Associated varieties and finite extensions of vertex algebras

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joint work in progress with Jethro Van Ekeren (IMPA)

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- **The associated variety X_V captures important properties of V.**

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- **The structure of the extension $\varphi(R_V) \hookrightarrow R_W$ might be related to the structure of the fusion ring of V.**

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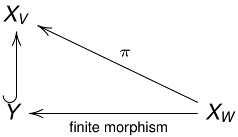

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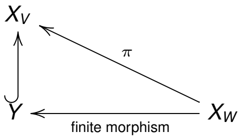

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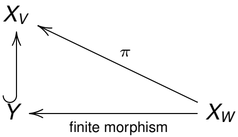

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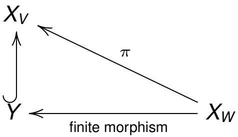

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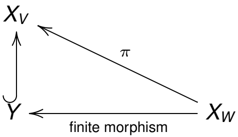

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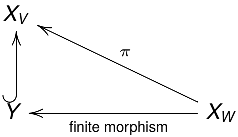

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$\rightsquigarrow$ *From now on we will focus on finite extensions of admissible simple affine vertex algebras which are simple W-algebras.*

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$$k = -h_{\mathfrak{g}}^{\vee} + p/q, \text{ with } (p,q) = 1 \text{ and } \begin{cases} p \geqslant h_{\mathfrak{g}}^{\vee} & \text{if } (q, r^{\vee}) = 1, \\ p \geqslant h_{\mathfrak{g}} & \text{if } (q, r^{\vee}) \neq 1, \end{cases}$$

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Simple affine vertex algebras at admissible levels

Consider the universal affine vertex algebra $V^k(\mathfrak{g})$ associated with a simple Lie algebra $\mathfrak{g}$ at level $k \in \mathbb{C}$.

We have

$$X_{V^k(\mathfrak{g})} = \mathfrak{g}^* \cong \mathfrak{g}.$$

The associated variety $X_{L_k(\mathfrak{g})} \subset \mathfrak{g}$ of the simple quotient $L_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$ is G-invariant and conic.

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For example, if $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathbb{C}$, then k is collapsing.

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► Subsequently, there were several works on collapsing non-admissible levels

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1. Pick $f \in \mathbb{O}_k$ so that $\mathcal{W}_k(\mathfrak{sl}_n, f)$ is lisse (and even rational).
   - if $n \equiv \pm 1 \mod q$, then $\mathcal{W}_k(\mathfrak{sl}_n, f) \cong \mathbb{C}$.
   - if $n \equiv 0 \mod q$, then $\mathcal{W}_{-n+(n+1)/q}(\mathfrak{sl}_n, f) \cong L_1(\mathfrak{sl}_{m_0})$.
2. Pick $f \in \mathbb{O}_{(q^m, 1^s)} \in \overline{\mathbb{O}_k}$ with $s \neq 0$. Then
$$\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f) \cong L_{-s+s/q}(\mathfrak{sl}_s).$$
3. Assume that $n = qm_0 + (q-2)$ and pick $f \in \mathbb{O}_{(q^{m_0-1}, (q-1)^2)} \in \overline{\mathbb{O}_k}$. Then
$$\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f) \cong L_{-2+2/q}(\mathfrak{sl}_2).$$

► Subsequently, there were several works on collapsing non-admissible levels [ADAMOVIČ-MÖSENEDER-PAPI '22, ARAKAWA-CREUTZIG-LINSHAW-M. '22, FASQUEL '22, ETC.]

What is known about collapsing levels?

► [AKMPP '18] There is a full classification of collapsing levels for $\mathcal{W}_k(\mathfrak{g}, f_{min})$, including simple affine Lie superalgebras.

► [AvEM '21] We studied admissible collapsing levels: we provided a conjectural exhaustive list of such levels.

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For example [AMP '22], $\mathcal{W}_{-n+(n-1)/q}(\mathfrak{sl}_{qm_0}, f_{(q^{m_0})}) \cong L_{-1}(\mathfrak{sl}_{m_0})$, $m_0 \geqslant 3$.

Finite extensions of admissible simple vertex algebras

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- **We also have examples in types E_7, E_8 and in the classical types.**

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In particular, its image is contained in the nilpotent cone of $\mathfrak{g}^\natural$.

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- More recently, [FU-JUTEAU-LEVY-SOMMERS '17] determined the generic singularities in the exceptional types.

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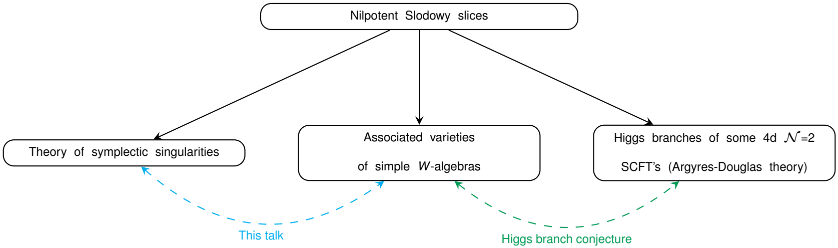

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⚠ Note that $\mathscr{S}_{\tilde{A}_1, A_1} = \overline{G^\natural . x}$, with $x := f + x_0 + 2e_5$, is not normal while $\mathcal{N}(\mathfrak{sl}_2)$ is [FJLS '17].

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Work in progress: $\mathscr{S}_{\tilde{A}_2+A_1,A_2}$ is isomorphic to the affinization of the 3 : 1 cover $(G^\natural)^0/(G^\natural)^0_x$ of the regular orbit in $\mathfrak{sl}_3$.

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$$H^0_{DS,A_2}(L_{-9+10/3}(F_4)) \cong L_{-3+5/3}(A_2) \oplus L_{-3+5/3}(A_2; \varpi_1) \oplus L_{-3+5/3}(A_2; \varpi_2).$$

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$$6 = \dim \mathcal{N}(\mathfrak{sl}_3) = \dim \mathscr{S}_{\tilde{A}_2+A_1, A_2} = 36 - 30.$$

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$$x_0 := 2e_8 + 2f_{12} \in \mathbb{O}_{reg} \subset \mathfrak{sl}_3$$

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$$\pi^{-1}(x_0) = \{f + x_0 + 6\omega e_2 + 6\omega^2 e_{10} \colon \omega^3 = 1\} \subset \tilde{A}_2 + A_1,$$

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$$\boldsymbol{g}_{L_k(\mathfrak{g})} = \begin{cases} \dim \mathfrak{g} \left(1 - \dfrac{h^\vee_{\mathfrak{g}}}{pq}\right) & \text{if } (q, r^\vee) = 1 \\ \dim \mathfrak{g} \left(1 - \dfrac{h^\vee_{{}^L\mathfrak{g}} r^\vee}{pq}\right) & \text{if } (q, r^\vee) = r^\vee. \end{cases}$$

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By [AvEM '21], we assert that if f is conformal then f factors through

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2. We also need a better understanding of the connection between the cardinality of the generic fiber of π and the decomposition of $H^0_{DS,f}(L_k(\mathfrak{g}))$ as $L_{k^\natural}(\mathfrak{g}^\natural)$-modules.