On the Classification of Holomorphic Vertex Operator Superalgebras

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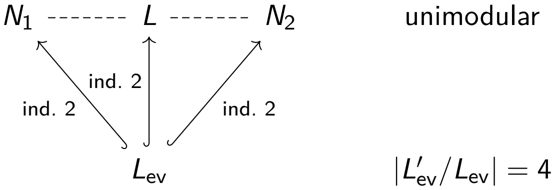
Section 1

Introduction

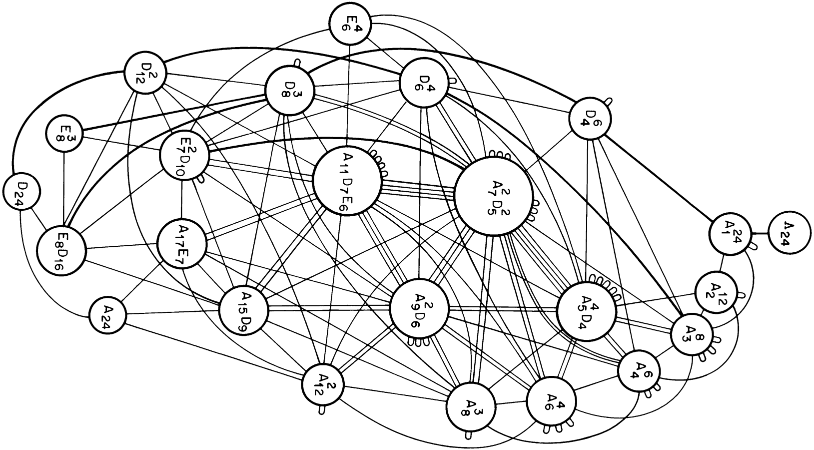
In 1985 Borcherds classified the positive-definite, odd, unimodular lattices of rank 24 [Bor85] using:

Let $8 \mid d$. There is a natural bijection between the positive-definite, odd, unimodular lattices L of rank d and the pairs of Kneser 2-neighbours of positive-definite, even, unimodular lattices N.



All positive-definite, odd, unimodular lattices of rank $\leq d$ may be obtained by splitting off copies of the standard lattice $\mathbb{Z}$.

The 2-neighbourhood graph in rank 24 (some loops omitted):



Goal: Classify the nice (i.e. strongly rational), holomorphic vertex operator superalgebras of central charge at most 24 by generalising Borcherds' neighbourhood method.

Indeed, Kneser 2-neighbours (of positive-definite, even, unimodular lattices) are generalised by the $\mathbb{Z}_2$-orbifold construction (for nice, holomorphic vertex operator algebras).

This approach was probably known to several people in the 90s. However, it has just recently become feasible thanks to

1. advancements in orbifold theory,
2. the classification of holomorphic vertex operator algebras of central charge 24,
3. the determination of their automorphism groups [BLS22].

(Classification up to central 12 obtained in [CDR18].)

Section 2

Vertex Operator Superalgebras

Vertex Operator Superalgebras

Assume "correct statistics": $V=\bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n = V^{\bar{0}}\oplus V^{\bar{1}}$ with

$$V^{\bar{0}}=\bigoplus_{n\in\mathbb{Z}} V_n \quad\text{and}\quad V^{\bar{1}}=\bigoplus_{n\in\frac{1}{2}+\mathbb{Z}} V_n\neq\{0\}.$$

Examples:

- The Clifford (or free fermion) vertex operator superalgebra $F=L_{1/2}(0)\oplus L_{1/2}(1/2)$ of central charge $1/2$.
(Note that $F^{\otimes 2}\cong\langle bc\rangle\cong V_{\mathbb{Z}}$.)
- The lattice vertex operator superalgebra V_L of central charge $\mathrm{rk}(L)$ for a positive-definite, odd, unimodular lattice L.
- In particular: $A^{f\natural}\cong V_{D_{12}^+}$ (super-moonshine for Co_0, [Dun07]).
- The shorter moonshine module $VB^\natural$ of central charge 23.5 [Höh95].

By results of Dong et al., Carnahan, Miyamoto and Huang we obtain:

Let V be a nice vertex operator superalgebra. Then

- the even part $V^{\bar{0}}$ is a nice vertex operator algebra,
- the central charge $c \in \mathbb{Q}_{>0}$,
- $\operatorname{Rep}(V^{\bar{0}})$ is a modular tensor category,
- the odd part $V^{\bar{1}}$ is an irreducible $V^{\bar{0}}$-module and a simple current of order 2,
- $\operatorname{Rep}(V^{\bar{0}})$ is a *minimal modular extension* of $\operatorname{Rep}^{\bar{0}}(V^{\bar{0}})$.

Now, let V be nice and holomorphic (or self-dual) vertex operator superalgebra. Using the "16-fold way" [DNR21] we show:

Theorem

Let V be a nice, holomorphic vertex operator superalgebra. Then the central charge $c \in (1/2)\mathbb{Z}_{>0}$, and $\operatorname{Rep}(V^{\bar{0}})$ is one of 16 minimal modular extensions of $\mathcal{C}((\mathbb{Z}_2, 0))$, depending only on $c \pmod 8$.
In fact, the modular tensor category $\operatorname{Rep}(V^{\bar{0}})$ is isomorphic to $\operatorname{Rep}((F^{\otimes d})^{\bar{0}})$ with $d = 2c \pmod{16}$.

In particular, $\operatorname{glob}(V_{\bar{0}}) = 4$, but $|\operatorname{Rep}(V_{\bar{0}})| = 3$ or 4.

If $c \in \mathbb{Z}$: four irreducible modules $[0], [1/2], c/8, c/8$ with fusion algebra $\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ or $\mathbb{C}[\mathbb{Z}_4]$ (all simple currents).

If $c \in (1/2)\mathbb{Z} \setminus \mathbb{Z}$: three irreducible modules $[0], [1/2], (c/8)$ with fusion rules of the Ising model $L_{1/2}(0)$.

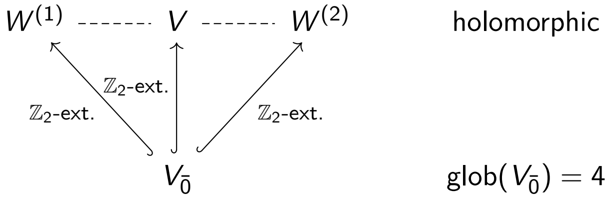
Section 3

Neighbourhood Graph

If $8 \mid c$, then $\mathrm{Rep}(V^{\bar{0}}) \cong \mathcal{C}(2_{II}^{+2})$, with fusion algebra $\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ and conformal weights $0, 1/2, 0, 0 \pmod 1$. Hence:

Theorem

Let $8 \mid c$. Then there is a natural bijection between the nice, holomorphic vertex operator superalgebras V of central charge c and the pairs of 2-neighbours ($= \mathbb{Z}_2$-orbifold construction) of nice, holomorphic vertex operator algebras W.



We shall use this to classify all nice, holomorphic vertex operator superalgebras of central charge 24.

Well-known splitting result:

For a vertex operator superalgebra V of central charge c, $\langle V_{1/2}\rangle \cong F^{\otimes\ell}$ where $\ell = \dim(V_{1/2})$, and V decomposes as $V \cong \bar{V} \otimes F^{\ell}$, where the *stump* $\bar{V}$ of V is a vertex operator (super)algebra of central charge $c - \ell/2$ with $\dim(\bar{V}_{1/2}) = 0$.

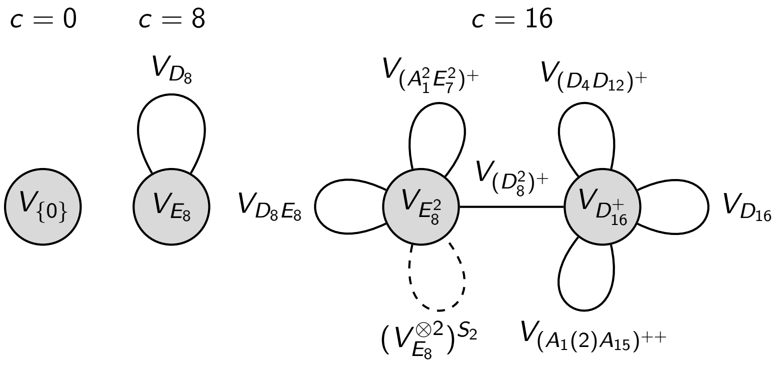
V is holomorphic or nice if and only if the stump $\bar{V}$ is.

Theorem

For every $c \in (1/2)\mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}_{\geq 0}$ the map $V \mapsto V \otimes F^{l}$ defines a bijection between the nice, holomorphic vertex operator (super)algebras V of central charge c with $\dim(V_{1/2}) = k$ and those of central charge $c + l/2$ with $\dim(V_{1/2}) = k + l$.

This allows us to classify all nice, holomorphic vertex operator superalgebras of central charge between 0 and 24.

Neighbourhood graph up to central charge 16:



For $c = 8$ there are only lattice nodes and edges.

For $c = 16$ there are only lattice nodes but non-lattice edges.

For $c = 24$ there will be non-lattice nodes and edges.

Classification of nice, holomorphic vertex operator superalgebras of central charge at most 16:

V	V_1	$V^{\bar{0}}$	$W^{(1)}$	$W^{(2)}$	type	c_{stump}
$F^{\otimes 32}$	D_{16}	$V_{D_{16}}$	$V_{D_{16}^+}$	$V_{D_{16}^+}$	I	0
$V_{D_{12}^+} \otimes F^{\otimes 8}$	$D_4 D_{12}$	$V_{(D_4 D_{12})^+}$	$V_{D_{16}^+}$	$V_{D_{16}^+}$	I	12
$V_{A_{15}^+} \otimes F^{\otimes 2}$	$A_{15}\mathbb{C}$	$V_{(A_1(2)A_{15})^{++}}$	$V_{D_{16}^+}$	$V_{D_{16}^+}$	I	15
$V_{(D_8^2)^{++}}$	D_8^2	$V_{(D_8^2)^+}$	$V_{D_{16}^+}$	$V_{E_8^2}$	I	16
$V_{E_8} \otimes F^{\otimes 16}$	$D_8 E_8$	$V_{D_8 E_8}$	$V_{E_8^2}$	$V_{E_8^2}$	I	8
$V_{(E_7^2)^+} \otimes F^{\otimes 4}$	$A_1^2 E_7^2$	$V_{(A_1^2 E_7^2)^+}$	$V_{E_8^2}$	$V_{E_8^2}$	I	14
$\bar{V} \otimes F$	$E_{8,2}$	$(V_{E_8}^{\otimes 2})^{S_2}$	$V_{E_8^2}$	$V_{E_8^2}$	III	15.5

We shall extend these results to central charge 24.

Section 4

Cartan Subalgebras of Vertex Operator Algebras

In a nice vertex operator algebra V, V_1 is a reductive Lie algebra of rank $r \leq c$ [DM04]. Let $\mathcal{H}$ be a choice of Cartan subalgebra. By [Mas14, CKLR19]:

Let V be a nice vertex operator algebra. Then $U := \mathrm{Com}_V(\langle\mathcal{H}\rangle)$ is a nice vertex operator algebra of central charge $c - r$ and $\mathrm{Com}_V(U) = V_L$ for some positive-definite, even lattice L of rank r.

We call L the *associated lattice* of V. It is unique up isomorphism.

Then $U \otimes V_L \subseteq V$ forms a dual pair and by the theory of mirror extensions [Lin17, CKM22] V is a simple-current extension

$$V \cong \bigoplus_{\alpha+L\in A} U^{\tau(\alpha+L)} \otimes V_{\alpha+L}$$

for some $A \leq L'/L$.

(This was used in [Höh17, Lam20] to classify the nice, holomorphic vertex operator algebras of central charge 24.)

We note that in the above situation $U \otimes V_L \subseteq V$, $\operatorname{Rep}(V)$ is pointed if and only if $\operatorname{Rep}(U)$ is [YY21].

We now apply this theory to the even part $V^{\bar{0}}$ of a nice, holomorphic vertex operator superalgebra V with $8 \mid c$ so that $\operatorname{Rep}(V^{\bar{0}}) \cong \mathcal{C}(2_{II}^{+2})$.
There are three cases for the dual pair $U \otimes V_K \subseteq V^{\bar{0}}$:

Case	Orb./inv. orb.	Cond. on inner aut.	Dual pair in $V^{\bar{0}}$
I	both inner	$L^h \neq L$	$\lvert K'/K\rvert = 4\,\lvert R(U)\rvert$
IIa		$L^h = L$, $\langle h,h\rangle/2 \notin \mathbb{Z}$	$\lvert K'/K\rvert = \lvert R(U)\rvert$
IIb	inner/non-inner	$L^h = L$, $\langle h,h\rangle/2 \in \mathbb{Z}$	
III	both non-inner		$4\,\lvert K'/K\rvert = \lvert R(U)\rvert$

Here, L is the associated lattice of the 2-neighbour vertex operator algebras $W^{(1)}$ and $W^{(2)}$, and $g = \mathrm{e}^{2\pi \mathrm{i} h_0}$ with $h \in L/2$ the corresponding inner automorphism.

Section 5

Classification Results

We now classify the nice, holomorphic vertex operator superalgebras of central charge 24 by studying

- $\mathbb{Z}_2$-orbifolds of nice, holomorphic vertex operator algebras,
- simple-current extensions of dual pairs $U \otimes V_K$ to $\mathcal{C}(2_{II}^{+2})$.

For this we need to know the involved automorphism groups [BLS22, BLS21].

Recall from [Höh17, Lam20] that for a nice, holomorphic vertex operator algebra of central charge 24, the commutant U is from a list of 12 vertex operator algebras:

- $V^\natural$ + possibly fake copies,
- $V_{\Lambda_\nu}^{\hat{\nu}}$ for 11 conjugacy classes ν $(1^{24}, 1^8 2^8, \ldots, 2^2 10^2)$ in $\mathrm{Co}_0 = \mathrm{O}(\Lambda)$.

The same is true for the commutant U of the even part $V^{\bar{0}}$ of a nice, holomorphic vertex operator superalgebra V of central charge 24 for cases I, IIa and IIb (inner orbifolds).

For case III (non-inner orbifolds), new commutants appear:

- $(V^\natural)^g$, $g \in M$ of class $2A$ (Fricke) + possibly fake copies,
- $V_{\Lambda_\nu}^{\hat{\nu}}$ for $\nu \in \mathrm{Co}_0$ of cycle shape $1^{-24}2^{24}$, $1^{-8}2^{16}$, $1^8 2^{-8} 4^8$, $1^4 2^1 3^{-4} 6^5$, $2^4 4^4$, $1^2 2^1 5^{-2} 10^3$, $2^1 4^1 6^1 12^1$.

Conjecture: All the commutants come from $V^\natural$ or V_Λ in this way (like for holomorphic vertex operator algebras).

Classification results (loops + proper edges):

Case	$c = 8$	$c = 16$	$c = 24$
I	$1 + 0$	$5 + 1$	$299 + 207$
IIa	0	0	$158 + 9$
IIb	0	0	$0 + 171$
III	0	1+0	≈ 50

It remains to complete case III.

Some examples:

V	V_1	$V^{\bar{0}}$	$W^{(1)}$	$W^{(2)}$	type	c_{stump}
$F^{\otimes 48}$	D_{24}	$V_{D_{24}}$	$V_{N(D_{24})}$	$V_{N(D_{24})}$	I	0
$V_{D_{12}^+} \otimes F^{\otimes 24}$	D_{12}^2	$V_{(D_{12}^2)^+}$	$V_{N(D_{12}^2)}$	$V_{N(D_{12}^2)}$	I	12
$\ldots \otimes F$	$A_{1,10}C_{3,10}$	$\ldots$	$Sch(C_{4,10})$	$Sch(C_{4,10})$	IIa	23.5
$\ldots$	$B_{2,10}^2$	$\ldots$	$Sch(A_{4,5}^2)$	$Sch(C_{4,10})$	IIb	24
$\ldots \otimes F$	$E_8E_{8,2}$	$V_{E_8} \otimes (V_{E_8}^{\otimes 2})^{S_2}$	$V_{E_8^3}$	$V_{E_8^3}$	III	23.5
$\ldots$	$A_{4,10}$	$\ldots$	$Sch(A_{4,5}^2)$	$Sch(A_{4,5}^2)$	III	24
$VB^\natural \otimes F$	0	$(V^\natural)^{2A}$	$V^\natural$	$V^\natural$	III	23.5
$VO^\natural$	0	$(V^\natural)^{2B} \cong V_\Lambda^+$	$V^\natural$	V_Λ	III	24


Thank you for your attention!
Hvala!

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