On Collapsing Levels (II)

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Goal

Conformal embeddings

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We shall consider conformal embeddings and collapsing levels associated to affine W–algebras for not necessarily admissible levels.

Focus of this talk

1. A criterion for detecting conformal embeddings and collapsing levels.
2. Hook type W–algebras and rectangular W–algebras for $sl(N)$.
3. Conformal vs collapsing levels.
4. Decomposition of conformal embeddings: hook W–algebra case.

Some history

- The first appearance of the notion of conformal embedding, still not embodied in vertex algebra language, occurs in papers by Kac-Peterson, and Kac-Wakimoto and in Physics literature (Arcuri-Gomez-Olive): the key results is the classification of conformal embeddings $V_{\mathbf{k}}(\mathfrak{k}) \hookrightarrow V_1(\mathfrak{g})$ where $\mathfrak{k}$ is a semisimple subalgebra of a simple Lie algebra $\mathfrak{g}$.
- In a series of papers in collaboration with V.G. Kac and O. Perše we have completely solved the problem for the embeddings $V_{\mathbf{k}'}(\mathfrak{k}) \hookrightarrow V_k(\mathfrak{g})$ corresponding to *maximally reductive embeddings* $\mathfrak{k} \hookrightarrow \mathfrak{g}$.
- We have then considered the case when $\mathfrak{g}$ is a basic Lie superalgebra (joint with O. Perše).
- In all these cases, it is important to have an explicit description of the decomposition of $V_k(\mathfrak{g})$ as a $V_{\mathbf{k}}(\mathfrak{k})$-module. This is a difficult problem which has been solved in many, but not all, cases. It yields applications to combinatorics and character theory.

Prototypical tool: the AP-criterion

$\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_t$ semisimple subalgebra of a simple Lie algebra $\mathfrak{g}$; let $\mathfrak{p}$ be the orthocomplement of $\mathfrak{k}$ in $\mathfrak{g}$ w.r.t. a non-degenerate invariant bilinear form and decompose, as $\mathfrak{k}$-modules

$$\mathfrak{p} = \bigoplus_{i=1}^{s} (V(\mu_i^1) \otimes \ldots \otimes V(\mu_i^t))$$

$$\widetilde{V}(k, \mathfrak{k}) = \text{ vertex subalgebra of } V_k(\mathfrak{g}) \text{ generated by } \{x_{(-1)}\mathbf{1} \mid x \in \mathfrak{k}\}.$$

Theorem (Adamović-Perše)

$\widetilde{V}(k, \mathfrak{k})$ is conformally embedded in $V_k(\mathfrak{g})$ if and only if, for any $i = 1, \ldots, s$,

$$\sum_{j=0}^{t} \frac{(\mu_i^j, \mu_i^j + 2\rho_0^j)_j}{2(k_j + h_j^\vee)} = 1$$

W-algebras

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The universal affine W-algebra $W^k(\mathfrak{g}, x, f)$ of central charge $c(\mathfrak{g}, k)$ is a vertex algebra obtained by quantum Hamiltonian reduction from the datum $(\mathfrak{g}, x, f)$, where $\mathfrak{g}$ is a basic Lie superalgebra, x is an ad–diagonalizable element of $\mathfrak{g}$ with eigenvalues in $\frac{1}{2}\mathbb{Z}$, f is an even element of $\mathfrak{g}$ such that $[x, f] = -f$ and the eigenvalues of $\operatorname{ad} x$ on the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ are non-positive.

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We will also assume that the datum $(\mathfrak{g}, x, f)$ is *Dynkin*, i.e. there is a $sl(2)$-triple $\{e, h, f\}$ and $x = \frac{1}{2}h$.

W-algebras and conformal embeddings

Let $\mathfrak{g}^\natural$ be the centralizer of this $sl(2)$-triple.

Basic Facts

It turns out that $W^k(\mathfrak{g}, x, f)$ contains an affine vertex subalgebra $V(\mathfrak{g}^\natural)$ and that $W^k(\mathfrak{g}, x, f)$ has a unique simple graded quotient $W_k(\mathfrak{g}, x, f)$ (at non critical level).

Theorem

The image $\mathcal{V}(\mathfrak{g}^\natural)$ of $V(\mathfrak{g}^\natural)$ into the simple affine W–algebra $W_k(\mathfrak{g}, x, f_\theta)$ attached to a minimal nilpotent element f_θ is conformal if and only if

$$c_{\mathfrak{g}^\natural} = c(\mathfrak{g}, k) = \frac{k \,\mathrm{sdim}\mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4, \tag{1.1}$$

where $c_{\mathfrak{g}^\natural}$ is the Sugawara central charge of the affine vertex subalgebra $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_\theta)$ and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

Comment

Relation (1.1) is clearly a necessary condition for conformal embedding. But proving that this condition is sufficient was highly non-trivial and it was a central part of our work. Indeed we proved that

- a sufficient condition for a conformal embedding is that the the Sugawara conformal weight of the G–generators of $W_k(\mathfrak{g}, f_\theta)$ is $\frac{3}{2}$;
- the previous condition implies that the conformal level k is either $-\frac{2h^\vee}{3}$ or $-\frac{h^\vee-1}{2}$.

This result easily implies that (1.1) is a sufficient condition for conformal embedding.

The first new result I would like to present is a generalization of criterion (1.1) for non-minimal affine W–algebras.

Setup

Definition 1.1

A *conformal* vertex algebra is a vertex algebra W with a Virasoro vector $L \in W_2$, such that $L(0)$ is diagonalizable and its eigenspace decomposition has the form

$$W = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} W(n), \tag{1.2}$$

$$\dim W(n) < \infty \text{ for all } n \text{ and } W(0) = \mathbb{C}\mathbf{1}. \tag{1.3}$$

Let c denote the central charge of W.

Setup

If $a \in W(\Delta_a)$, set

$$(-1)^{L(0)}a = e^{\pi\sqrt{-1}\Delta_a}a, \quad \sigma^{1/2}(a) = e^{\frac{\pi}{2}\sqrt{-1}p(a)}a.$$

If ϕ is a conjugate linear involution of W, set

$$g = ((-1)^{L(0)}\sigma^{1/2})^{-1}\phi.$$

Definition 2.1

Let ϕ be a conjugate linear involution of a conformal vertex algebra W such that $\phi(L) = L$. A Hermitian form $(\cdot\,,\cdot)$ on W is said to be *ϕ–invariant* if, for all $a \in W$,

$$(v, Y(a,z)u) = (Y(e^{zL(1)}z^{-2L(0)}g\ a, z^{-1})v, u), \quad u, v \in W.$$

Explanation of the definition

Remarks

- The motivation for this definition stems from the observation that, given a W–module M, one has (as in the Lie algebra case), a bijective correspondence between ϕ–invariant Hermitian forms $(\cdot,\cdot)$ on W and W–module linear homomorphisms $\Theta : M \to M^\dagger$, where $M^\dagger$ is the conjugate linear dual to M, with W–module structure defined by

$$\langle Y_{M^\dagger}(a,z)m', m\rangle = \langle m', Y_M(e^{zL_1}z^{-2L_0}g\,a, z^{-1})m\rangle$$

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- For a conformal vertex algebra the existence conditions reduce to

$$L(1)W(1) = \{0\} \tag{2.1}$$

In such a case we normalize the form by requiring $(\mathbf{1},\mathbf{1}) = 1$.

Setup

Recall that $\mathfrak{a} = W(1)$ is a Lie superalgebra with bracket defined by

$$[a, b] = a_0 b.$$

Let $\langle \cdot, \cdot \rangle$ be the bilinear form on $\mathfrak{a}$ defined by

$$\langle a, b \rangle = (g(a), b).$$

If $L(1)W(1) = \{0\}$, then $\mathfrak{a}$ is made of primary elements. It follows that

$$\langle a, b \rangle \mathbf{1} = (g(a), b)\mathbf{1} = (g(a)_{-1}\mathbf{1}, b_{-1}\mathbf{1})\mathbf{1} = (\mathbf{1}, a_1 b)\mathbf{1} = a_1 b,$$

so that we can write

$$[a_\lambda b] = [a, b] + \lambda \langle a, b \rangle \mathbf{1}. \tag{2.2}$$

Note that the form $\langle \cdot, \cdot \rangle$ is a supersymmetric invariant form.

Therefore, by (2.2), the vertex subalgebra generated by $\mathfrak{a}$ is an affine vertex algebra that we denote by $V(\mathfrak{a})$.

Setup

We assume that $\mathfrak{a} = \bigoplus_{i=0}^{r} \mathfrak{a}_i$ with $\mathfrak{a}_0$ an even abelian Lie algebra (possibly zero) and the ideals $\mathfrak{a}_i$ are simple Lie algebras or basic Lie superalgebras.

Setup

We assume that $\mathfrak{a} = \bigoplus_{i=0}^{r} \mathfrak{a}_i$ with $\mathfrak{a}_0$ an even abelian Lie algebra (possibly zero) and the ideals $\mathfrak{a}_i$ are simple Lie algebras or basic Lie superalgebras.

- $\langle \cdot, \cdot \rangle_i$ be the nondegenerate invariant form on $\mathfrak{a}_i$, suitably normalized.
- Assume k_i are non-critical, so $V^{k_i}(\mathfrak{a}_i)$ admits a Virasoro vector $L^{\mathfrak{a}_i}$.
- Set $L^{\mathfrak{a}} \in W$ to be the image of $\sum_{i\geq 0} L^{\mathfrak{a}_i}$ in $V(\mathfrak{a})$. (This definition needs fine tuning is $\langle \cdot, \cdot \rangle$ is degenerate).
- $$L^{\mathfrak{a}}(2)L^{\mathfrak{a}} = \frac{1}{2}c_{\mathfrak{a}}\mathbf{1}.$$

A criterion for conformal embedding of affine vertex algebras into conformal vertex algebras

Let W be a conformal vertex algebra with conformal vector L. Assume:

$$\text{there is a conjugate linear involution } \phi \text{ of } W \text{ with } \phi(L) = L; \tag{3.1}$$

$$W(\tfrac{1}{2}) = \{0\}; \tag{3.2}$$

$$\mathfrak{a} = W(1) \text{ consists of } L\text{–primary elements.} \tag{3.3}$$

By a previous remark, there is a ϕ-invariant Hermitian form $(\cdot,\cdot)$ on W such that $(\mathbf{1},\mathbf{1}) = 1$.
We say that a subset $\mathfrak{V}$ of a vertex algebra W, homogeneous with respect to parity, strongly generates W if

$$W = span(:T^{j_1}(w_1)\cdots T^{j_r}(w_r):\mid r \in \mathbb{Z}_+,\ w_i \in \mathfrak{V}).$$

A criterion for conformal embedding of affine vertex algebras into conformal vertex algebras

Let $\overline{W} = W/I$ be a nonzero quotient of W and set

$$\widetilde{V}(\mathfrak{a}) = V(\mathfrak{a})/(I \cap V(\mathfrak{a})).$$

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Our first result is a general criterion for the existence of a conformal embedding

$$\widetilde{V}(\mathfrak{a}) \hookrightarrow \overline{W}.$$

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Theorem 1

Assume that W is strongly generated by $\mathfrak{a}$ and by $\{L - L^{\mathfrak{a}}\} \cup S$ with S homogeneous with respect to the gradation $W = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+} W(n)$ and such that $(L - L^{\mathfrak{a}})(2)X = 0$ for $X \in S \cap W(2)$. Then $(L - L^{\mathfrak{a}})$ generates a proper ideal I in W if and only if $c = c_{\mathfrak{a}}$.

Sketch of proof

Set $U = span(X \in S \mid X \in W(2))$. Observe that

$$W(2) = \mathbb{C}(L - L^{\mathfrak{a}}) + U + V(\mathfrak{a}) \cap W(2). \tag{3.4}$$

We have

$$(L - L^{\mathfrak{a}}, U) = (\mathbf{1}, (L - L^{\mathfrak{a}})(2)U) = 0 \tag{3.5}$$

by our assumption that $(L - L^{\mathfrak{a}})(2)U = 0$. If $c = c_{\mathfrak{a}}$ then

$$(L - L^{\mathfrak{a}}, L - L^{\mathfrak{a}}) = ((L - L^{\mathfrak{a}})(2)(L - L^{\mathfrak{a}}), \mathbf{1}) = \frac{1}{2}\overline{(c - c_{\mathfrak{a}})} = 0.$$

Since $L - L^{\mathfrak{a}} \in Com(V(\mathfrak{a}), W)$, we have

$$(L - L^{\mathfrak{a}}, V(\mathfrak{a}) \cap W(2)) = 0. \tag{3.6}$$

From (3.4)-(3.6) it follows that $(L - L^{\mathfrak{a}}, W(2)) = 0$. Since $(W(i), W(j)) = 0$ if $i \neq j$ we see that $L - L^{\mathfrak{a}}$ is in the kernel of the form $(\cdot, \cdot)$, which is a proper ideal.

Criterion

We retain the setting of the above Theorem. Furthermore we assume that

- S is a space stable under the action of $\mathfrak{a}$;
- the action of $\mathfrak{a}$ on W is completely reducible, thus we can choose S so that $(\mathbb{C}L + \mathfrak{a}) \oplus S$ is a set of strong generators for W.

Decomposing S as $\mathfrak{a}$-module, we can write $S = \oplus_{i \in \mathcal{J}} S_i$ with S_i irreducible and $\mathcal{J}$ some index set. Write

$$S_i = \bigotimes_j S_i^j$$

with S_i^j irreducible $\mathfrak{a}_j$–modules. Set

$$C_i = \sum_{j \geq 1} \frac{c_j^{(i)}}{2(k_j + h_j^\vee)} + (1 - \delta_{k_0,0}) \frac{c_0^{(i)}}{2k_0}, \tag{3.7}$$

where $c_j^{(i)}$ is the eigenvalue of the Casimir operator $C_{\mathfrak{a}_j}$ on S_i^j.

Continuation

Since the actions of $\mathfrak{a}$ and $L(0)$ commute, we can assume that

$$S_i, i \in \mathcal{J} \text{ are homogeneous.}$$

Set

$$\Delta_i = \text{ eigenvalue of } L(0) \text{ on } S_i.$$

Assume $\mathbf{c} = \mathbf{c}_{\mathfrak{a}}$, set $I = W(L - L^{\mathfrak{a}})$, $\pi_I : W \to W/I = \overline{W}$ quotient map.

$$(\mathbb{C}(L - L^{\mathfrak{a}}) + \mathfrak{a}) \oplus \sum_{i \in \mathcal{J}} S_i \text{ strong generators for } W$$

$$\rightsquigarrow$$

$$\mathfrak{a} \oplus \sum_{i \in \mathcal{K}} \pi_I(S_i), \mathcal{K} \text{ minimal, strong generators for } \overline{W}$$

Theorem 2

$C_j = \Delta_j$ for all $j \in \mathcal{K}$.

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Theorem 2

$C_j = \Delta_j$ for all $j \in \mathcal{K}$. So, $\boxed{C_i \neq \Delta_i \,\forall\, i \in \mathcal{J} \implies \overline{W} \cong \widetilde{V}(\mathfrak{a})}$.

Hook type W-algebras

We shall consider the special case $\mathfrak{g} = sl(m+n)$, and the nilpotent element $f = f_{m,n}$ determined by the partition $(m, 1^n)$. Then we can choose

$$f = \left(\begin{array}{c|c} J_m & 0 \\ \hline 0 & 0 \end{array}\right), \quad x = \left(\begin{array}{cccc|c} \frac{m-1}{2} & 0 & \cdots & 0 & \\ 0 & \frac{m-3}{2} & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & -\frac{m-1}{2} & \\ \hline & & 0 & & 0 \end{array}\right).$$

Also,

$$\mathfrak{g}^\natural = \left\{ \left(\begin{array}{c|c} -\frac{tr(A)}{m} I_m & 0 \\ \hline 0 & A \end{array}\right) \mid A \in gl(n) \right\} \cong gl(n),$$

and the 1-dimensional center is spanned by

$$\varpi = \left(\begin{array}{c|c} -\frac{n}{m} I_m & 0 \\ \hline 0 & I_n \end{array}\right).$$

Hook type W-algebras

If m is odd

$$\mathfrak{g}^f_j = \mathbb{C}\left(\begin{array}{c|c} J^j_m & 0 \\ \hline 0 & 0 \end{array}\right),\ 1 \leq j \leq m-1, j \neq \frac{m-1}{2},$$

and

$$\mathfrak{g}^f_{(m-1)/2} = \mathbb{C}\left(\begin{array}{c|c} J^{(m-1)/2}_m & 0 \\ \hline 0 & 0 \end{array}\right) \oplus \left\{\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & v \\ \hline {}^t w & 0 & 0 \end{array}\right) \mid v, w \in \mathbb{C}^n\right\}.$$

If m is even

$$\mathfrak{g}^f_j = \mathbb{C}\left(\begin{array}{c|c} J^j_m & 0 \\ \hline 0 & 0 \end{array}\right),\ 1 \leq j \leq m-1,$$

and

$$\mathfrak{g}^f_{(m-1)/2} = \left\{\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & v \\ \hline {}^t w & 0 & 0 \end{array}\right) \mid v, w \in \mathbb{C}^n\right\}.$$

Structure

Theorem 3

Assume that k is not critical, i.e. $k \neq -n-m$. One can choose strong generators for the vertex algebra $W^k(\mathfrak{g}, x, f_{m,n})$ as follows:

1. $J^{\{a\}}$, $a \in \mathfrak{g}^\natural \cong gl(n)$*; these generators are primary for L of conformal weight 1;*
2. *the Virasoro field L;*
3. *fields W_i, $i = 3, \ldots, m$, of conformal weight i;*
4. *fields $G_i^\pm$, $i = 1, \ldots, n$ of conformal weight $\frac{m+1}{2}$.*

The fields $G_i^\pm$, $i = 1, \ldots, n$ are primary for both L and $V(\mathfrak{g}^\natural)$. As $sl(n)$–modules,

$$span_{\mathbb{C}}\{G_i^+, i = 1, \ldots, n\} \cong \mathbb{C}^n, \quad span_{\mathbb{C}}\{G_i^-, i = 1, \ldots, n\} \cong (\mathbb{C}^n)^*.$$

Finally, $gl(n)$ acts trivially on W_i.

Idea of proof

KW proved the existence of a $\mathfrak{g}^\natural$–module isomorphism

$$\Psi : S(\widehat{\mathfrak{g}^f}) \to W^k(\mathfrak{g}, x, f).$$

We observe that one can choose Ψ so that $\Psi(\mathfrak{g}^f)$ strongly and freely generates $W^k(\mathfrak{g}, x, f)$ and $\Psi(f) = L$.
We set

$$W_i = \Psi(\left(\begin{array}{c|c} J_m^{i-1} & 0 \\ \hline 0 & 0 \end{array}\right)),\ 3 \le i \le m,$$

and

$$G_i^+ = \Psi(E_{m+i,1}),\ 1 \le i \le n,\ \ G_i^- = \Psi(E_{m,m+i}),\ 1 \le i \le n.$$

Conformal levels

Let $\widetilde{W}_k(\mathfrak{g}, x, f)$ be a quotient of $W^k(\mathfrak{g}, x, f)$. We have the following homomorphisms:

$$V(\mathfrak{g}^\natural) \to W^k(\mathfrak{g}, x, f) \to \widetilde{W}_k(\mathfrak{g}, x, f). \tag{4.1}$$

Let us denote the image of the resulting homomorphism by $\mathcal{V}(\mathfrak{g}^\natural)$.

Conformal levels

Theorem 4

The embedding $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_{m,n})$ *is conformal if and only if*

$$k = k^{(i)}_{m,n}, \quad 1 \le i \le 4,$$

where

(1) $k^{(1)}_{m,n} = -n - m + \frac{n+m}{m+1} = -\frac{m}{m+1} h^\vee$ *and* $n > 1$,

(2) $k^{(2)}_{m,n} = -n - m + \frac{1+m+n}{m} = -\frac{(m-1)h^\vee - 1}{m}$ *and* $n \ge 1$,

(3) $k^{(3)}_{m,n} = \frac{-1+2m-m^2+2n-mn}{m-1} = -\frac{(m-2)h^\vee + 1}{m-1} = -h^\vee + \frac{h^\vee - 1}{m-1}$ *and* $n \ge 1$, $m > 1$,

(4) $k^{(4)}_{m,n} = -\frac{(m-1)h^\vee}{m} = -h^\vee + \frac{h^\vee}{m}$.

Conformal levels

Remark 4.1

T. Creutzig conjectured that the embedding $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_{m,n})$ is conformal when $k = k^{(1)}_{m,n}$.

Conformal levels

Remark 4.1

Known results (cf. [AKMPP-CMP], [AKMPP-JA], [AKMPP-JJM]).

- $m = 1,\ k = k^{(1)}_{m,n} = -\frac{n+1}{2}$, since then $W_k(\mathfrak{g}, x, f_{m,n}) = V_k(sl(n+1))$
- $m = 1,\ k = k^{(2)}_{m,n} = 1$, since then $W_k(\mathfrak{g}, x, f_{m,n}) = V_1(sl(n+1))$.
- $m = 2,\ k = k^{(1)}_{m,n} = -\frac{2h^\vee}{3}$ since then $W_k(\mathfrak{g}, f_{m,n}) = W_k(sl(n+2), f_\theta)$ and k is a conformal level.
- $m = 2,\ k = k^{(2)}_{m,n} = -\frac{h^\vee - 1}{2}$ since then $W_k(\mathfrak{g}, f_{m,n}) = W_k(sl(n+2), f_\theta)$ and k is conformal level.
- $m = 2,\ k = k^{(3)}_{m,n} = -1$ since then $W_k(\mathfrak{g}, f_{m,n}) = W_k(sl(n+2), f_\theta)$ and k is a collapsing level.
- $m = 2,\ k = k^{(4)}_{m,n} = -\frac{h^\vee}{2}$ is collapsing level for $W_k(\mathfrak{g}, f) = W_k(sl(n+2), f_\theta)$.

Sketch of proof

We apply Theorem 1. Recall that $\mathfrak{g}^\natural = \mathbb{C}\varpi \oplus sl(n)$.

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Verification of the preliminary assumptions.

We choose the bilinear form on $\mathfrak{g}_0^\natural$ setting $\langle \varpi, \varpi \rangle_0 = (\frac{n}{m})^2 + 1$. We choose the conjugate linear involution ϕ to be the conjugation with respect to $sl(m+n, \mathbb{R})$, so that $\phi(x) = x, \phi(L) = L$.

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$$S = span\left(\{W_i \mid 3 \leq i \leq m-1\} \cup \{G_i^{\pm} \mid 1 \leq i \leq n\}\right). \qquad (4.1)$$

If $(m+1)/2 \neq 2$, i.e. $m \neq 3$, then $S \cap W^k(\mathfrak{g}, x, f_{m,n})(2) = \emptyset$, hence the hypotheses of Theorem 1 are vacuously verified. If $m = 3$, we have to check that $(L - L^{\mathfrak{a}})(2)G_i^{\pm} = 0$: this readily follows from Theorem 3.

Sketch of proof

By Theorem 1, it is enough to equate the central charge of $W_k(\mathfrak{g}, x, f_{m,n})$ (known by [KW]) and the central charge of $\mathcal{V}(\mathfrak{g}^\natural)$. We have that $\mathcal{V}(\mathfrak{g}^\natural)$ is a quotient of $V^{k_0}(\mathbb{C}\varpi) \otimes V^{k_1}(sl(n))$, where

$$k_0 = k + \frac{(m-1)(m+n)}{m}, \quad k_1 = k + m - 1.$$

Moreover $h_1^\vee = n, h_0^\vee = 0$. We obtain the equations

$$c_{m,n}(k) = \frac{k_1(n^2-1)}{k_1+n} + (1 - \delta_{k_0,0}). \tag{4.1}$$

Solving for k, we get $k = k_{m,n}^{(i)}, 1 \le i \le 3$ if $k_0 \neq 0$. If instead $k_0 = 0$, then $k = k_{m,n}^{(4)}$ and one readily verifies that this value of k satisfies (4.1).

Superization

We shall consider the case $\mathfrak{g} = sl(m|n)$, $m \neq n$, $m \geq 2$ and $n \geq 1$ and the case $\mathfrak{g} = psl(m|m)$, $m \geq 2$. We choose $f = f_{m|n} = \left(\begin{array}{c|c} J_m & 0 \\ \hline 0 & 0 \end{array}\right)$

Theorem 5

Assume $m \neq n$. The embedding $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_{m|n})$ is conformal if and only if

$$k = k^{(i)}_{m,n}, \quad 1 \leq i \leq 4,$$

where

(1) $k^{(1)}_{m,n} = -\frac{m}{m+1}(m-n) = -h^\vee + \frac{h^\vee}{m+1}$ *and* $n > 1$,

(2) $k^{(2)}_{m,n} = -\frac{(m-1)(m-n)-1}{m} = -h^\vee + \frac{h^\vee - 1}{m}$,

(3) $k^{(3)}_{m,n} = -\frac{(m-2)(m-n)+1}{m-1} = -h^\vee + \frac{h^\vee - 1}{m-1}$, $m > 1$,

(4) $k^{(4)}_{m,n} = -\frac{(m-1)(m-n)}{m} = -h^\vee + \frac{h^\vee}{m}$.

Collapsing levels

Recall from [AKMPP] the following definition.

Definition 4.1

We say a level k is *collapsing* if $\mathcal{V}(\mathfrak{g}^\natural) = W_k(\mathfrak{g}, x, f)$.

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Remark 4.2

1. The collapsing levels for minimal W-algebras have been classified in [AKMPP-JA]. There we proved that they correspond to the roots of an explicit monic quadratic polynomial $p(k)$.
2. The notion has remarkable applications to the representation theory of affine algebras. For instance, at a collapsing level k, we have proved semisimplicity of the category of ordinary modules KL_k for a simple affine vertex algebra $V_k(\mathfrak{g})$ when $\mathfrak{g}$ is a Lie algebra [AKMPP-IMRN], and, very recently, of KL_k^{fin} be the subcategory of KL_k consisting of ordinary modules on which a Cartan subalgebra acts semisimply when $\mathfrak{g}$ is a basic Lie superalgebra [AMP-AdvMath2022].

Collapsing levels

Recall from [AKMPP] the following definition.

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One more Remark

The notion of collapsing level has been extended beyond the minimal case by Arakawa, Moreau, Van Ekeren. They use geometric methods and Kac's notion of asymptotic growth towards a full classification in the admissible case.

Strongly collapsing levels

Assume that k is a conformal level. Then $\bar{L} = L - L^{\mathfrak{g}^\natural}$ belongs to the maximal ideal of $W^k(\mathfrak{g}, x, f)$. We define

$$\overline{W}_k(\mathfrak{g}, x, f) = W^k(\mathfrak{g}, x, f)/W^k(\mathfrak{g}, x, f) \cdot (L - L^{\mathfrak{g}^\natural}).$$

Definition 4.2

A level k is called *strongly collapsing* if $\mathcal{V}(\mathfrak{g}^\natural) = \overline{W}_k(\mathfrak{g}, x, f)$ (see (4.1)).

Remark 4.2

Clearly, any strongly collapsing level is also a collapsing. We have produced examples collapsing levels which are not strongly collapsing. In particular, this holds for minimal affine W–algebras in the following cases:

- $\mathfrak{g} = sl(3)$, $k = -1$;
- $\mathfrak{g} = sp(4)$, $k = -2$.

Strongly collapsing levels

Theorem 6

For the hook W-algebra $W(sl(n+m), x, f_{m,n})$ the following are strongly collapsing levels:

1. $k^{(3)}_{m,n}, \; n \neq m-1,$
2. $k^{(4)}_{m,n}.$

Strongly collapsing levels

Theorem 6

For the hook W-algebra $W(sl(n+m), x, f_{m,n})$ the following are strongly collapsing levels:

1. $k^{(3)}_{m,n}$, $n \neq m-1$,
2. $k^{(4)}_{m,n}$.

We apply Theorem 2. By Theorem 3, we can choose S as above:

$$S = span\left(\{W_i \mid 3 \leq i \leq m-1\} \cup \{G_i^{\pm} \mid 1 \leq i \leq n\}\right).$$

Let $\mathbb{C}_t$ be the 1-dimensional representation of $\mathbb{C}\varpi$ with ϖ acting by $\varpi \cdot 1 = t$. Then one sees that, as a $(\mathbb{C}\varpi \oplus sl(n))$–module,

$$S = S_0 \oplus (\mathbb{C}_{\frac{n+m}{m}} \otimes \mathbb{C}^n) \oplus (\mathbb{C}_{-\frac{n+m}{m}} \otimes (\mathbb{C}^n)^*), \tag{4.2}$$

where S_0 is the isotypic component of the trivial representation of $\mathfrak{g}^\natural$.

Strongly collapsing levels

Theorem 6

For the hook W-algebra $W(sl(n+m), x, f_{m,n})$ the following are strongly collapsing levels:

1. $k^{(3)}_{m,n}$, $n \neq m-1$,
2. $k^{(4)}_{m,n}$.

We now compute $C_1, \dots, C_m$, where $C_1, \dots, C_{m-2}$ correspond to the $m-2$ copies of the trivial representation occurring in S_0 and C_{m-1}, C_m to the other two factors. We have

$$C_1 = \cdots = C_{m-2} = 0, \quad C_{m-1} = C_m = (1 - \delta_{k_0,0})\frac{m+n}{2mnk_0} + \frac{n^2-1}{2n(k_1+n)}.$$

Strongly collapsing levels

Theorem 6

For the hook W-algebra $W(sl(n+m), x, f_{m,n})$ the following are strongly collapsing levels:

1. $k^{(3)}_{m,n}$, $n \neq m-1$,
2. $k^{(4)}_{m,n}$.

A direct verification shows that if $k = k^{(3)}_{m,n}$ then

$$C_{m-1} = C_m = \frac{(m-1)\left(m+n^2+n-1\right)}{2n^2},$$

hence $C_{m-1} = C_m = \frac{m+1}{2}$ if and only if $n = m-1$.

If $k = k^{(4)}_{m,n}$ then $C_{m-1} = C_m = \frac{m\left(n^2-1\right)}{2n^2}$ hence, $C_{m-1} = C_m < \frac{m+1}{2}$.

Non-collapsing conformal levels

Theorem 7

For the hook W-algebra $W(sl(n+m), x, f_{m,n})$ the conformal levels $k^{(1)}_{m,n}$ for $n > 1$ and $k^{(2)}_{m,n}$ are not strongly collapsing.

Non-collapsing conformal levels

Proof.

One readily computes that, if $k = k^{(i)}_{m,n}$, $i = 1, 2$, then $C_{m-1} = C_m = \frac{m+1}{2}$. We need only to check whether

$$G_i^{\pm} \notin I := W^k(\mathfrak{g}, x, f).(L - L^{\mathfrak{g}^\natural}).$$

Non-collapsing conformal levels

Proof.

Set $\bar{L} = W_2 = L - L^{\mathfrak{g}^\natural}$. Let $\tau \in \mathrm{span}_{\mathbb{C}}\{G_1^{\pm}, \ldots, G_n^{\pm}\}$. Then for each $i \in \mathbb{Z}_{\geq 1}$:

$$\tau_i \bar{L} = 0.$$

Non-collapsing conformal levels

Proof.

For $2 \leq p \leq m$, we set

$$W_p(s) = (W_p)_{s+p-1}.$$

Then for $s \geq 1$ we have

$$W_p(s)\bar{L} = 0.$$

Non-collapsing conformal levels

Proof.

Moreover

$$W_p(0)\bar{L} = a\bar{L} + u$$

for certain $a \in \mathbb{C}$ and $u \in V(\mathfrak{g}^\natural) \cap I$. We prove that

$$\bar{L} \text{ is a singular vector in } W^k(\mathfrak{g}, x, f).$$

In particular, I is a quotient of a Verma module, hence it is linearly spanned by monomials

$$U_s = y_1(-q_1)\cdots y_t(-q_t)x_1(-n_1-1)\cdots x_r(-n_r-1)(g^1)_{-m_1}\cdots(g^s)_{-m_s}\bar{L},$$

where $y_i \in \{W_p, p = 2, \ldots, m\}$, $x^i \in \mathfrak{g}^\natural$, $g^j \in \{G_1^{\pm}, \ldots, G_n^{\pm}\}$, $q_i, n_i \in {}_{\geq 0}$, and $m_i \in \mathbb{Z}$, $m_i \geq -\frac{m-1}{2}$, $m_1 \geq m_2 \geq \ldots \geq m_s$. □

Non-collapsing conformal levels

Proof.

Since $\tau_i \bar{L} = 0$, we have that $m_s \geq 0$, hence all m_i are non-negative integers. Denote the conformal weight of U_s by $\mathrm{wt}(U_s)$. Then we have

$$\mathrm{wt}(U_s) \geq 2 + s\frac{m+1}{2} + (m_1 + \cdots + m_s) - s \geq 2 + s\frac{m-1}{2}.$$

This implies that if $s \geq 1$, then $\mathrm{wt}(U_s) \geq \frac{m+3}{2}$. If $G_i^{\pm} \in I$, then it can be written as a a linear combination of elements U_{s_i} with $s_i \geq 1$ of conformal weight $\frac{m+1}{2}$. This is not possible. The claim follows. □

Results on decompositions

Using techniques derived from our previous papers on conformal embeddings, we prove

Results on decompositions

Theorem 7

Let $k = k^{(i)}_{m,n}$ for $i = 1, 2$ and assume that k is non-collapsing. Assume also that $\frac{m+1}{n-1} \notin \mathbb{Z}$ if $i = 1$ and $\frac{m}{n+1} \notin \mathbb{Z}$ if $i = 2$. Then

$$W_k(\mathfrak{g}, x, f_{m,n}) = \bigoplus_{i \in \mathbb{Z}} W_k^{(i)}$$

and each $W_k^{(i)} = \{v \in W_k \mid J(0)v = iv\}$ is an irreducible $V(\mathfrak{g}^\natural)$–module. The summands in the r.h.s. of the previous equation have the form:

- $W_k^{(i)} = L_{k_1}^{sl(n)}(i\omega_1) \otimes M(k_0, i)$ *if* $i \geq 0$,
- $W_k^{(i)} = L_{k_1}^{sl(n)}(-i\omega_{n-1}) \otimes M(k_0, i)$ *if* $i < 0$.

In particular, $\mathcal{V}(\mathfrak{g}^\natural) \cong W_k(\mathfrak{g}, x, f_{m,n})^{(0)} = V(sl(n)) \otimes V^{k_0}(\mathbb{C}J)$ is a simple vertex algebra which is conformally embedded in $W_k(\mathfrak{g}, x, f_{m,n})$.

The admissible case

Recall the following definition.

Definition 8

A level k for $\mathfrak{g} = sl(n+m)$ is said to be admissible if $k + h^\vee = \frac{p'}{p}$, $p, p' \in \mathbb{Z}_{\geq 1}$, $(p, p') = 1$ and $p' \geq h^\vee = n + m$.

The admissible case

Recall the following definition.

Definition 8

A level k for $\mathfrak{g} = sl(n+m)$ is said to be admissible if $k + h^\vee = \frac{p'}{p}$, $p, p' \in \mathbb{Z}_{\geq 1}$, $(p, p') = 1$ and $p' \geq h^\vee = n + m$.

Recall that if k is an admissible level, then

$$V_k(\mathfrak{g}) = V^k(\mathfrak{g})/J^k(\mathfrak{g}),$$

where the maximal ideal $J^k(\mathfrak{g}) = V^k(\mathfrak{g}).\Omega_k$ is generated by the singular vector Ω_k, which is the unique (up to a scalar factor) singular vector in $V^k(\mathfrak{g})$ of $\mathfrak{g}$–highest weight $\mu^{(k)} = (p' + 1 - h^\vee)\theta$, and conformal weight $d_{(k)} = p(p' + 1 - h^\vee)$.

The admissible case

Denote by H_f the quantum Hamiltonian reduction functor. Kac and Wakimoto states the following conjecture:

$$H_f(V_k(\mathfrak{g})) \text{ is either zero or isomorphic to } W_k(\mathfrak{g}, x, f). \qquad (6.1)$$

If $k = -h^\vee + p'/p$ is admissible, then the associated variety of the simple affine vertex algebra $V_k(\mathfrak{g})$ is the closure of a nilpotent orbit $\mathbb{O}_k$, depending just on p. Moreover, if we set $N_p = \{x \in \mathfrak{g} \mid \mathrm{ad}(x)^{2p} = 0\}$, then

$$\overline{\mathbb{O}}_k = N_p.$$

Theorem 9 (Arakawa-Van Ekeren)

Assume that k is an admissible level, $f \in \overline{\mathbb{O}}_k$ and f admits an even good grading. Then (6.1) holds.

The admissible case

We note that $k^{(1)}_{m,n}$ is admissible if and only if $(n-1, m+1) = 1$, and $k^{(2)}_{m,n}$ is admissible if and only if $(n+1, m) = 1$. In both cases, if k is admissible we have $d^W_k = 2$.

Theorem 10

Assume that $k = k^{(1)}_{m,n}$ or $k = k^{(2)}_{m,n}$ and that k is admissible. We have:

1. $W_k(\mathfrak{g}, x, f_{m,n}) = \overline{W}_k(\mathfrak{g}, x, f_{m,n})$.
2. *Level k is not collapsing.*
3. $W_k(\mathfrak{g}, x, f_{m,n})$ *admits the decomposition given in Theorem 7 provided that* $k \neq k^{(1)}_{p-1,2}$.
4. *If* $k = k^{(1)}_{p-1,2}$ *the decomposition can be explicitly described.*

The rectangular case and the Arakawa-Van Ekeren-Moreau question

Let $f = f_{[m,q]} = diag(\underbrace{J_q, \dots, J_q}_{m \text{ times}})$ be the nilpotent element of $sl(qm)$ with Jordan block decomposition corresponding to the partition (q^m). Let $x = diag(\underbrace{\frac{q-1}{2}, \frac{q-3}{2}, \dots, \frac{1-1}{2}}_{m \text{ times}})$ be the corresponding Dynkin element. One has

$$\mathfrak{g}^f = \left(\begin{array}{c|c|c} A_{11} & \dots & A_{1m} \\ \hline \vdots & \dots & \vdots \\ \hline A_{m1} & \dots & A_{mm} \end{array}\right)$$

where

$$A_{ij} = \alpha_0^{ij} Id + \alpha_1^{ij} J_q + \dots + \alpha_{q-1}^{ij} J_q^{q-1}.$$

The rectangular case and the Arakawa-Van Ekeren-Moreau question

Arguing as in the hook case, we obtain

Theorem 11

One can choose strong generators for the vertex algebra $W^k(\mathfrak{g}, x, f_{[m,q]})$ as follows:

(1) $J^{\{a\}}$, $a \in \mathfrak{g}^\natural \cong sl(m)$; *these generators are primary for L of conformal weight 1;*

(2) *the Virasoro field L;*

(3) *fields W_i, $3 \leq i \leq q$, of conformal weight i;*

(4) *fields $G_i^{j,s}$, $2 \leq i \leq q$, $1 \leq s, j \leq m^2 - 1$, of conformal weight i.*

The action of $\mathfrak{g}^\natural$ on W_i (resp. $G_i^{j,s}$) is trivial (resp. adjoint).

Conformal levels

According to [KW], the central charge for L in $W^k(sl(mq), f_{[m,q]}, x)$ is

$$C(k) = \frac{k\left(m\left(q-q^3\right)(k+mq)+m^2q^2-1\right)}{k+mq} - m^2q\left(q^3-2q^2+1\right).$$

The hypothesis of Theorem 1 are satisfied since we can choose $S \cap W^k(\mathfrak{g}, f_{[m,q]}, x)(2)$ to be $\Psi(sl(m))$. Thus the embedding $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_{[m,q]})$ is conformal if and only if k is a solution of the equation $C(k) = c_{\mathfrak{g}^\natural}$. One readily computes that

$$c_{\mathfrak{g}^\natural} = \frac{\left(m^2-1\right)q(k+mq-m)}{q(k+mq-m)+m},$$

hence the conformal levels are

$$k^{[1]}_{m,q} = -\frac{mq^2}{q+1}, \quad k^{[2]}_{m,q} = \frac{-mq^2+mq-1}{q}, \quad k^{[3]}_{m,q} = \frac{-mq^2+mq+1}{q}. \tag{7.1}$$

Collapsing levels

Theorem 12

Levels $k_{m,q}^{[i]}$, $i = 1, 2, 3$ *are collapsing for all* $q \geq 2$ *and* $m \geq 2$, *except possibly level* $k_{2,q}^{(2)}$. *More precisely:*

$$W_{k_{m,q}^{[1]}}(sl(mq), x, f_{[m,q]}) = V_{-\frac{mq}{q+1}}(sl(m)), \tag{7.2}$$

$$W_{k_{m,q}^{[2]}}(sl(mq), x, f_{[m,q]}) = V_{-1}(sl(m)), \ m \geq 3, \tag{7.3}$$

$$W_{k_{m,q}^{[3]}}(sl(mq), x, f_{[m,q]}) = V_{1}(sl(m)). \tag{7.4}$$

Collapsing levels

Theorem 12

Levels $k^{[i]}_{m,q}$, $i = 1, 2, 3$ are collapsing for all $q \geq 2$ and $m \geq 2$, except possibly level $k^{(2)}_{2,q}$. More precisely:

$$W_{k^{[1]}_{m,q}}(sl(mq), x, f_{[m,q]}) = V_{-\frac{mq}{q+1}}(sl(m)), \quad (7.2)$$

$$W_{k^{[2]}_{m,q}}(sl(mq), x, f_{[m,q]}) = V_{-1}(sl(m)), \ m \geq 3, \quad (7.3)$$

$$W_{k^{[3]}_{m,q}}(sl(mq), x, f_{[m,q]}) = V_1(sl(m)). \quad (7.4)$$

- $k^{[1]}_{m,q}$ is admissible if and only if $(m, q+1) = 1$.
- $k^{[2]}_{m,q}$ is never admissible. The result (7.3) gives a positive answer to a question by T. Arakawa, J. van Ekeren and A. Moreau in the case $m \geq 3$.
- $k^{[3]}_{m,q}$ is always admissible. The result (7.4) is proved by different methods in by Arakawa-Van Ekeren-Moreau.

Collapsing levels

Theorem 12 cannot be applied for $m = 2$. In the case $m = q = 2$, $f_{[m,q]}$ coincides with the short nilpotent element f_{sh}. Adamović, Milas and Penn prove, using explicit OPE formulas, that $W_{-\frac{5}{2}}(sl(4), f_{sh})$ is isomorphic to an orbifold of the rank two Weyl vertex algebra. So $k_{2,2}^{[2]} = -5/2$ is not collapsing.

Conjecture

Level $k = k_{2,q}^{[2]} = \frac{-2q^2+2q-1}{q}$ is not collapsing for $q \geq 2$ and

$$W_k(sl(2q), f_{[m,q]}, x) = W_{-\frac{5}{2}}(sl(4), f_{sh}).$$