Weight one elements of vertex operator algebras and automorphisms of categories of generalized twisted modules

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Based on joint work with Yi-Zhi Huang

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Outline

- Definitions of twisted modules

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- Results on gradings
- Applications to affine vertex operator algebras.

Twisted modules for finite order automorphisms

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Let $(V, Y, \mathbf{1}, \omega)$ by a vertex operator algebra and $g \in \mathrm{Aut}(V)$ be an automorphism of order k.

Then V can be decomposed as

$$V = \coprod_{j \in \mathbb{Z}/k\mathbb{Z}} V^j$$

where

$$V^j = \{v \in V | gv = \eta^j v\}$$

for $j \in \mathbb{Z}/k\mathbb{Z}$, where $\eta = e^{\frac{2\sqrt{-1}\pi}{k}}$

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Definition A *g-twisted V-module* is a $\mathbb{C}$-graded vector space $W = \coprod_{n \in \mathbb{C}} W_{(n)}$ be a $\mathbb{C}$-graded vector space equipped with a linear map

$$Y_W^g : V \otimes W \to W[[x^{1/k}, x^{-1/k}]],$$
$$v \otimes w \mapsto Y_W^g(v, x)w = \sum_{n \in \frac{1}{k}\mathbb{Z}} (Y_W^g)_n(v) w x^{-n-1}$$

satisfying the following axioms:

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satisfying the following axioms:

- The *grading-restriction condition*: For each $n \in \mathbb{C}$ we have $\dim W_{(n)} < \infty$ and $W_{n+l/k} = 0$ for sufficiently negative integers l.

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satisfying the following axioms:

- The *grading-restriction condition*: For each $n \in \mathbb{C}$ we have $\dim W_{(n)} < \infty$ and $W_{n+l/k} = 0$ for sufficiently negative integers l.
- The *formal monodromy condition*: For $j \in \mathbb{Z}/k\mathbb{Z}$ and $v \in V^j$ we have that

$$Y^g(v,x) = \sum_{n\in j/k+\mathbb{Z}} (Y_W^g)_n(v)x^{-n-1}$$

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- The *identity property*:
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- The *Jacobi identity*: For $u, v \in V$ and $w \in W$:

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W^g(u, x_1)Y_W^g(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W^g(v, x_2)Y_W^g(u, x_1)$$
$$= \frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}} x_2^{-1}\delta\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}}\right) Y_W^g(Y(g^j u, x_0)v, x_2)$$

Twisted modules for finite order automorphisms

And finally some conditions regarding ω: Let $L_W^g(n) = (Y_W^g)_{n+1}(\omega)$ i.e.

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- The *$L(-1)$-derivative condition*:

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In order to generalize the definition of twisted module to *arbitrary* automorphisms of V we need to somehow deal with the appearance of k in both the formal monodromy condition and in the Jacobi identity.

Notation and complex variables

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$$\log z = \log |z| + i \arg z$$

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$$l_p(z) = \log z + 2p\sqrt{-1}\pi$$

for $p \in \mathbb{Z}$.

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$$\begin{aligned} X : & W_1 \otimes W_2 \to W_3[[x^{1/k}, x^{-1/k}]], \\ & w_1 \otimes w_2 \mapsto X(w_1, x)w_2 = \sum_{n \in \frac{1}{k}\mathbb{Z}} X_n(w_1)w_2 x^{-n-1} \end{aligned}$$

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be a linear map preserving the gradings.Define for $w_1 \in W_1$ and $w_2 \in W_2$ the map

$$X^p(w_1, z)w_2 = X(w_1, x)w_2|_{x^n = e^{n l_{\tilde{p}}(z)}} \in \overline{W_3} = \prod_{n \in \mathbb{C}} W_{(n)}$$

where for $p \in \mathbb{Z}/k\mathbb{Z}$ we have $\tilde{p}$ is the integer satisfying $0 \leq \tilde{p} < k$.

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$$\begin{aligned} X : & \mathbb{C}^{\times} \to Hom(W_1 \otimes W_2, \overline{W_3}) \\ & z \mapsto X(\cdot, z) \cdot \end{aligned}$$

which we call the p-th analytic branch of X.

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which we call the p-th analytic branch of X. We have the following theorem by Huang:

Theorem (Huang 2010)

The formal monodromy condition in the definition of twisted module can be replaced by the following property, which we call **equivariance***:*
For $p \in \mathbb{Z}/k\mathbb{Z}$, $z \in \mathbb{C}^\times$, and $v \in V$,

$$Y^{g;p+1}(gv, z) = Y^{g;p}(v, z)$$

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$$f(z_1, z_2) = \sum_{r,s=N_1}^{N_2} a_{rs} z_1^{r/k} z_2^{s/k} (z_1 - z_2)^{-N}$$

for $N_1, N_2 \in \mathbb{Z}$ *and* $N \in \mathbb{Z}_+$

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for $N_1, N_2 \in \mathbb{Z}$ and $N \in \mathbb{Z}_+$such that the series

$$\langle w', Y^{g;p}(u, z_1) Y^{g;p}(v, z_2) w \rangle = \sum_{n \in \mathbb{C}} \langle w', Y^{g;p}(u, z_1) \pi_n Y^{g;p}(v, z_2) w \rangle$$

along with $\langle w', Y^{g;p}(v, z_2) Y^{g;p}(u, z_1) w \rangle$ and $\langle w', Y^{g;p}(Y(u, z_1 - z_2) v, z_2) w \rangle$ are absolutely convergent to the branch

$$\sum_{r,s=N_1}^{N_2} a_{rs} e^{(r/k) l_p(z_1)} e^{(s/k) l_p(z_2)} (z_1 - z_2)^{-N}$$

of $f(z_1, z_2)$ in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, and $|z_2| > |z_1 - z_2| > 0$, respectively.

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Using these theorems as motivation, we generalize the above definition to arbitrary automorphisms of V.

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is a grading-preserving linear map(we take the degree of x to be -1 and the degree of the formal variable $\log x$ to be 0), we can define the map

$$X^p : W_1 \otimes W_2 \to \overline{W_3}$$

by

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A $\mathbb{C}/\mathbb{Z}$-graded generalized g-twisted V-module is a $\mathbb{C} \times \mathbb{C}/\mathbb{Z}$-graded vector space $W = \coprod_{n \in \mathbb{C}, \alpha \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$ equipped with an action of g and a linear map

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- The *duality property*: Let $W' = \coprod_{n\in\mathbb{C},\alpha\in\mathbb{C}/\mathbb{Z}} (W_{[n]}^{[\alpha]})^*$ and, for $n \in \mathbb{C}$, $\pi_n : W \to W_{[n]}$ be the projection. For any $u, v \in V$, $w \in W$ and $w' \in W'$, there exists a multivalued analytic function of the form

$$f(z_1, z_2) = \sum_{i,j,k,l=0}^{N} a_{ijkl} z_1^{m_i} z_2^{n_i} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t}$$

for $N \in \mathbb{N}$, $m_1, \ldots, m_N, n_1, \ldots, n_N \in \mathbb{C}$ and $t \in \mathbb{Z}_+$, such that the series

$$\langle w', Y^{g;p}(u, z_1) Y^{g;p}(v, z_2) w\rangle = \sum_{n\in\mathbb{C}} \langle w', Y^{g;p}(u, z_1) \pi_n Y^{g;p}(v, z_2) w\rangle, \qquad (1)$$

$$\langle w', Y^{g;p}(v, z_2) Y^{g;p}(u, z_1) w\rangle = \sum_{n\in\mathbb{C}} \langle w', Y^{g;p}(v, z_2) \pi_n Y^{g;p}(u, z_1) w\rangle, \qquad (2)$$

$$\langle w', Y^{g;p}(Y(u, z_1 - z_2) v, z_2) w\rangle = \sum_{n\in\mathbb{C}} \langle w', Y^{g;p}(\pi_n Y(u, z_1 - z_2) v, z_2) w\rangle$$

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along with the *lower truncation property*:

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along with the *lower truncation property*:
For all $v\in V$ and $w\in W$, $Y_W^g(v,x)w$ is lower truncated, that is, $(Y_W^g)_n(v)w=0$ when $\Re(n)$ is sufficiently negative.

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along with the *lower truncation property*:
For all $v\in V$ and $w\in W$, $Y_W^g(v,x)w$ is lower truncated, that is, $(Y_W^g)_n(v)w=0$ when $\Re(n)$ is sufficiently negative.

Here, we define $\mathcal{L}_g=\mathcal{S}_g+\mathcal{N}_g$ to be a linear operator such that $g=e^{2\pi i\mathcal{L}_g}=e^{2\pi i\mathcal{S}_g}e^{2\pi i\mathcal{N}_g}$ where $\mathcal{S}_g$ is semisimple on V and $\mathcal{N}_g$ is locally nilpotent on V.

Twisted modules for arbitrary automorphisms

We note that for $\mathcal{L}_g = \mathcal{S}_g + \mathcal{N}_g$ we define

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In particular, if v is a generalized eigenvector of $\mathcal{L}_g$ with eigenvalue λ, we have that

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We define

$$P_V^g := \{\alpha \in [0,1) + i\mathbb{R} | e^{2\pi i \alpha} \text{ is an eigenvalue of } g\}.$$

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- We call a lower bounded generalized g-twisted V-module *strongly* $\mathbb{C}/\mathbb{Z}$-graded if

$$\dim W_{[n]}^{[\alpha]} < \infty$$

and

$$W_{[n+l]}^{[\alpha]} = 0$$

for sufficiently negative real l.

Some more definitions

We say that V has a $\mathbb{C}$-*graded vertex operator algebra structure compatible with* g if V has an additional grading

$$V = \coprod_{\alpha \in \mathbb{C}} V^{[\alpha]} = \coprod_{n \in \mathbb{Z}, \alpha \in \mathbb{C}} V_{(n)}^{[\alpha]}$$

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For such a VOA V a $\mathbb{C}$*-graded generalized* g*-twisted* V*-module* is a $\mathbb{C} \times \mathbb{C}$-graded vector space $W = \coprod_{n, \alpha \in \mathbb{C}} W_{[n]}^{[\alpha]}$ equipped with an action of g and a vertex operator map as before satisfying all the axioms above where $\mathbb{C}/\mathbb{Z}$ is replaced by $\mathbb{C}$

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- The *grading compatibility condition*: for $\alpha, \beta \in \mathbb{C}$, $v \in V^{[\alpha]}$, and $w \in W^{[\beta]}$ we have

$$Y_W^g(v, x)w \in W^{[\alpha+\beta]}\{x\}[\log x]$$

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and also assume that Spec $Y_0(u) \subset \frac{1}{T}\mathbb{Z}$ for some $T \in \mathbb{Z}_+$.

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Theorem (Li 1995)

Let (M, Y_M^g) be a g-twisted V-module and consider

$$\Delta(u,z) = x^{Y_0(u)} \exp\left(\sum_{n\geq 1} \frac{Y_n(u)}{-n}(-z)^{-n} \right)$$

Then $(M, Y_M^{gg_u})$ is a weak gg_u-twisted V-module, where we define

$$Y_M^{gg_u}(v,x) = Y_M^g(\Delta(u,x)v,x).$$

Generalizing $\Delta_V^{(u)}(x)$

Throughout we let V be a vertex operator algebra, $u \in V_{(1)}$ such that $L(1)u = 0$.

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Here we define

$$x^{-Y_0(u)} = x^{-Y_0(u)_S} \exp\left(e^{Y_0(u)_N \log x} \right)$$

where $Y_0(u)_S$ and $Y_0(u)_N$ are the semisimple and nilpotent parts of $Y_0(u)$, respectively.

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Using these properties, it is now just a matter of some direct calculation to prove our main theorem.

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Let (W, Y_W^g) *be a* $\mathbb{C}/\mathbb{Z}$*-graded (or* $\mathbb{C}$*-graded) generalized* g*-twisted* V*-module and* $g_u = e^{2\pi i Y_0(u)} \in Aut(V)$.

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$$Y_W^{gg_u} : V \otimes W \to W\{x\}[\log x]$$

defined by

$$Y_W^{gg_u}(v, x) = Y_W^g(\Delta_V^{(u)}(x)v, x)$$

satisfies the identity property, the lower truncation property, the $L(-1)$*-derivative property, the equivariance property and the Jacobi identity.*

Some theorems about $\mathbb{C}/\mathbb{Z}$-gradings

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The proof of this result uses the fact that

$$L_W^{gg_u}(0) = L_W^g(0) - (Y_W^g)_0(u) + \frac{1}{2}\mu$$

where μ is a constant determined by $Y_1(u)u = \mu\mathbf{1}$ and...

Some theorems about $\mathbb{C}/\mathbb{Z}$-gradings

By assumption, we have that

$$W = \coprod_{n \in \mathbb{C}, \alpha \in P_W^g} W_{[n]}^{[\alpha]}$$

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which we use to define the $L_W^{gg_u}(0)$ generalized eigenspaces with eigenvalue n:

$$W_{\langle n \rangle}^{[\alpha],[\beta]} = \coprod_{\tilde{\beta} \in \beta + \mathbb{Z}} W_{[n+\tilde{\beta}-\frac{1}{2}\mu]}^{[\alpha],[\tilde{\beta}]}$$

This leaves us with a triple grading:

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and finally we define a new $\mathbb{C}/\mathbb{Z}$-grading on W by:

$$W = \coprod_{n\in\mathbb{C},\gamma\in P_W^{ggu}} W_{\langle n\rangle}^{\langle\gamma\rangle}$$

where

$$W_{\langle n\rangle}^{\langle\gamma\rangle} = \coprod_{\alpha\in P_W^{g},\beta\in P_W^{gu},\alpha+\beta\in\gamma+\mathbb{Z}} W_{\langle n\rangle}^{[\alpha],[\beta]}$$

which gives us our desired $\mathbb{C}/\mathbb{Z}$-grading.

Some results on $\mathbb{C}$-gradings

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Suppose V, g, and u are as in the main theorem and let W be a $\mathbb{C}$-graded generalized g-twisted V-module. Assume that there is a semisimple operator $\tilde{\mathcal{S}}_g$ on V such that $g = e^{2\pi i(\tilde{\mathcal{S}}_g + \mathcal{N}_g)}$ and that $Y_V(u, x)v \in V^{[\tilde{\alpha_1}+\tilde{\alpha_2}]}[[x, x^{-1}]]$ for $u \in V^{[\tilde{\alpha}_1]}$ and $v \in V^{[\tilde{\alpha}_2]}$ where for $\alpha \in P_V^g$ we have that $V^{[\tilde{\alpha}]}$ is the $\tilde{\mathcal{S}}_g$-eigenspace with eigenvalue $\tilde{\alpha}$.

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Assume also that $\tilde{\mathcal{S}}_g$ acts on W semisimply and the actions of $e^{2\pi i \mathcal{S}_g}$ and $e^{2\pi i \tilde{\mathcal{S}}_g}$ are equal on W and that $Y_W^g(v,x)w \in W^{[\tilde{\alpha}_1+\tilde{\alpha}_2]}$ for $u \in V^{[\tilde{\alpha_1}]}$ and $w \in W^{[\tilde{\alpha_2}]}$.

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Then $(W, Y_W^{gg_u})$ is a $\mathbb{C}$-graded generalized gg_u-twisted module.

Strongly graded modules

Our final general result gives conditions which ensure a strongly $\mathbb{C}$-graded generalized g-twisted V-module gets mapped to a strongly $\mathbb{C}$-graded generalized gg_u-twisted V-module.

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- The corresponding $\mathbb{C}/\mathbb{Z}$-grading on the g-twisted module

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  gives a strong $\mathbb{C}/\mathbb{Z}$-grading on W.
- A condition which ensures that the graded pieces in new grading on the gg_u-twisted module W are made up of a finite direct sum of g- and g_u-generalized eigenspaces.

An Open Question

We will see later that, in general, $\Delta_V^{(u)}(x)$ does not map a lower-bounded, grading restricted generalized g-twisted module map to a lower-bounded, grading restricted generalized gg_u-module

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Question: Under what conditions does a lower-bounded, grading restricted generalized g-twisted module map to a lower-bounded, grading restricted generalized gg_u-module?

A bit on categories, from the title of the talk

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Let $\mathcal{C}$ be the category of all generalized twisted V-modules.Let $u \in V_{(1)}$ and G^u be the subgroup of $\mathrm{Aut}(V)$ which fix u. Let $\mathcal{C}^u$ be the subcategory of $\mathcal{C}$ consisting of generalized g-twisted modules for $g \in G^u$.

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$$\Delta^u \circ \Delta^{-u} = \Delta^{-u} \circ \Delta^u = 1_u$$

so that Δ^u is an automorphism of the category $\mathcal{C}^u$.

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Application to affine vertex operator algebras

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$$\sigma := e^{2\pi i \mathcal{S}_g} = \tau_\sigma \mu e^{2\pi i \mathrm{ad}_h} \tau_\sigma^{-1}$$

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We also have that $\mathcal{N}_g = ad_{a_{\mathcal{N}_g}}$ where $a_{\mathcal{N}_g}$ is fixed by g.

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A straightforward calculation now gives us:

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Using a result of Huang, there is an invertible functor ϕ_{τ_σ} from the category of generalized g_σ-twisted modules to the category of g-twisted modules (category isomorphism). In particular, to construct g-twisted modules we need only construct g_σ-twisted modules and then apply this functor.

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Our goal is to construct a g_σ-twisted module for any automorphism g of $\mathfrak{g}$

Moving between twisted modules

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Moving between twisted modules

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Our automorphisms g_σ and $g_{\sigma,s}$ give automorphisms of V and we retain their notations. Moreover, ad_h and $ad_{{\tau_\sigma}^{-1} a_{\mathcal{N}_g}}$ will act as $h(0)$ and $\left({\tau_\sigma}^{-1} a_{\mathcal{N}_g}\right)(0)$

Moving between twisted modules

We have the following application of our earlier theorem:

Theorem

Let (W, Y_W^{μ}) be a $\mathbb{C}/\mathbb{Z}$-graded generalized μ-twisted V-module. Assume that $h_W(0) = Res_x Y_W^{\mu}(h(-1)\mathbf{1}, x)$ acts on W semisimply. Define

$$Y_W^{g_{\sigma,s}}(v, x) = Y_W^{\mu}(\Delta_V^{(h)}(x)v, x)$$

Then $(W, Y_W^{g_{\sigma,s}})$, equipped with the earlier gradings, is a $\mathbb{C}/\mathbb{Z}$-graded generalized $g_{\sigma,s}$-twisted V-module.

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Then $(W, Y_W^{g_{\sigma,s}})$, equipped with the earlier gradings, is a $\mathbb{C}/\mathbb{Z}$-graded generalized $g_{\sigma,s}$-twisted V-module.

Conversely, if $(W, Y^{g_{\sigma,s}})$ is a $\mathbb{C}/\mathbb{Z}$-graded generalized $g_{\sigma,s}$-twisted V-module, then we may apply the earlier theorem to obtain a $\mathbb{C}/\mathbb{Z}$-graded generalized μ-twisted module (W, Y_W^{μ}) where

$$Y_W^{\mu}(v, x) = Y_W^{g_{\sigma,s}}(\Delta_V^{(-h)}(x)v, x)$$

Moving between twisted modules

One utility of this construction is that the underlying vector space W stays the same, but one can explicitly compute the actions of elements of V on twisted modules.

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For example, if we take $b \in \mathfrak{g}$ such that $\mu(b) = e^{\frac{2\pi ij}{k}}$ we have that

$$Y_W^{g_{\sigma,s}}(b(-n-1)\mathbf{1}, x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{-\lambda}{k}k!\left(\left(\frac{\partial}{\partial x}\right)^{n-k} Y_W^{\mu}(b(-1)\mathbf{1}, x)\right)x^{-\lambda-k} \tag{3}$$

where we have $[h, b] = \lambda b$ and $\lambda \neq 0$ and where

$$Y_W^{\mu}(b(-1)\mathbf{1}, x) = \sum_{m \in \frac{j}{k}+\mathbb{Z}} b_W^{\mu}(m)x^{-m-1}$$

Now, we move from a $\mathbb{C}/\mathbb{Z}$-graded generalized $g_{\sigma,s}$-twisted V-module to a $\mathbb{C}/\mathbb{Z}$-graded generalized g_{σ}-twisted V-module. We have the following application of our earlier theorems:

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Theorem

Let $(W, Y^{g_{\sigma,s}})$ be a $\mathbb{C}/\mathbb{Z}$-graded generalized $g_{\sigma,s}$-twisted V-module. Assume that $g_{\sigma,s}$ acts on W semisimply and $\tau^{-1}a_{\mathcal{N}_g}(0)$ on W is locally nilpotent. Let $u = \tau_\sigma^{-1}a_{\mathcal{N}_g}(-1)\mathbf{1}$. Then the pair $(W, Y_W^{g_\sigma})$ has the structure of a $\mathbb{C}/\mathbb{Z}$-graded generalized g_σ-twisted V-module where

$$Y_W^{g_\sigma}(v, x) := Y_W^{g_{\sigma,s}}(\Delta_V^{(u)}(x)v, x)$$

for $v \in V$.

Moving between twisted modules

Adding this unipotent part of the automorphism now introduces powers of $\log x$. For example, take $b \in \mathfrak{g}$ to be a generalized eigenvector of g_σ with eigenvalue λ. We have

$$
\begin{aligned}
&Y_W^{g_\sigma}(b(-1)\mathbf{1}, x) \\
&= \sum_{j=0}^{M} \sum_{m \in \lambda + \mathbb{Z}} \frac{(-1)^j}{j!} (\mathrm{ad}^j_{(\tau_\sigma^{-1} a_{\mathcal{N}_g})}(b))_W^{g_{\sigma_s}}(m) x^{-m-1} (\log x)^j - (\tau_\sigma^{-1} a_{\mathcal{N}_g}, b) \ell x^{-1}
\end{aligned}
$$

and we can use the $L(-1)$-derivative property to compute $Y_W^{g_\sigma}(b(-n-1)\mathbf{1}, x)$.

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One can, of course, start with a (untwisted) V-module W and construct modules twisted by inner automorphisms the same way.

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In earlier work, Huang showed that $\Delta_V^{(u)}(x)$ maps a strongly $\mathbb{C}$-graded V-module W to a strongly $\mathbb{C}$-graded generalized $e^{2\pi i Y_0(u)}$-twisted V-module.

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It is unclear, however, what conditions are needed to move from one lower-bounded, grading restricted twisted module to another.

Moving between twisted modules

We have some more results when starting with a strongly $\mathbb{C}$-graded generalized V-module (W, Y_W). Let $a \in \mathfrak{g}$ with Jordan-Chevalley decomposition $a = s + n$. Let $g_a = e^{2\pi i a(0)}$ and $g_s = e^{2\pi i s(0)}$

Theorem

Assume $s_W(0)$ acts semisimply on W. Then, the pair $(W, Y_W^{g_s})$ can be given the structure of a strongly $\mathbb{C}$-graded generalized g_s-twisted module where

$$Y_W^{g_s}(v, x) = Y_W(\Delta_V^{(s(-1)\mathbf{1})}(x)v, x)$$

Theorem

Let $(W, Y_W^{g_s})$ be a $\mathbb{C}$-graded generalized g_s-twisted V-module. The pair $(W, Y_W^{g_a})$ can be given the structure of a $\mathbb{C}$-graded generalized g_a-twisted V-module. Moreover, if (W, Y^{g_s}) is grading restricted, the $(W, Y_W^{g_a})$ is also grading restricted.

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Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Consider $V = M(\ell, 0)$, the generalized Verma module and $W = M(\ell, 0)$. W is lower bounded and grading restricted by definition.

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Let $\{e_\alpha, f_\alpha, h_\alpha\}$ be an $\mathfrak{sl}(2)$-triple where

$$[e_\alpha, f_\alpha] = h_\alpha, \quad [h_\alpha, e_\alpha] = 2e_\alpha \quad [h_\alpha, f_\alpha] = -2f_\alpha.$$

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Then, the set

$$\{e_\alpha(-1)^k \mathbf{1} | k \geq 0\}$$

is an infinite linearly independent subset of $W_{\langle \frac{1}{2}(s,s)\ell \rangle}$ and so this twisted module structure doesn't satisfy the grading-restriction condition.

Thank you!