Unitary forms for holomorphic vertex operator algebras of central charge 24

Ching Hung Lam

Academia Sinica

October 7 , 2022

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   There is $f \in \mathrm{Aut}(V)$ such that $V^f = V_N^g$.

   Try to find another automorphism h such that h is conjugate to f and $[f, g] = 1$.

$$V = \bigoplus_{i,j} V^{i,j} \quad V_N^g = V^f \oplus V^{i,0} \quad V^h = \oplus V^{0,i}.$$

Unitary VOA and unitary modules

Definition

Let $(V, Y, \mathbb{1}, \omega)$ be a vertex operator algebra. An isomorphism $\phi : V \to V$ is called an *anti-linear automorphism* of V if $\phi(\lambda x) = \bar{\lambda}\phi(x)$, $\phi(\mathbb{1}) = \mathbb{1}, \phi(\omega) = \omega$ and $\phi(u_n v) = \phi(u)_n \phi(v)$ for any $u, v \in V$ and $n \in \mathbb{Z}$.

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Definition ([DLin14])

Let $(V, Y, \mathbb{1}, \omega)$ be a vertex operator algebra and let $\phi : V \to V$ be an anti-linear involution of V. Then (V, ϕ) is said to be unitary if there exists a positive-definite Hermitian form $(\ ,\ )_V : V \times V \to \mathbb{C}$, which is $\mathbb{C}$-linear on the first vector and anti-$\mathbb{C}$-linear on the second vector,

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$$(Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})u, v)_V = (u, Y(\phi(a), z)v)_V,$$

where $L(n)$ is defined by $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$.

Remark

Let (V, ϕ) be a simple unitary VOA with an inv. Hermitian form $(\cdot, \cdot)_V$. Then V is self-dual and of CFT-type ([CKLW, Proposition 5.3]) and V has a unique invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$, up to scalar ([Li94]).

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Definition ([DLin14])

Let (V, ϕ) be a unitary VOA and g a finite order automorphism of V. An (ordinary) g-twisted V-module (M, Y_M) is called a unitary g-twisted V-module if there exists a positive-definite Hermitian form $(\ ,\ )_M : M \times M \to \mathbb{C}$ such that the following invariant property holds for $a \in V$ and $w_1, w_2 \in M$:

$$(Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})w_1, w_2)_M = (w_1, Y_M(\phi(a), z)w_2)_M.$$

Lemma (cf. [FHL93, Remark 5.3.3])

Let (V, ϕ) be a unitary VOA. Let M be a V-module and M' the contregredient module of M with a natural pairing $\langle \cdot, \cdot \rangle$ between M and M'.

1. *If M has a non-degenerate invariant sesquilinear form $(\cdot, \cdot)$, which is linear on the first vector and anti-$\mathbb{C}$-linear on the second vector and satisfies the invariant property, then the map $\Phi : M \to M'$ defined by $(u, v) = \langle u, \Phi(v) \rangle$, $u, v \in M$, is an anti-linear bijective map and $\Phi(a_n u) = \phi(a)_n \Phi(u)$ for $a \in V$ and $u \in M$.*
2. *If there exists an anti-linear bijective map $\Phi : M \to M'$ such that $\Phi(a_n u) = \phi(a)_n \Phi(u)$ for $a \in V$ and $u \in M$, then $(u, v) = \langle u, \Phi(v) \rangle$, $u, v \in M$, is a non-degenerate invariant sesquilinear form on M.*

Let (V, ϕ) be a unitary VOA and $(\ , \ )$ the corresponding positive definite invariant Hermitian form. Define

$$\mathrm{Aut}_{(\ , \ )}(V) = \{g \in \mathrm{Aut}(V) \mid (gx, gy) = (x, y) \text{ for all } x, y \in V\}.$$

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Let (V, ϕ) be a unitary VOA. Then
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(1) $g \in \mathrm{Aut}_{(\ ,\ )}(V)$ if and only if $g^{-1}\phi g = \phi$.

(2) For any $H < \mathrm{Aut}_{(\ ,\ )}(V)$, (V^H, ϕ) is also a unitary VOA.

Lattice VOA

Let $V_L = M(1) \otimes_{\mathbb{C}} \mathbb{C}\{L\}$ be the lattice VOA.

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$$(\mathbb{1}, \mathbb{1}) = 1, \quad (\alpha(n)u, v) = (u, \alpha(-n)v)$$

for $\alpha \in L$ and for any $u, v \in M(1)$.

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There also exists a positive-definite Hermitian form on $\mathbb{C}\{L\} = \mathrm{Span}_{\mathbb{C}}\{e^{\alpha} \mid \alpha \in L\}$ determined by $(e^{\alpha}, e^{\beta}) = \delta_{\alpha,\beta}$.

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Then a positive-definite Hermitian form on V_{L^*} can be defined by

$$(u \otimes e^\alpha, v \otimes e^\beta) = (u, v) \cdot (e^\alpha, e^\beta),$$

where $u, v \in M(1)$ and $\alpha, \beta \in L$.

Let $\phi : V_L \to V_L$ be an anti-linear map determined by:

$$\alpha_1(-n_1)\cdots\alpha_k(-n_k)\otimes e^{\alpha} \mapsto (-1)^k\alpha_1(-n_1)\cdots\alpha_k(-n_k)\otimes e^{-\alpha},$$

where $\alpha_1,\dots,\alpha_k \in L, \alpha \in L$.

Theorem

Let L be a positive-definite even lattice and let ϕ be the anti-linear map of V_L defined as above.
Then the lattice vertex operator algebra (V_L, ϕ) is a unitary VOA.

$\mathrm{Aut}_{(\,,\,)}(V_L)$

Theorem ([DN99])

Let L be a positive definite even lattice. Then

$$\mathrm{Aut}(V_L) = N(V_L)\, O(\hat{L})$$

and the quotient $\mathrm{Aut}(V_L)/N(V_L)$ *is isomorphic to a quotient group of* $O(L)$.

Lemma

Let $g \in O(\hat{L})$. *Then* $g \in \mathrm{Aut}_{(\;,\;)}(V_L)$.

Lemma

Let $\beta \in L^*$ *and n a positive integer. Then*

$$h = \exp(2\pi i \frac{\beta(0)}{n}) \in \mathrm{Aut}_{(\;,\;)}(V_L).$$

Twisted modules

Let τ be an isometry of L. Let p be a positive integer such that $\tau^p = 1$. Define $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the $\mathbb{Z}$-form $\langle \cdot | \cdot \rangle$ $\mathbb{C}$-linearly to $\mathfrak{h}$. Denote

$$\mathfrak{h}_{(n)} = \{\alpha \in \mathfrak{h} \mid \tau\alpha = \xi^n \alpha\} \quad \text{for } n \in \mathbb{Z},$$

where $\xi = \exp(2\pi\sqrt{-1}/p)$.

Let $\hat{\mathfrak{h}}[\tau] = \coprod_{n\in\mathbb{Z}} \mathfrak{h}_{(n)} \otimes t^{n/p} \oplus \mathbb{C}c$ be the τ-twisted affine Lie algebra of $\mathfrak{h}$. Denote

$$\hat{\mathfrak{h}}[\tau]^+ = \coprod_{n>0} \mathfrak{h}_{(n)} \otimes t^{n/p}, \quad \hat{\mathfrak{h}}[\tau]^- = \coprod_{n<0} \mathfrak{h}_{(n)} \otimes t^{n/p}, \quad \text{and} \quad \hat{\mathfrak{h}}[\tau]^0 = \mathfrak{h}_{(0)} \oplus \mathbb{C}c,$$

and

$$S[\tau] = S(\hat{\mathfrak{h}}[\tau]^-).$$

Set $s = p$ if p is even and $s = 2p$ if p is odd.
Define a τ-invariant alternating $\mathbb{Z}$-bilinear map c^τ from $L \times L$ to $\mathbb{Z}_s$ by

$$c^\tau(\alpha, \beta) = \sum_{i=0}^{p-1} (s/2 + si/p)\langle \tau^i(\alpha)|\beta\rangle + s\mathbb{Z}.$$

Consider the central extension

$$1 \longrightarrow \langle \kappa_s \rangle \longrightarrow \hat{L}_\tau \overset{-}{\longrightarrow} L \longrightarrow 1$$

such that $aba^{-1}b^{-1} = \kappa_s^{c^\tau(\bar{a},\bar{b})}$ for $a, b \in \hat{L}_\tau$.

Let $\beta \in \mathbb{Q} \otimes L^\tau$ such that $p\langle \beta | L\rangle \in \mathbb{Z}$.
Then $g = \hat{\tau} \exp(2\pi i \beta(0))$ also defines an automorphism of V_L and $g^p = 1$.
An irreducible g-twisted module is then given by

$$V_L^\chi(g) = S[\tau] \otimes e^{-\beta} \otimes U_\chi,$$

as a vector space.

We define a Hermitian form on $V_L^\chi(g)$ as follows.
For any $a, b \in e^{-\beta}\hat{L}_\tau$, define

$$(t(a), t(b)) = \begin{cases} 0 & \text{if } b^{-1}a \notin \mathcal{A}, \\ \chi(b^{-1}a) & \text{if } b^{-1}a \in \mathcal{A}, \end{cases} \tag{0-1}$$

where $t(a) = a \otimes 1 \in e^{-\beta} \otimes U_\chi$;
there ia positive-definite Hermitian form (,) on $S[\tau]$ such that

$$(1,1) = 1,$$
$$(\alpha(n) \cdot u, v) = (u, \alpha(-n) \cdot v),$$

for any $u, v \in S[\tau]$ and $\alpha \in L$.
Then one can define a positive-definite Hermitian form on $V_L^\chi(g)$ by

$$(u \otimes r, v \otimes s) = (u, v) \cdot (r, s), \quad \text{where } u, v \in S[\tau], r, s \in e^{-\beta} \otimes U_\chi.$$

Lemma

For any χ, $V_L^\chi(g)$ is a unitary g-twisted module of (V_L, ϕ).

Orbifold construction from Niemeier lattice VOAs

Theorem (cf. Proposition 5.7 and Remark 5.8 of Höhn-Möller)

Let V be a holomorphic VOA of central charge 24 with $V_1 \neq 0$. Then there exist a Niemeier N and an automorphism $g = \hat{\tau}\exp(2\pi i\beta(0)) \in Aut(V_N)$ such that $V \cong \widetilde{V_N}(g)$. Moreover,

1. *τ has the same frame shape as one of the 11 conjugacy classes of Co_0 as discussed in [Hö2].*
2. *$L \cong N_\beta^\tau$ and $V_{N_\tau}^{\hat{\tau}} \cong V_{\Lambda_\tau}^{\hat{\tau}}$; in particular, $V_N^g > V_L \otimes V_{\Lambda_\tau}^{\hat{\tau}}$.*
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Example: $V_1 \cong A_{2,3}^6$.

$N = N(A_1^{24})$;
τ acts a permutation of the 24 copies of A_1's with the cycle shape $1^6 3^6$; and

$\beta = \frac{1}{6}(0^{12}, \alpha^{12})$, where $\mathbb{Z}\alpha \cong A_1$, i.e, $\langle \alpha, \alpha \rangle = 2$.

In this case, $V = \widetilde{V_N}(g)$ and $(V_N(A_1^{24})^g)_1 \cong A_{1,3}^6 U(1)^6$.

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Assume that M has an anti-linear map ψ such that $\psi(v_n w) = \varphi(v)_n \psi(w)$ and $\psi^2 = id$, $(\psi(w_1), \psi(w_2))_M = (w_1, w_2)_M$ and the Hermitian form $(\ ,\ )_V$ on V has the property that $(\varphi(v_1), \varphi(v_2))_V = (v_1, v_2)_V$.

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Let $U = V \oplus M$. Then (U, φ_U) has a unique unitary vertex operator algebra structure, where $\varphi_U : U \to U$ is the anti-linear involution defined by $\varphi_U(v, w) = (\varphi(v), \psi(w))$, for $v \in V, w \in M$. Furthermore, U is rational and C_2-cofinite.

As a consequences, we have the following result.

Theorem

Let V be a holomorphic VOA of central charge 24 with the weight one Lie algebra isomorphic to one of the Lie algebras in the following Table. Then V is unitary.

Class	# of V	Weight one Lie algebra structures
$2A$	17	$A_{1,2}^{16}$, $A_{3,2}^{4}A_{1,1}^{4}$, $D_{4,2}^{2}B_{2,1}^{4}$, $A_{5,2}^{2}C_{2,1}A_{2,1}^{2}$, $D_{5,2}^{2}C_{2,1}A_{2,1}^{2}$, $A_{7,2}C_{3,1}^{2}A_{3,1}$, $C_{4,1}^{4}$, $D_{6,2}C_{4,1}B_{3,1}^{2}$, $A_{9,2}A_{4,1}B_{3,1}$, $E_{6,2}C_{5,1}A_{5,1}$, $D_{8,2}B_{4,1}^{2}$, $C_{6,1}^{2}B_{4,1}$, $D_{9,2}A_{7,1}$, $C_{8,1}F_{4,1}^{2}$,, $E_{7,2}B_{5,1}F_{4,1}$ $C_{10,1}B_{6,1}$, $B_{8,1}E_{8,2}$
$2C$	9	$A_{1,4}^{12}$, $B_{2,2}^{6}$, $B_{3,2}^{4}$, $B_{4,2}^{3}$, $B_{6,2}^{2}$, $B_{12,2}$, $D_{4,4}A_{2,2}^{4}$, $C_{4,2}A_{4,2}^{2}$, $A_{8,2}F_{4,2}$

Next we consider other orbifold constructions.

Theorem

Let V be a self-dual, simple VOA of CFT-type. Assume that V has two commuting automorphisms f and h of order p. For $i, j \in \mathbb{Z}$, set $V^{i,j} = \{v \in V \mid f(v) = \xi^i v,\ h(v) = \xi^j v\}$, where $\xi = \exp(2\pi\sqrt{-1}/p)$. Set $V^i = \bigoplus_{j=0}^{p-1} V^{i,j}$. Assume the following:

- (A) *There exists an anti-linear involution ϕ of V^0 such that (V^0, ϕ) is a unitary VOA;*
- (B) *For $i \in \{1, \dots, p-1\}$, V^i is a unitary (V^0, ϕ)-module;*
- (C) *There exists an automorphism $\psi \in \mathrm{Aut}(V)$ such that $\psi^{-1} f \psi = h$;*
- (D) *$\psi(V^{0,0}) = V^{0,0}$ and $\psi\phi\psi^{-1} = \phi$ on $V^{0,0}$;*

Then there exist an anti-linear involution Φ of V such that (V, Φ) is a unitary VOA.

Note: $V = \bigoplus_{0 \le i,j \le p-1} V^{i,j}$ is $\mathbb{Z}_p^2$-graded.

By the assumption (C), $\psi(V^0) = V^{0,0} \oplus V^{1,0} \oplus \cdots \oplus V^{p-1,0}$ is also a unitary VOA with the anti-linear automorphism $\psi\phi\psi^{-1}$ and a positive-definite invariant Hermitian form defined by

$$(a, b)_{\psi(V^0)} = (\psi^{-1}(a), \psi^{-1}(b))_{V^0} \quad \text{for} \quad a, b \in \psi(V^0).$$

Note that $\psi\phi\psi^{-1} = \phi$ on $V^{0,0}$ by Assumption (D).

The invariant Hermitian form on the unitary $(V^{0,0}, \phi)$-module $V^{i,0}$ is unique up to scalar for each $i = 1, \ldots, p-1$.

We may choose a positive-definite invariant Hermitian form $(\cdot, \cdot)_{V^i}$ on V^i so that

$$(u, v)_{V^i} = (u, v)_{\psi(V^0)} \quad \text{for} \quad u, v \in V^{i,0}.$$

By Lemma 5, there exists an anti-linear bijective map $\Phi^i : V^i \to V^{p-i}$ such that

$$\Phi^i(a_n v) = \phi(a)_n \Phi^i(v) \quad \text{for } a \in V^0, v \in V^i$$

and

$$(u, v)_{V^i} = \langle u, \Phi^i(v) \rangle \quad \text{for} \quad u, v \in V^i.$$

For any $u, v \in V^{i,0}$, we have

$$\begin{aligned} \langle u, \Phi^i(v) \rangle = (u, v)_{V^i} &= (\psi^{-1}(u), \psi^{-1}(v))_{V^0} \\ &= \langle \psi^{-1}(u), \phi\psi^{-1}(v) \rangle = \langle u, \psi\phi\psi^{-1}(v) \rangle. \end{aligned}$$

Hence

$$\psi\phi\psi^{-1} = \Phi^i \quad \text{on} \quad V^{i,0}.$$

Define the anti-linear map $\Phi : V \to V$ so that

$$\Phi(u) = \begin{cases} \phi(u) & \text{for } u \in V^0, \\ \Phi^i(u) & \text{for } u \in V^i,\ i = 1, \dots, p-1, \end{cases}$$

and the positive-definite Hermitian form $(\cdot, \cdot)$ on V by

$$(u, v) = \begin{cases} (u, v)_{V^i} & \text{if } u, v \in V^i,\ i = 0, 1, \dots, p-1, \\ 0 & \text{if } u \in V^i,\ v \in V^j,\ i \neq j. \end{cases}$$

Clearly, Φ is bijective.

Remark: Since the order of ϕ is 2, both the composition maps $\Phi^{p-i} \circ \Phi^i$ and $\Phi^i \circ \Phi^{p-i}$ are the identity map on $V^{i,0}$. Viewing V^{p-i} as an irreducible unitary (V^0, ϕ)-module, we have $\Phi^{p-i} = (\Phi^i)^{-1}$ on V^{p-i}, also.

Therefore, $\Phi \circ \Phi$ is the identity of V.

Lemma

1. *For* $i, j \in \{0, 1, \ldots, p-1\}$, $\Phi(V^{i,j}) = V^{p-i,p-j}$.
2. *For* $u, v \in V$, $(u, v) = \langle u, \Phi(v) \rangle$.

Proposition

The anti-linear map Φ *is an anti-linear involution of* V.

Proof: Since ϕ is an anti-linear automorphism of V^0, Φ fixes the vacuum vector and the conformal vector of V. Since (V^0, ϕ) is unitary, the equation

$$\Phi(u_n v) = \Phi(u)_n \Phi(v) \tag{0-2}$$

holds for $u, v \in V^0$ and $n \in \mathbb{Z}$.

By the definition of Φ^i for $i = 1, 2$, (0-2) holds for $u \in V^0$ and $v \in V^i$.

By the skew symmetry, we have

$$u_n v = (-1)^{n+1} v_n u + \sum_{i \geq 1} \frac{(-1)^{n+i+1}}{i!} L(-1)^i (v_{n+1} u)$$

for $u, v \in V$ and $n \in \mathbb{Z}$.

Hence the equation (0-2) also holds for $u \in V^i$ and $v \in V^0$.

Let $x \in V^{0,j}$, $y \in V^{i,0}$ and $u \in V^{k,\ell}$. By Borcherds' identity, for $r, q \in \mathbb{Z}$,

$$(x_r y)_q u = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \left(x_{r-i}(y_{q+i} u) - (-1)^r y_{q+r-i}(x_i u) \right).$$

By the assumptions on x and y and the identity above, we have

$$\Phi((x_r y)_q u) = (\Phi(x)_r \Phi(y))_q \Phi(u) = \Phi(x_r y)_q \Phi(u).$$

Thus, we obtain $\Phi(u_n v) = \Phi(u)_n \Phi(v)$ for all $x, y \in V$ and $n \in \mathbb{Z}$.

Proposition

The positive-definite Hermitian form $(\ ,\ )$ on V satisfies the invariant property for (V, Φ).

Unitary forms

Every holomorphic VOA of central charge 24 with $V_1 \neq 0$ can be constructed by a single orbifold construction from a Niemeier lattice VOA. Let (N, g) be a pair of a Niemeier lattice and an automorphism of V_N such that $V \cong \widetilde{V_N}(g)$. Then

$$V = V_N^g \oplus V_N[g]_0 \oplus \cdots \oplus V_N[g^{p-1}]_0,$$

where $V_N[g^i]$ denotes the irreducible g^i-twisted module of V_N.

Let L be the even lattice such that $V_L \cong \mathrm{Com}_V(\mathrm{Com}_V(M(\mathfrak{h})))$, where $\mathfrak{h}$ is a Cartan subalgebra of V_1 and suppose $g = \hat{\tau}\exp(2\pi i\beta(0) \in \mathrm{Aut}(V_N)$. Then

$$L \cong N_\beta^\tau \quad \text{and} \quad V_N^g > V_L \otimes V_{\Lambda_\tau}^{\hat{\tau}}.$$

Set

$$V_N = \bigoplus_{\lambda + N^\tau \in (N^\tau)^*/N^\tau} V_{\lambda+N^\tau} \otimes V_{\lambda'+N_\tau}.$$

Then

$$V_N^g = \bigoplus_{\lambda + N^\tau \in (N^\tau)^*/N^\tau} (V_{\lambda+N^\tau} \otimes V_{\lambda'+N_\tau})^g = \bigoplus_{\lambda + L \in (N^\tau)^*/L} V_{\lambda+L} \otimes W_\lambda < V.$$

Define $f \in \mathrm{Aut}(V)$ so that f acts on $V_N[g^i]_0$ as a multiplication of the scalar ξ^i. Then $V^f = V_N^g$ and there is a $\gamma \in \mathbb{Q} \otimes_{\mathbb{Z}} N^\tau$ such that $\langle \gamma | \beta \rangle \notin \mathbb{Z}$ and $f = \exp(2\pi i \gamma(0))$.

By our choices of (N, g), there is always a root of V_1 and a lift $\psi_\alpha \in \mathrm{Stab}_{\mathrm{Aut}(V)}(V_L \otimes W)$ of a reflection $s_\alpha \in W(V_1)$ such that $\psi_\alpha((V_N^g)_1) \neq (V_N^g)_1$ and $\psi_\alpha^2 = 1$.
For simplicity, we use w and ψ to denote s_α and ψ_α, respectively.

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Define $h = \psi f \psi^{-1}$. Then $h = \exp(2\pi i w(\gamma)(0))$ and it is clear that both f and h fix $V_L \otimes V_{\Lambda_\tau}^{\hat{\tau}}$ point-wisely.

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Since all irreducible modules for $V_L \otimes V_{\Lambda_\tau}^{\hat{\tau}}$ are simple current modules, the subgroup of $\mathrm{Aut}(V)$ that fixes $V_L \otimes V_{\Lambda_\tau}^{\hat{\tau}}$ point-wisely is a finite abelian group.

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In particular, $[f, h] = 1$.

Moreover, we have

$$V^{0,0} = V^{<f,h>} = \bigoplus_{\lambda+L\in J/L} V_{\lambda+L} \otimes W_\lambda,$$

where $J = \{\lambda \in L^* \mid \langle \lambda, \gamma \rangle \in \mathbb{Z}, \langle \lambda, w(\gamma) \rangle \in \mathbb{Z}\}$.

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Lemma

We have $w(J) = J$ and $\psi(V^{0,0}) = V^{0,0}$.

Note that $w^2 = 1$

Lemma

Let X be a sublattice of N such that $P_0(X) = J$. Then $V^{0,0} < V_X$ and ψ can be considered as a lift of an isometry of X in $\mathrm{Aut}(V_X)$. In particular, we have $\psi\phi\psi^{-1} = \phi$ on $V^{0,0} < V_X$.

Lemma

Let X be a sublattice of N such that $P_0(X) = J$. Then $V^{0,0} < V_X$ and ψ can be considered as a lift of an isometry of X in $\mathrm{Aut}(V_X)$*. In particular, we have $\psi\phi\psi^{-1} = \phi$ on $V^{0,0} < V_X$.*

Therefore, V, f and h satisfy the conditions in Theorem 15 and the main theorem follows.

Thank You