1,1}$ is

$$\eta(\tau)^{\mathrm{rk}(K)} \prod_{\alpha \in R^+} \frac{\vartheta((z,\alpha),\tau)}{\eta(\tau)}$$

where R is the root system of $\mathfrak{g}$. We can also recover the scalings.

Summary

Automorphic forms for $O_{n,2}(\mathbb{Z})$ are natural generalizations of the classical modular forms for $SL_2(\mathbb{Z})$.

Reflective automorphic forms of singular weight can be classified by using the Riemann-Roch Theorem and obstruction theory.

Under certain assumptions there are exactly 11 reflective automorphic forms of singular weight. They have exactly 70 generic 1-dimensional cusps. The expansions at these cusps correspond to the 70 Lie algebras on Schellekens' list.

Summary

A holomorphic vertex operator algebra of central charge 24 gives rise to a reflective automorphic form Ψ of singular weight. The first non-vanishing coefficient of the expansion of Ψ at the generic 1-dimensional cusp is the denominator function of $\hat{V}_1$ if V_1 is semisimple or $\Delta(\tau)$.

This gives a new approach to classifying holomorphic vertex operator algebras of central charge 24.