Permutation Orbifolds of Virasoro Vertex Algebras and W-algebras.

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Outline

- Classical invariant theory
- Sources of VOAs
- $\mathbb{Z}_2 \cong S_2$ orbifolds of $V(c,0)^{\otimes 2}$ and connections to $\mathcal{W} - algebras$.
- $\mathbb{Z}_3 \subset S_3$ orbifolds of $V(c,0)^{\otimes 3}$
- S_3 orbifolds of $V(c,0)^{\otimes 3}$ and a connection to $W_k(\mathfrak{g}_2)$.

Symmetric Polynomials

S_n has an "obvious" action on the algebra of polynomials $k[x_1, \ldots, x_n]$ by

$$\sigma \cdot p(x_1, \ldots, x_n) = p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \text{ for } \sigma \in S_n.$$

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Theorem

(Newton? Waring?) If char$(k) = 0$, we have

$$k[x_1, \ldots, x_n]^{S_n} \cong k[p_1, \ldots, p_n],$$

where

$$p_m = x_1^m + \cdots x_n^m.$$

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This "freeness" does not remain if we look at more than one copy of $k[x_1, \cdots, k_n]$.

Example

In the case of $k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]$ *we can note that*

$$k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}] = k[s_1, s_2, t_1, t_2],$$

where $s_i = x_{1,i} + x_{2,i}$ *and* $t_i = x_{1,i} - x_{2,i}$

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$$k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]^{S_2}$$

is generated by s_1, s_2 *and quadratics* $t_1^2, t_2^2, t_1 t_2$*, but not freely.*

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is generated by s_1, s_2 *and quadratics* $t_1^2, t_2^2, t_1 t_2$, *but not freely. Observe that*

$$(t_1^2)(t_2^2) - (t_1 t_2)^2 = 0.$$

Symmetric Polynomials

For our purposes (to study VOAs), we will be interested in the polynomial algebra

$$\mathbb{C}[x_{i,j} | 1 \leq i \leq n, j \geq 0]$$

where

$$\sigma \cdot x_{i,j} = x_{\sigma(i),j}.$$

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This algebra is of particular interest as

$$\mathbb{C}[x_{i,j} | 1 \leq i \leq n, j \geq 0] \cong \mathrm{gr}(V^{\otimes n}),$$

where V is $\mathcal{H}$, $\mathcal{F}$, $V_{\mathrm{Vir}}(c, 0)$, etc...

Symmetric Polynomials

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For S_n, writing down a complete set of relations seems hard – is this known? Luckily as we will see for small values of n we can hack together enough by taking advantage of the internal structure of VOAs.

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  - $\beta - \gamma$ system
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- Orbifolds: V is a VOA and $G \subset \mathrm{Aut}(V)$

$$V^G = \{v \in V | g \cdot v = v \text{ for all } g \in G\}$$

Sources of VOAs

- Cosets: V is a VOA and $W \subset V$

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- $\mathcal{W}$ algebras:
  - Start with a Lie superalgebra $\mathfrak{g}$ and a nilpotent $f \in \mathfrak{g}$.
  - Find an $\mathfrak{sl}(2)$ triple in $\mathfrak{g}$ associated to f: (h, e, f).
  - Decompose $\mathfrak{g}$ by eigenvalues of ad h.
  - Form a free field VOA, $\mathcal{F}(A_{\mathrm{ch}}) \otimes \mathcal{F}(A_{\mathrm{ne}})$, related to this decomposition.
  - Consider $\mathcal{C}(\mathfrak{g}, f) = V^k(\mathfrak{g}) \otimes \mathcal{F}(A_{\mathrm{ch}}) \otimes \mathcal{F}(A_{\mathrm{ne}})$ and a certain vertex algebra homomorphism D.
  - $\mathcal{W}^k(\mathfrak{g}, f)$ is the homology of the related complex.

$V_{\mathrm{Vir}}(c, 0)$

Consider the universal Virasoro vertex algebra of central charge $c \in \mathbb{C}$

$$V_{\mathrm{Vir}}(c, 0) = \langle \omega \rangle \text{ with } Y(\omega, z) = L(z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$$

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where

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

equivalently

$$L(z)L(w) \sim \frac{c/2}{(z-w)^4} + \frac{L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}.$$

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We consider

$$(V_{\mathrm{Vir}}(c,0))^{\otimes n} = \langle \omega_1, \ldots, \omega_n \rangle$$

The general case and a conjecture

We have

$$\mathrm{ch}\left(\mathcal{V}_c^{\otimes n}\right)^{S_n} \sim \prod_{i\geq 0} \frac{1}{(1-q^{i+2})(1-q^{i+3})^{n_3}\cdots(1-q^{i+k})^{n_k}} + O(q^m)$$

and $(\mathcal{V}_c^{\otimes n})^{S_n}$ is of type $(2, 3^{n_3}, \cdots, k^{n_k})$.

Set

$$\omega = \omega_1 + \omega_2$$
$$w_{m+4} = L_1(-2-m)L_1(-2)\mathbb{1} + L_2(-2-m)L_2(-2)\mathbb{1}$$

for $m \geq 0$, and

$$L(z) = Y(\omega, z)$$
$$W^k(z) = Y(w_k, z)$$

for $k \geq 0$.

$(L_{Vir}(c,0)^{\otimes 2})^{S_2}$

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$$L(z) = Y(\omega, z)$$
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for $k \geq 0$. To get to this point, we use

$$L = L_1 + L_2 \text{ and } U = L_1 - L_2$$

forming

$$W^{a,b} = {}^{\circ}_{\circ}\partial^a U \partial^b U^{\circ}_{\circ}.$$

$(L_{Vir}(c,0)^{\otimes 2})^{S_2}$

We make use of the following expressions of weight $m+9$

$$C_1(m) = {}^{\circ}_{\circ} W^{m,0} W^{1,0} {}^{\circ}_{\circ} - {}^{\circ}_{\circ} W^{m,1} W^{0,0} {}^{\circ}_{\circ}$$
$$C_2(m) = {}^{\circ}_{\circ} W^{m-1,1} W^{1,0} {}^{\circ}_{\circ} - {}^{\circ}_{\circ} W^{m-1,0} W^{1,1} {}^{\circ}_{\circ}$$

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Observe the following

- These overlap at $m=1$, but give use two independent expressions for $m \geq 2$.
- The expressions are set up so the arbitrary derivative stays "to the left", keeping the OPE calculations more manageable.

$(L_{Vir}(c,0)^{\otimes 2})^{S_2}$

Setting $m = 1$ in the above expressions allows to to write

$$(47c - 256)W^{6,0}$$

as a combination of lower weight fields, thus eliminating the need for $W^{6,0}$ as a generator unless $c = \frac{256}{47}$.

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as a combination of lower weight fields, thus eliminating the need for $W^{6,0}$ as a generator unless $c = \frac{256}{47}$. For $m \geq 2$ we have independent expression for

$$p_1(m,c)W^{m+5,0} \qquad p_2(m,c)W^{m+5,0}$$

in terms of lower weight fields, where $p_1(m,c)$ and $p_2(m,c)$ are polynomials that are never simultaneously zero. This eliminates the need for

$$W^{8,0}, W^{10,0}, W^{12,0}, \ldots$$

$(L_{Vir}(c,0)^{\otimes 2})^{S_2}$

Theorem (Milas-P.-Sadowski)

For all $c \neq \frac{128}{47}$ the orbifold $\left(\mathcal{V}_c^{\otimes 2}\right)^{S_2}$ is strongly generated by ω, w_4, w_6, and w_8, and is of type $(2,4,6,8)$. Further $(\mathcal{V}_{\frac{128}{47}}^{\otimes 2})^{S_2}$ is generated these five vectors with the addition of w_6 and is of type $(2,4,6,8,10)$.

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Remark Here we use the shortened notation

$$\mathcal{V}_c = V_{\mathrm{Vir}}(c,0) \text{ and, upcoming } \mathcal{L}_c = L_{\mathrm{Vir}}(c,0).$$

$(L_{Vir}(c,0)^{\otimes 2})^{S_2}$

Linshaw and Kanade rigorously constructed the universal two parameter algebra, $\mathcal{W}^{\mathrm{ev}}(c,\lambda)$, of type $\mathcal{W}(2,4,6,\dots)$. This algebra is (weakly) generated by the weight 4 field, W^4_∞ and the parameters are determined by

$$\begin{aligned}\left(W^4_\infty\right)_{(3)} W^4_\infty =& 32\lambda W^4_\infty - \frac{128(49\lambda^2(2c-1)(2c-25)-1)}{63(2c-1)(2c+24)(4c-1)} {:}LL{:} \\ &- \frac{32(2c-4)(49\lambda^2(2c-1)(2c-25))}{441(2c-1)(2c+24)(4c-1)}\partial^2 L.\end{aligned}$$

In the orbifold we have

$$\begin{aligned}\left(\widetilde{W}^4\right)_{(3)} \widetilde{W}^4 =& \frac{2(5c^2+33c-44)}{5c+11}\mu \widetilde{W}^4 + \frac{21c(5c+22)}{(5c+11)^2}\mu^2 {:}LL{:} \\ &+ \frac{3c(c-2)(5c+22)}{2(5c+11)^2}\mu^2\partial^2 L.\end{aligned}$$

Equating these give the following coincidental isomorphisms

$$(\mathcal{L}_{c'} \otimes \mathcal{L}_{c'})^{S_2} \cong \mathcal{W}_{k'}(\mathfrak{so}_{2n}, f_{\text{princ}})^{\mathbb{Z}_2}.$$

where

$$c' = -\frac{n(4n-5)}{n+1} \text{ and } k' = -\frac{4n^2-2n-3}{2n+2},$$

or

$$c' = -\frac{n(4n-5)}{n+1} \text{ and } k' = -\frac{4n(n-2)}{2n-1}.$$

As well as,

$$(\mathcal{L}_{-2} \otimes \mathcal{L}_{-2})^{S_2} \cong N_{-1}(\mathfrak{sl}_2)^{\mathbb{Z}_2} \text{ and } (\mathcal{L}_{7/10} \otimes \mathcal{L}_{7/10})^{S_2} \cong N_8(\mathfrak{sl}_2)^{\mathbb{Z}_2}.$$

$(L_{Vir}(c,0)^{\otimes 2})^{S_2}$

We have $(\mathcal{L}_c \otimes \mathcal{L}_c)^{S_2} \cong \mathcal{W}_k(\mathfrak{sp}(2m))$ for the following values of c and k

$$c = -\frac{12m^2+10m}{2m+3} = c_{2,2m+3} \text{ and } k = -\frac{4m^2+8m+5}{4m+6},$$

$$c = -\frac{3m^2-m-2}{m+2} = c_{2,m+2} \text{ and } k = -\frac{2m^2+m-2}{2m}$$

$$c = \frac{2m^2-5m}{2m^2-5m+3} = c_{2m-3,2m-2} \text{ and } k = -\frac{2m^2-2m-2}{2m-3},$$

where by $c_{p,q}$ we denote the appropriate minimal model. As such, we have established the rationality of these $\mathcal{W}$-algebras.

$(L_{Vir}(c,0)^{\otimes 2})^{S_2}$

Some other cases

- $\left(\mathcal{L}_{-\frac{22}{5}} \otimes \mathcal{L}_{-\frac{22}{5}}\right)^{S_2} \cong \mathcal{L}_{-\frac{44}{5}}$.
- $\left(\mathcal{L}_{-\frac{68}{7}} \otimes \mathcal{L}_{-\frac{68}{7}}\right)^{S_2}$ is of type $(2,4)$.
- $\left(\mathcal{L}_{\frac{1}{2}} \otimes \mathcal{L}_{\frac{1}{2}}\right)^{S_2}$ is of type $(2,4,8)$ and is of extension of $M(1)^+$.
- $(\mathcal{L}_{-12} \otimes \mathcal{L}_{-12})^{S_2}$ is of type $(2,4,8)$.
- $\left(\mathcal{L}_{-\frac{3}{5}} \otimes \mathcal{L}_{-\frac{3}{5}}\right)^{S_2}$ and $\left(\mathcal{L}_{-\frac{46}{3}} \otimes \mathcal{L}_{-\frac{46}{3}}\right)^{S_2}$ are of type $(2,4,6)$.

$(V_{Vir}(c,0)^{\otimes 3})^{\mathbb{Z}_3}$ and $(L_{Vir}(c,0)^{\otimes 3})^{\mathbb{Z}_3}$

Using similar techniques to the above cases, we have

Theorem

For generic c, the orbifold $(\mathcal{V}_c^{\otimes 3})^{\mathbb{Z}_3}$ is of type $(2,4,5,6^3,7,8^3,9^3,10^2)$.

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The orbifold $\left(\mathcal{L}_{-\frac{22}{5}} \otimes \mathcal{L}_{-\frac{22}{5}} \otimes \mathcal{L}_{-\frac{22}{5}}\right)^{\mathbb{Z}_3}$ is of type $(2,5,6,9)$ and the orbifold $\left(\mathcal{L}_{\frac{1}{2}} \otimes \mathcal{L}_{\frac{1}{2}} \otimes \mathcal{L}_{\frac{1}{2}}\right)^{\mathbb{Z}_3}$ is of type $(2,4,5,6,6,7,8,9)$.

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Moving on to S_3.

S_3 – the large c limit

Define $\alpha_i(z) = \sqrt{\frac{2}{c}} L_i(z)$, giving us

$$\alpha_i(z)\alpha_i(w) \sim \frac{1}{(z-w)^4} + \frac{2\sqrt{\frac{2}{c}}\alpha_i(w)}{(z-w)^2} + \frac{\sqrt{\frac{2}{c}}\partial\alpha_i(w)}{z-w}.$$

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For $c \to +\infty$, we obtain a well defined algebra, we denote by $\mathcal{V}_\infty := \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ where

$$\alpha_i(z)\alpha_j(w) \sim \frac{\delta_{i,j}}{(z-w)^4}.$$

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This is sometimes known as a generalized free field algebra, but also it is a subalgebra of the Heisenberg algebra generated by derivatives of the basic fields.

S_3 – the large c limit

Proposition

(adapted from Linshaw 2013) Let u_i $i \in I$ be a strong set of generators of $(\mathcal{V}_\infty^{\otimes 3})^{S_3}$ then for at most countably many values c of the central charge, there is a strong generating set t_i, $i \in I$ of $(\mathcal{V}_c^{\otimes 3})^{S_3}$ with $deg(t_i) = deg(u_i)$.

S_3 – computations

Define elements in $\mathcal{V}_\infty^{\otimes 3} = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$:

$$\begin{aligned} T_0 &= \frac{1}{\sqrt{3}}(\alpha_1 + \alpha_2 + \alpha_3) \\ T_1 &= \frac{1}{\sqrt{3}}(\alpha_1 + \eta\alpha_2 + \eta^2\alpha_3) \\ T_2 &= \frac{1}{\sqrt{3}}(\alpha_1 + \eta^2\alpha_2 + \eta\alpha_3), \end{aligned} \tag{1}$$

where η is a primitive third root of unity. Under this operation the algebra generated from T_0, T_1 and T_2 has the following nontrivial OPE

$$\begin{aligned} T_0(z)T_0(w) &\sim \frac{1}{(z-w)^4} \\ T_1(z)T_2(w) &\sim \frac{1}{(z-w)^4}. \end{aligned} \tag{2}$$

S_3 – computations

Consider the following quadratic and cubic fields

$$
\begin{aligned}
W_{m,n} &= {}^{\circ}_{\circ}(\partial^m T_1)(\partial^n T_2){}^{\circ}_{\circ} + {}^{\circ}_{\circ}(\partial^n T_1)(\partial^m T_2){}^{\circ}_{\circ} \\
C_{\ell,m,n} &= {}^{\circ}_{\circ}(\partial^\ell T_1)(\partial^m T_1)(\partial^n T_1){}^{\circ}_{\circ} + {}^{\circ}_{\circ}(\partial^\ell T_2)(\partial^m T_2)(\partial^n T_2){}^{\circ}_{\circ}
\end{aligned}
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First notice that

$$W_{m,n} = \sum_{i=0}^{n} (-1)^{n-k} \binom{n}{k} \partial^i W_{m+n-k,0}$$

$$C_{\ell,m,n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} C_{m+k,n+\ell-k,0}.$$

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$$C_{\ell,m,n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} C_{m+k,n+\ell-k,0}.$$

So we only need $W_{m,0}$ and $C_{m,n,0}$ for $m, n \geq 0$.

S_3 – computations

Next consider the following expressions

$$
\begin{aligned}
R_1(\mathbf{a},\mathbf{m}) = {}& (a_1+a_2+a_3) {}^\circ_\circ W_{m_4,m_5} C_{m_1,m_2,m_3} {}^\circ_\circ + (a_1+a_3+a_5) {}^\circ_\circ W_{m_3,m_5} C_{m_1,m_2,m_4} {}^\circ_\circ \\
& - (a_1+a_2+a_3+a_4+a_5) {}^\circ_\circ W_{m_3,m_4} C_{m_1,m_2,m_5} {}^\circ_\circ + a_5 {}^\circ_\circ W_{m_2,m_4} C_{m_1,m_3,m_5} {}^\circ_\circ \\
& - (a_1+a_4+a_5) {}^\circ_\circ W_{m_2,m_5} C_{m_1,m_3,m_4} {}^\circ_\circ - (a_1+a_2+a_3) {}^\circ_\circ W_{m_1,m_5} C_{m_2,m_3,m_4} {}^\circ_\circ \\
& + a_4 {}^\circ_\circ W_{m_2,m_3} C_{m_1,m_4,m_5} {}^\circ_\circ + a_3 {}^\circ_\circ W_{m_1,m_4} C_{m_2,m_3,m_5} {}^\circ_\circ \\
& + a_1 {}^\circ_\circ W_{m_1,m_2} C_{m_3,m_4,m_5} {}^\circ_\circ + a_2 {}^\circ_\circ W_{m_1,m_3} C_{m_2,m_4,m_5} {}^\circ_\circ ,
\end{aligned}
$$

Next consider the following expressions

$$\begin{aligned}
R_1(\mathbf{a},\mathbf{m}) = {} & (a_1+a_2+a_3) {}^\circ_\circ W_{m_4,m_5} C_{m_1,m_2,m_3} {}^\circ_\circ + (a_1+a_3+a_5) {}^\circ_\circ W_{m_3,m_5} C_{m_1,m_2,m_4} {}^\circ_\circ \\
& - (a_1+a_2+a_3+a_4+a_5) {}^\circ_\circ W_{m_3,m_4} C_{m_1,m_2,m_5} {}^\circ_\circ + a_5 {}^\circ_\circ W_{m_2,m_4} C_{m_1,m_3,m_5} {}^\circ_\circ \\
& - (a_1+a_4+a_5) {}^\circ_\circ W_{m_2,m_5} C_{m_1,m_3,m_4} {}^\circ_\circ - (a_1+a_2+a_3) {}^\circ_\circ W_{m_1,m_5} C_{m_2,m_3,m_4} {}^\circ_\circ \\
& + a_4 {}^\circ_\circ W_{m_2,m_3} C_{m_1,m_4,m_5} {}^\circ_\circ + a_3 {}^\circ_\circ W_{m_1,m_4} C_{m_2,m_3,m_5} {}^\circ_\circ \\
& + a_1 {}^\circ_\circ W_{m_1,m_2} C_{m_3,m_4,m_5} {}^\circ_\circ + a_2 {}^\circ_\circ W_{m_1,m_3} C_{m_2,m_4,m_5} {}^\circ_\circ ,
\end{aligned}$$

The fact that $R(\mathbf{a},\mathbf{m}) = 0$ in the associated graded algebras allows us to create certain quantum corrections in order to write the C_{m_1,m_2,m_3} generators in terms of lower weight generators.

S_3 – computations

If we set $R_2(\mathbf{m}) = R_1(1,0,0,0,0,\mathbf{m})$, we can find $b_1,\ldots,b_8$ such that

$$\begin{aligned} C_{16,0,0} =& b_1 R_2(10,2,0,0,0) + b_2 R_2(9,2,1,0,0) + b_3 R_2(8,3,1,0,0) \\ &+ b_4 R_2(7,3,2,0,0) + b_5 R_2(6,5,1,0,0) + b_6 R_2(5,4,2,1,0) \\ &+ b_7 R_2(4,3,2,2,1) + b_8 R_2(4,4,4,0,0). \end{aligned}$$

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with

$$b_1 = \frac{179048401021754539228 8}{168520823757097513517}, b_2 = \frac{179580948755993608824 0}{168520823757097513517}, \dots$$
$$b_8 = -\frac{146489450195468612446 2}{168520823757097513517}.$$

S_3 – computations

To finish reducing the cubic generating set, we construct an explicit family of quantum corrections at arbitrary weights higher than 23 in order to write these generators in terms of the cubic generators from weight 6 to 22.

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From weight 12 to 22 the process isn't quite as systematic as evidenced by the equation of on the previous slide.

In the end, we only require

$$C_{0,0,0},\, C_{2,0,0},\, C_{3,0,0},\, C_{4,0,0},\, C_{5,0,0},\, C_{6,0,0},\ \text{and}\ C_{3,3,0},$$

at weight $6, 8, 9, 10, 11, 12, 12$.

S_3 – computations

A different strategy is used for the quadratic generators

- $W_{10,0}$ and $W_{12,0}$ can be removed with relations involving $W_{2k,0}$ and $C_{m,n,0}$.

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In the end we only require

$$W_{0,0}, W_{2,0}, W_{4,0}, W_{6,0}, W_{8,0},$$

at weight $4, 6, 8, 10, 12$.

S_3 – generic theorem

Theorem

For any generic c, including a suitably defined $c \to \infty$ limit, the S_3-orbifold subalgebra $(\mathcal{V}_c^{\otimes n})^{S_3}$ is strongly generated by vectors of of weight $2, 4, 6, 6, 8, 8, 9, 10, 10, 11, 12, 12, 12$. Moreover, this is also a minimal generating set.

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Proof.

Generation is given by our previous calculations and minimality is given by a comparison with a suitable "free" character.

$\square$

S_3 – simple $c = \frac{1}{2}$

Here we can work directly inside the algebra rather than passing to the limit.

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$$
\begin{aligned}
L &= \frac{1}{\sqrt{3}}(L_1 + L_2 + L_3) \\
U_1 &= \frac{1}{\sqrt{3}}(L_1 + \eta L_2 + \eta^2 L_3) \\
U_2 &= \frac{1}{\sqrt{3}}(L_1 + \eta^2 L_2 + \eta L_3),
\end{aligned}
\tag{3}
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$$

We also diagonalize the three weight 6 singular vectors as well, for example

$$
S = 128{}^\circ_\circ LLL{}^\circ_\circ + 768{}^\circ_\circ LU_1U_2{}^\circ_\circ + \cdots
$$

S_3 – simple $c = \frac{1}{2}$

Next, we set

$$W_{m+4} = {:}(\partial^m U_1)U_2{:} + (-1)^m {:}(\partial^m U_2)U_1{:}$$

and

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Using similar strategies as to those described before, we have

Theorem

The simple orbifold $(L_{\frac{1}{2}}^{\otimes 3})^{\mathbb{Z}_3}$ is of type 2,4,5,6,6,7,8,9 and is strongly generated by L, together with $W_4, W_5, W_6, W_7, W_8, W_9$, and C_6^-.

S_3 – simple $c = \frac{1}{2}$

- The generators described above we have $L, W_4, W_6, W_8 \in (L_{\frac{1}{2}}^{\otimes 3})^{S_3}$ while $C_6^- \mapsto -C_6^-$, $W_5 \mapsto -W_5$, $W_7 \mapsto -W_7$, and $W_9 \mapsto -W_9$ under the additional $(12) \cong \mathbb{Z}_2$ action.

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- Between quantum corrections and decedents of the singular vectors we are able to removed the need for any quadratic (in the non-fixed fields) generators from the generating set of the S_3 orbifold.

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Theorem

The orbifold algebra $\left(\mathcal{L}_{\frac{1}{2}}^{\otimes 3}\right)^{S_3}$ *is strongly generated by the fields* L, W_4, W_6, W_8 *and is thus of type* $2, 4, 6, 8$.

S_3 – simple $c = \frac{1}{2}$

- As the orbifold algebra $\left(\mathcal{L}_{\frac{1}{2}}^{\otimes 3}\right)^{S_3}$ is of type (2,4,6,8) it is reasonable to guess that it may be isomorphic to other known algebras of this type – the most likely being $(N_{10}(\mathfrak{sl}_2))^{\mathbb{Z}_2}$ which also has central charge $c = 3/2$.

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- Using the early described strategy one can check that this is not the case and thus this orbifold is a new example of a unitary $\mathcal{W}$-algebra of this type.

S_3 – simple $c = -\frac{22}{5}$

Using strategies similar to those above we have

Theorem

The orbifold algebra $\left(\mathcal{L}_{-22/5}^{\otimes 3}\right)^{S_3}$ *is strongly generated by the fields* L, W_6 *and is thus of type* $2, 6$.

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We have

$$\left(\mathcal{L}_{-22/5}^{\otimes 3}\right)^{S_3} \cong W_{-16/5}(\mathfrak{g}_2, f_{princ})$$

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In fact, the story is a bit more interesting than “just” this.

$W^k(\mathfrak{g}_2, f)$

- Consider the exceptional Lie algebra $\mathfrak{g} = \mathfrak{g}_2$ and a subregular nilpotent f_{sub} and complete the $\mathfrak{sl}_2$ triple. The corresponding W algebra and its OPE as constructed by J. Fasquel
- If we set $k = -\frac{16}{5}$ the weight two generators can be taken to be commuting Virasoro fields of central charge $c = -\frac{22}{5}$.

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- If we set $k = -\frac{16}{5}$ the weight two generators can be taken to be commuting Virasoro fields of central charge $c = -\frac{22}{5}$.
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Proposition

We have an isomorphism of simple vertex operator algebras

$$W_{-\frac{16}{5}}(\mathfrak{g}_2, f_{sub}) \cong \mathcal{L}_{-22/5}^{\otimes^3}.$$

Putting this all together:

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Theorem

We have an isomorphism of rational vertex algebras:

$$W_{-\frac{16}{5}}(\mathfrak{g}_2, f_{sub})^{S_3} \cong W_{-\frac{19}{6}}(\mathfrak{g}_2, f_{prin}).$$

A bit more about $W_k(\mathfrak{g}_2, f_{\mathrm{prin}})$

Proposition

The simple affine W-algebras $W_k(\mathfrak{g}_2, f_{prin})$ collapses to a (simple) Virasoro vertex algebra if and only if $k \in \{-\frac{34}{9}, -\frac{7}{2}, -\frac{10}{3}, -\frac{5}{2}, -\frac{22}{7}, -\frac{65}{18}, -\frac{40}{11}, -\frac{37}{12}, -\frac{27}{8}, -\frac{52}{15}, -\frac{53}{15}, -\frac{23}{7}\}$.

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Proof.

- This holds for generic central charges for $k \in \{-\frac{34}{9}, -\frac{7}{2}, -\frac{10}{3}, -\frac{5}{2}\}$ by only looking at the OPE.

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Proof.

- This holds for generic central charges for $k \in \{-\frac{34}{9}, -\frac{7}{2}, -\frac{10}{3}, -\frac{5}{2}\}$ by only looking at the OPE.
- The remaining cases correspond to central charges from the list

$$\{-\frac{22}{5}, -\frac{3}{5}, -\frac{46}{3}, -\frac{232}{11}\}$$

which gives us Virasoro singular vectors to complete the result.

□

$W^k(\mathfrak{g}_2, f)$

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- This resolves an old conjecture in the physics literature by Blumenhagen et al.

Conjecture

We have an isomorphism

$$W_k(\mathfrak{f}_4, f_{s_4}) \cong \mathcal{L}_{-22/5}^{\otimes^4},$$

for some nilpotent, and moreover

$$W_k(\mathfrak{f}_4, f_{s_4})^{S_4} \cong W_{k'}(\mathfrak{f}_4, f_{prin})$$

where k is a certain level such that $c(k) = -\frac{4\cdot 22}{5}$.

Conjecture

We have an isomorphism

$$W_k(\mathfrak{e}_8, f_{s_5}) \cong \mathcal{L}_{-22/5}^{\otimes^5},$$

for some nilpotent, and moreover

$$W_k(\mathfrak{e}_8, f_{s_5})^{S_5} \cong W_{-\frac{144}{5}}(\mathfrak{e}_8, f_{sub})$$

Thank You!