Induction and Wakimoto functors

Libor Křižka

October 5, 2022
Dubrovnik

Positive energy modules

- $\mathcal{V}$ – a $\mathbb{Z}$-graded vertex algebra
- $\mathcal{E}(\mathcal{V})$ – the category of graded $\mathcal{V}$-modules, $M \in \mathcal{E}(\mathcal{V})$ provided
  - M is a $\mathcal{V}$-module
  - M is a $\mathbb{C}$-graded vector space
  - $Y_M(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)}^M z^{-n-1}$ has conformal dimension m for $a \in \mathcal{V}_m$, i.e. $\deg a_{(n)}^M = -n+m-1$
- $\mathcal{E}_+(\mathcal{V})$ – the category of positive energy $\mathcal{V}$-modules, $M \in \mathcal{E}_+(\mathcal{V})$ provided
  - M belongs to $\mathcal{E}(\mathcal{V})$
  - $M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$ with $M_\lambda \neq 0$
- the top degree component

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda+n} \mapsto M_{\mathrm{top}} = M_\lambda$$

Zhu's correspondence

- $A(\mathcal{V})$ – the Zhu's algebra of $\mathcal{V}$
  - $A(\mathcal{V})$ is a unital associative algebra
  - $\pi_{\mathrm{Zhu}} \colon \mathcal{V} \to A(\mathcal{V})$ – a canonical surjective mapping
  - $\mathcal{M}(A(\mathcal{V}))$ – the category of $A(\mathcal{V})$-modules
  - $M \in \mathcal{E}_+(\mathcal{V}) \implies M_{\mathrm{top}} \in \mathcal{M}(A(\mathcal{V}))$, the action of $\pi_{\mathrm{Zhu}}(a) \in A(\mathcal{V})$ on M_{top} is given through $a^M_{(\deg a - 1)}$ for $a \in \mathcal{V}$
- Zhu's correspondence

$$M \mapsto M_{\mathrm{top}}$$

$$\left\{\begin{matrix}\text{simple positive energy} \\ \mathcal{V}\text{-modules}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\text{simple } A(\mathcal{V})\text{-modules}\right\}$$

Topological Lie algebra $U(\mathcal{V})$

- Borcherds (1986)

$$U(\mathcal{V}) = (\mathcal{V} \otimes_{\mathbb{C}} \mathbb{C}(\!(t)\!))/\operatorname{im}\partial, \qquad \partial = T \otimes \mathrm{id} + \mathrm{id} \otimes \partial_t$$

  - $U(\mathcal{V})$ is a complete topological Lie algebra
  - $a_{[n]}$ – the projection of $a \otimes t^n \in \mathcal{V} \otimes_{\mathbb{C}} \mathbb{C}(\!(t)\!)$ onto $U(\mathcal{V})$
  - $[a_{[m]}, b_{[n]}] = \sum_{k=0}^{\infty} \binom{m}{k} (a_{(k)}b)_{[m+n-k]}$ for $a, b \in \mathcal{V}$, $m, n \in \mathbb{Z}$
  - $\deg a_{[n]} = -n + \deg a - 1$ for $a \in \mathcal{V}$
- $U(\mathcal{V}) = U(\mathcal{V})_- \oplus U(\mathcal{V})_0 \oplus U(\mathcal{V})_+$ – a triangular decomposition of the Lie algebra $U(\mathcal{V})$
- $U(\mathcal{V})_0 \to A(\mathcal{V})$ – a canonical surjective homomorphism of Lie algebras

$$a_{[\deg a - 1]} \mapsto \pi_{\mathrm{Zhu}}(a)$$

Restriction functor

- $M \in \mathcal{E}(\mathcal{V}) \implies M$ is a $U(\mathcal{V})$-module

$$a_{[n]} \mapsto a^M_{(n)}$$

- $\Omega_\mathcal{V} \colon \mathcal{E}_+(\mathcal{V}) \to \mathcal{M}(A(\mathcal{V}))$ – the restriction functor

$$\Omega_\mathcal{V}(M) = \{v \in M;\, U(\mathcal{V})_- v = 0\}$$

for $M \in \mathcal{E}_+(\mathcal{V})$
  - $\Omega_\mathcal{V}(M)$ – the vector subspace of lowest weight vectors
  - $\Omega_\mathcal{V}(M)$ is a $U(\mathcal{V})_0$-module
- $\Omega_\mathcal{V}(M)$ is an $A(\mathcal{V})$-module

$$\pi_{\mathrm{zhu}}(a) \mapsto a^M_{(\deg a - 1)}$$

- $M \in \mathcal{E}_+(\mathcal{V}) \implies \Omega_\mathcal{V}(M) \supset M_{\mathrm{top}}$

Induction functor

- $\mathbb{M}_{\mathcal{V}} \colon \mathcal{M}(A(\mathcal{V})) \to \mathcal{E}_+(\mathcal{V})$ – the induction functor
  - the left adjoint functor to $\Omega_{\mathcal{V}}$

$$\mathrm{Hom}(\mathbb{M}_{\mathcal{V}}(E), M) \simeq \mathrm{Hom}(E, \Omega_{\mathcal{V}}(M))$$

  - $\mathbb{M}_{\mathcal{V}}(E)_{\mathrm{top}} \simeq E$ as $A(\mathcal{V})$-modules
- $\mathbb{M}_{\mathcal{V}}$ has the universal property

> For $M \in \mathcal{E}(\mathcal{V})$ and a morphism $\varphi \colon E \to \Omega_{\mathcal{V}}(M)$ of $A(\mathcal{V})$-modules, there exists a unique morphism $\widetilde{\varphi} \colon \mathbb{M}_{\mathcal{V}}(E) \to M$ of $\mathcal{V}$-modules which extends φ

Affine vertex algebras

- $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ – a simple Lie algebra
- $\kappa_{\mathfrak{g}}$ – the Cartan–Killing form on $\mathfrak{g}$, $\kappa_{\mathfrak{g}} = 2h^\vee \kappa_0$
- κ – a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$, $\kappa = k\kappa_0$ for $k \in \mathbb{C}$
- $\widehat{\mathfrak{g}}_\kappa$ – the affine Kac–Moody algebra associated to $\mathfrak{g}$ of level κ
  - $\mathfrak{g}_\kappa = \mathfrak{g}(\!(t)\!) \oplus \mathbb{C}c$
  - $[a_m, b_n] = [a,b]_{m+n} + \kappa(a,b)\delta_{m,-n}\, c$, $a_n = a \otimes t^n$ for $a \in \mathfrak{g}$, $n \in \mathbb{Z}$
  - c is the cental element of $\widehat{\mathfrak{g}}_\kappa$
- $\mathcal{V}^\kappa(\mathfrak{g})$ – the universal affine vertex algebra
  - $\mathcal{V}^\kappa(\mathfrak{g}) = U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}c)} \mathbb{C} \simeq U(\mathfrak{g} \otimes_{\mathbb{C}} t^{-1}\mathbb{C}[t^{-1}]) \otimes_{\mathbb{C}} \mathbb{C}$
  - $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$ for $a \in \mathfrak{g}$
  - $[a(z), b(w)] = [a,b](w)\delta(z-w) + \kappa(a,b)\partial_w \delta(z-w)$ for $a, b \in \mathfrak{g}$
- $\mathcal{V}^\kappa(\mathfrak{g})$ is an $\mathbb{N}_0$-graded vertex algebra

Induction functor

- $\mathcal{M}(\mathfrak{g})$ – the category of $\mathfrak{g}$-modules
- $\mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$ – the category of smooth $\widehat{\mathfrak{g}}_\kappa$-modules on which c acts as the identity, $M \in \mathcal{E}(\widehat{\mathfrak{g}}_\kappa)$ provided
  - $cv = v$ for $v \in M$
  - for $v \in M$, there exists $N_v \in \mathbb{N}$ such that $(\mathfrak{g} \otimes t^{N_v}\mathbb{C}[[t]])v = 0$
- $\mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$ – the category of positive energy $\widehat{\mathfrak{g}}_\kappa$-modules on which c acts as the identity
- $\mathbb{M}_{\kappa,\mathfrak{g}} \colon \mathcal{M}(\mathfrak{g}) \to \mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$

$$\mathbb{M}_{\kappa,\mathfrak{g}}(E) = U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]]\oplus\mathbb{C}c)} E$$

- $\mathbb{M}_{\kappa,\mathfrak{g}}(E)_{\mathrm{top}} \simeq E$ as $\mathfrak{g}$-modules
- $\mathbb{M}_{\kappa,\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))$ – the Verma $\widehat{\mathfrak{g}}_\kappa$-module for $\lambda \in \mathfrak{h}^*$
- $\mathbb{M}_{\kappa,\mathfrak{g}}(L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda))$ – the Weyl $\widehat{\mathfrak{g}}_\kappa$-module for $\lambda \in P_+$

Feigin–Frenkel homomorphism

- $w^e_{\kappa,\mathfrak{g}} \colon \mathcal{V}^\kappa(\mathfrak{g}) \to \mathcal{M}_{\bar{\mathfrak{n}}} \otimes_{\mathbb{C}} \mathcal{V}^{\kappa-\kappa_c}(\mathfrak{h})$

$$\begin{array}{ccc} \mathcal{V}^\kappa(\mathfrak{g}) & \xrightarrow{\ w^e_{\kappa,\mathfrak{g}}\ } & \mathcal{M}_{\bar{\mathfrak{n}}} \otimes_{\mathbb{C}} \mathcal{V}^{\kappa-\kappa_c}(\mathfrak{h}) \\ \Big\downarrow{\scriptstyle \pi_{\mathrm{Zhu}}} & & \Big\downarrow{\scriptstyle \pi_{\mathrm{Zhu}}} \\ U(\mathfrak{g}) & \xrightarrow[\ \pi_{e,\mathfrak{g}}\ ]{} & \mathcal{A}_{\bar{\mathfrak{n}}} \otimes_{\mathbb{C}} U(\mathfrak{h}) \end{array}$$

- $\mathbb{C}[\bar{\mathfrak{n}}^*]$ – an $\mathcal{A}_{\bar{\mathfrak{n}}}$-module, $\mathbb{C}_{\lambda+2\rho}$ – a 1-dim $\mathfrak{h}$-module given by $\lambda \in \mathfrak{h}^*$
- $\mathbb{C}[\bar{\mathfrak{n}}^*] \otimes_{\mathbb{C}} \mathbb{C}_{\lambda+2\rho} \simeq M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ as $\mathfrak{g}$-modules
- $\mathbb{M}_{\mathcal{M}_{\bar{\mathfrak{n}}}}(\mathbb{C}[\bar{\mathfrak{n}}^*]) \otimes_{\mathbb{C}} \mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(\mathbb{C}_{\lambda+2\rho}) \simeq \mathbb{W}_{\kappa,\mathfrak{g}}(\lambda)$ – the Wakimoto $\widehat{\mathfrak{g}}_\kappa$-module for $\lambda \in \mathfrak{h}^*$

Beilinson–Bernstein correspondence

- $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ – a simple Lie algebra
- G – a connected semisimple algebraic group with its Lie algebra $\mathfrak{g}$
- B – the Borel subgroup of G
- $X = G/B$ – the flag variety for G

$$\lambda \in \mathfrak{h}^* \quad \Longrightarrow \quad \mathcal{D}_X^\lambda \quad \Longrightarrow \quad \Phi_X^\lambda \colon U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\lambda)$$

- $w \in W \simeq N_G(H)/H$ – the Weyl group of $\mathfrak{g}$

$$\pi_{w,\mathfrak{g}}^\lambda \colon U(\mathfrak{g}) \xrightarrow{\Phi_X^\lambda} \Gamma(X, \mathcal{D}_X^\lambda) \to \Gamma(U_w, \mathcal{D}_X^\lambda) \simeq \Gamma(U_w, \mathcal{D}_X) \simeq \mathcal{A}_{\bar{\mathfrak{n}}}$$

$U_w = \dot{w}\overline{N}B \subset G/B$, $U_w \simeq \mathbb{C}^{\dim \mathfrak{n}}$, $\dot{w} \in N_G(H)$ – a Tits lift of w

- $\pi_{w,\mathfrak{g}}^\lambda$ – a homomorphism of associative algebras

Sheaf of rings of twisted differential operators

- $\mathfrak{g} = \mathfrak{sl}_2$, $X = G/B \simeq \mathbb{CP}^1$, $X = U_e \cup U_s$
- $W = \{e, s\}$, $\lambda \in \mathbb{C} \simeq \mathfrak{h}^*$ $(\lambda \mapsto \lambda\omega)$

$$\begin{matrix} \pi^\lambda_{e,\mathfrak{g}}\colon \\ \pi^\lambda_{s,\mathfrak{g}}\colon \end{matrix} \quad U(\mathfrak{g}) \xrightarrow{\Phi^\lambda_X} \Gamma(X, \mathcal{D}^\lambda_X) \to \begin{cases} \Gamma(U_e, \mathcal{D}^\lambda_X) \simeq \Gamma(U_e, \mathcal{D}_X) \simeq \mathcal{A}_{\mathbb{C}} \\ \Gamma(U_s, \mathcal{D}^\lambda_X) \simeq \Gamma(U_s, \mathcal{D}_X) \simeq \mathcal{A}_{\mathbb{C}} \end{cases}$$

- $$\begin{aligned} \pi^\lambda_{e,\mathfrak{g}}(f) &= -\partial_x, & \pi^\lambda_{s,\mathfrak{g}}(f) &= -y^2\partial_y - (\lambda+1)y, \\ \pi^\lambda_{e,\mathfrak{g}}(h) &= 2x\partial_x + \lambda + 1, & \pi^\lambda_{s,\mathfrak{g}}(h) &= -2y\partial_y - \lambda - 1, \\ \pi^\lambda_{e,\mathfrak{g}}(e) &= x^2\partial_x + (\lambda+1)x, & \pi^\lambda_{s,\mathfrak{g}}(e) &= \partial_y, \end{aligned}$$
- gluing rules

$$\begin{aligned} x &= -\frac{1}{y}, & y &= -\frac{1}{x}, \\ \partial_x &= y^2\partial_y + (\lambda+1)y, & \partial_y &= x^2\partial_x + (\lambda+1)x \end{aligned}$$

Universal sheaf of rings of twisted differential operators

- $\mathfrak{g} = \mathfrak{sl}_2$, $X = G/B \simeq \mathbb{CP}^1$, $X = U_e \cup U_s$
- $W = \{e, s\}$

$$\begin{matrix} \pi_{e,\mathfrak{g}} \colon \\ \pi_{s,\mathfrak{g}} \colon \end{matrix} \quad U(\mathfrak{g}) \xrightarrow{\Phi_X} \Gamma(X, \widetilde{\mathcal{D}}_X) \to \begin{cases} \Gamma(U_e, \widetilde{\mathcal{D}}_X) \simeq \mathcal{A}_{\mathbb{C}} \otimes_{\mathbb{C}} U(\mathfrak{h}) \\ \Gamma(U_s, \widetilde{\mathcal{D}}_X) \simeq \mathcal{A}_{\mathbb{C}} \otimes_{\mathbb{C}} U(\mathfrak{h}) \end{cases}$$

-

$$\pi_{e,\mathfrak{g}}(f) = -\partial_x, \qquad \pi_{s,\mathfrak{g}}(f) = -y^2\partial_y - yh,$$
$$\pi_{e,\mathfrak{g}}(h) = 2x\partial_x + h, \qquad \pi_{s,\mathfrak{g}}(h) = -2y\partial_y - h,$$
$$\pi_{e,\mathfrak{g}}(e) = x^2\partial_x + xh, \qquad \pi_{s,\mathfrak{g}}(e) = \partial_y,$$

- gluing rules

$$x = -\frac{1}{y}, \qquad y = -\frac{1}{x},$$
$$\partial_x = y^2\partial_y + yh, \qquad \partial_y = x^2\partial_x + xh$$

Wakimoto free field realization

- Wakimoto (1986), Feigin–Frenkel (1988)

$$\pi_{e,\mathfrak{g}} \colon U(\mathfrak{g}) \to \mathcal{A}_{\bar{\mathfrak{n}}} \otimes_{\mathbb{C}} U(\mathfrak{h})$$

$\Downarrow$ chiralization

$$w^{e}_{\kappa,\mathfrak{g}} \colon \mathcal{V}^{\kappa}(\mathfrak{g}) \to \mathcal{M}_{\bar{\mathfrak{n}}} \otimes_{\mathbb{C}} \mathcal{V}^{\kappa-\kappa_c}(\mathfrak{h})$$

- κ_c – the critical level

$$\kappa_c(a,b) = -\operatorname{tr}_{\mathfrak{g}/\mathfrak{b}}(\operatorname{ad}(a)\operatorname{ad}(b)), \quad a,b \in \mathfrak{b}$$

- $\mathfrak{g} = \mathfrak{sl}_n$, $\kappa_0(a,b) = \operatorname{tr}(ab)$ for $a,b \in \mathfrak{g}$, $\kappa_c = -n\kappa_0$, $n = h^\vee$

Wakimoto free field realization

- $\mathcal{D}_X^\kappa$ – a sheaf of vertex algebras over X

$$\begin{matrix} w^e_{\kappa,\mathfrak{g}}: \\ w^s_{\kappa,\mathfrak{g}}: \end{matrix} \quad \mathcal{V}^\kappa(\mathfrak{g}) \xrightarrow{w_{\kappa,\mathfrak{g}}} \Gamma(X, \mathcal{D}_X^\kappa) \to \begin{cases} \Gamma(U_e, \mathcal{D}_X^\kappa) \simeq \mathcal{M}_\mathbb{C} \otimes_\mathbb{C} \mathcal{V}^{\kappa-\kappa_c}(\mathfrak{h}) \\ \Gamma(U_s, \mathcal{D}_X^\kappa) \simeq \mathcal{M}_\mathbb{C} \otimes_\mathbb{C} \mathcal{V}^{\kappa-\kappa_c}(\mathfrak{h}) \end{cases}$$

- $$\begin{aligned} w^e_{\kappa,\mathfrak{g}}(f(z)) &= -a_x(z), \\ w^e_{\kappa,\mathfrak{g}}(h(z)) &= 2{:}a_x^*(z)a_x(z){:} + b_x(z), \\ w^e_{\kappa,\mathfrak{g}}(e(z)) &= {:}a_x^*(z)^2a_x(z){:} - k\partial_z a_x^*(z) + a_x^*(z)b_x(z) \end{aligned}$$

$$\begin{aligned} w^s_{\kappa,\mathfrak{g}}(f(z)) &= -{:}a_y^*(z)^2a_y(z){:} + k\partial_z a_y^*(z) - a_y^*(z)b_y(z) \\ w^s_{\kappa,\mathfrak{g}}(h(z)) &= -2{:}a_y^*(z)a_y(z){:} - b_y(z), \\ w^s_{\kappa,\mathfrak{g}}(e(z)) &= a_y(z), \end{aligned}$$

Wakimoto free field realization

- $\mathcal{V}^\kappa(\mathfrak{g})$ – the universal affine vertex algebra

$$[h(z), e(w)] = 2e(w)\delta(z-w), \qquad [h(z), f(w)] = -2f(w)\delta(z-w),$$
$$[e(z), f(w)] = h(w)\delta(z-w) + k\partial_w\delta(z-w),$$
$$[h(z), h(w)] = 2k\partial_w\delta(z-w)$$

- $\mathcal{M}_\mathbb{C}$ – the Weyl vertex algebra

$$[a_x(z), a_x^*(w)] = \delta(z-w), \qquad [a_y(z), a_y^*(w)] = \delta(z-w)$$

- $\mathcal{V}^{\kappa-\kappa_c}(\mathfrak{h})$ – the Heisenberg vertex algebra

$$[b_x(z), b_x(w)] = 2(k+2)\partial_w\delta(z-w),$$
$$[b_y(z), b_y(w)] = 2(k+2)\partial_w\delta(z-w)$$

Wakimoto free field realization

- $\pi_{e,\mathfrak{g}} \colon U(\mathfrak{g}) \to \mathcal{A}_{\bar{\mathfrak{n}}} \otimes_{\mathbb{C}} U(\mathfrak{h})$

$$\begin{aligned} \pi_{e,\mathfrak{g}}(f) &= -\partial_x, \\ \pi_{e,\mathfrak{g}}(h) &= 2x\partial_x + h, \\ \pi_{e,\mathfrak{g}}(e) &= x^2\partial_x + xh, \end{aligned}$$

- $w^e_{\kappa,\mathfrak{g}} \colon \mathcal{V}^\kappa(\mathfrak{g}) \to \mathcal{M}_{\bar{\mathfrak{n}}} \otimes_{\mathbb{C}} \mathcal{V}^{\kappa-\kappa_c}(\mathfrak{h})$

$$\begin{aligned} w^e_{\kappa,\mathfrak{g}}(f(z)) &= -a_x(z), \\ w^e_{\kappa,\mathfrak{g}}(h(z)) &= 2{:}a_x^*(z)a_x(z){:} + b_x(z), \\ w^e_{\kappa,\mathfrak{g}}(e(z)) &= {:}a_x^*(z)^2 a_x(z){:} - k\partial_z a_x^*(z) + a_x^*(z) b_x(z) \end{aligned}$$

Sheaf of differential operators $\mathcal{D}_{\mathbb{C}}$

- $\mathcal{A}_{\mathbb{C}}$ – the Weyl algebra

$$\mathbb{C} \quad \Longrightarrow \quad \mathcal{A}_{\mathbb{C}} = \Gamma(\mathbb{C}, \mathcal{D}_{\mathbb{C}})$$

- $f \in \mathcal{O}_{\mathbb{C}}(\mathbb{C})$ – a polynomial $\quad \Longrightarrow \quad U_f = \{x \in \mathbb{C};\, f(x) \neq 0\}$ – a principal open subset of $\mathbb{C}$

$$\mathcal{D}_{\mathbb{C}}(U_f) \simeq \mathcal{O}_{\mathbb{C}}(\mathbb{C})_f \otimes_{\mathcal{O}_{\mathbb{C}}(\mathbb{C})} \mathcal{A}_{\mathbb{C}}$$

- The sheaf $\mathcal{D}_{\mathbb{C}}$ is uniquely determined by the global sections, i.e by the Weyl algebra $\mathcal{A}_{\mathbb{C}}$

Sheaf of chiral differential operators $\mathcal{D}^{\mathrm{ch}}_{\mathbb{C}}$

- $\mathcal{M}_{\mathbb{C}}$ – the Weyl vertex algebra (the algebra of chiral differential operators)

$$\mathbb{C} \quad \Longrightarrow \quad \mathcal{M}_{\mathbb{C}} = \Gamma(\mathbb{C}, \mathcal{D}^{\mathrm{ch}}_{\mathbb{C}})$$

- $f \in \mathcal{O}_{\mathbb{C}}(\mathbb{C})$ – a polynomial $\quad \Longrightarrow \quad U_f = \{x \in \mathbb{C};\ f(x) \neq 0\}$ – a principal open subset of $\mathbb{C}$

$$\mathcal{D}^{\mathrm{ch}}_{\mathbb{C}}(U_f) \simeq \mathcal{O}_{\mathbb{C}}(\mathbb{C})_f \otimes_{\mathcal{O}_{\mathbb{C}}(\mathbb{C})} \mathcal{M}_{\mathbb{C}}$$

- The sheaf $\mathcal{D}^{\mathrm{ch}}_{\mathbb{C}}$ is uniquely determined by the global sections, i.e by the Weyl vertex algebra $\mathcal{M}_{\mathbb{C}}$

Weyl vertex algebra $\mathcal{M}_{\mathbb{C}}$

- $x\colon \mathbb{C} \to \mathbb{C}$ – the canonical linear coordinate function
- $\mathcal{M}_{\mathbb{C}} = \mathbb{C}[\dots, \partial_{x_{-2}}, \partial_{x_{-1}}, x_0, x_1, \dots]$
- fields

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad a^*(z) = \sum_{n\in\mathbb{Z}} a^* n z^{-n},$$

  where $a_n = \partial_{x_n}$ and $a_n^* = x_{-n}$ for $n \in \mathbb{Z}$
- $|0\rangle = 1$ – the vacuum
- commutation relations

$$[a(z), a(w)] = 0, \quad [a(z), a^*(w)] = \delta(z-w), \quad [a^*(z), a^*(w)] = 0$$

Sheaf of chiral differential operators $\mathcal{D}^{\mathrm{ch}}_{\mathbb{C}}$

- Example

$$\mathcal{D}^{\mathrm{ch}}_{\mathbb{C}}(\mathbb{C}^*) \simeq \mathbb{C}[\dots, \partial_{x_{-2}}, \partial_{x_{-1}}, x_0^{\pm 1}, x_1, \dots]$$

- $Y(x_0^{-N}|0\rangle, z) = a^*(z)^{-N}$ for $N \in \mathbb{N}$ ...???
- Feigin's trick

$$\begin{aligned} a^*(z)^{-N} &= \frac{1}{(x_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} x_n z^n)^N} \\ &= x_0^{-N} \sum_{j=0}^{\infty} \binom{-N}{j} x_0^{-j} \Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} x_n z^n \Big)^j \\ &= x_0^{-N} \sum_{j=0}^{\infty} (-1)^j \binom{N+j-1}{j} x_0^{-j} \Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} x_n z^n \Big)^j \end{aligned}$$

Sheaf of twisted chiral differential operators $\mathcal{D}_X^\kappa$

- $X = U_e \cup U_s$

$$\Gamma(U_e, \mathcal{D}_X^\kappa) \simeq \mathcal{M}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{V}^{\kappa-\kappa_c}(\mathfrak{h}),\ \Gamma(U_s, \mathcal{D}_X^\kappa) \simeq \mathcal{M}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{V}^{\kappa-\kappa_c}(\mathfrak{h})$$

- gluing rules

$$\begin{aligned} a_x^*(z) &= -a_y^*(z)^{-1}, \\ a_x(z) &= :a_y^*(z)^2 a_y(z): - k\partial_z a_y^*(z) + a_y^*(z) b_y(z), \\ b_x(z) &= b_y(z) - 2(k+2) a_y^*(z)^{-1} \partial_z a_y^*(z) \end{aligned}$$

$$\begin{aligned} a_y^*(z) &= -a_x^*(z)^{-1}, \\ a_y(z) &= :a_x^*(z)^2 a_x(z): - k\partial_z a_x^*(z) + a_x^*(z) b_x(z), \\ b_y(z) &= b_x(z) - 2(k+2) a_x^*(z)^{-1} \partial_z a_x^*(z) \end{aligned}$$

- $\kappa = k\kappa_0$, $\kappa_c = -2\kappa_0$, $\kappa_0(a,b) = \operatorname{tr}(ab)$ for $a, b \in \mathfrak{g}$

Chiral differential operators

- $\mathcal{D}_X^\kappa$ – a sheaf of twisted chiral differential operators

$$\kappa \quad \Longrightarrow \quad \mathcal{D}_X^\kappa \quad \Longrightarrow \quad w_{\kappa,\mathfrak{g}} \colon \mathcal{V}^\kappa(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\kappa)$$

- $\mathcal{D}_X^{\kappa_c}$ [Arakawa, Chebotarov, Malikov 2011]
- $\mathcal{D}_X^\kappa$ – a $\mathbb{N}_0$-graded sheaf of vertex algebras over X
- $A(\mathcal{D}_X^\kappa)$ – a sheaf of associative algebras over X

$$U \subset X \mapsto A(\mathcal{D}_X^\kappa(U))$$

- $A(\mathcal{D}_X^\kappa) \simeq \widetilde{\mathcal{D}}_X$ as sheaves of associative algebras over X
- $\mathcal{E}(\mathcal{D}_X^\kappa)$, $\mathcal{E}_+(\mathcal{D}_X^\kappa)$

Wakimoto functor

- $\Omega_{\mathcal{D}_X^\kappa} : \mathcal{E}_+(\mathcal{D}_X^\kappa) \to \mathcal{M}(\widetilde{\mathcal{D}}_X)$ – the restriction functor
- $\mathbb{M}_{\mathcal{D}_X^\kappa} : \mathcal{M}(\widetilde{\mathcal{D}}_X) \to \mathcal{E}_+(\mathcal{D}_X^\kappa)$ – the induction functor
- $\mathcal{M}(\mathfrak{g}, \chi)$ – the category of $\mathfrak{g}$-modules with central character χ
- $\mathbb{W}_{\kappa,\mathfrak{g}}^\lambda : \mathcal{M}(\mathfrak{g}, \chi_\lambda) \to \mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$ for $\lambda \in \mathfrak{h}^*$

$$\mathcal{M}(\mathfrak{g}, \chi_\lambda) \xrightarrow{\Delta} \mathcal{M}(\mathcal{D}_X^\lambda) \longrightarrow \mathcal{M}(\widetilde{\mathcal{D}}_X) \xrightarrow{\mathbb{M}} \mathcal{E}_+(\mathcal{D}_X^\kappa) \xrightarrow{\Gamma(X,?)} \mathcal{E}_+(\widehat{\mathfrak{g}}_\kappa)$$

- $w_{\kappa,\mathfrak{g}} : \mathcal{V}^\kappa(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\kappa)$
- $\mathbb{W}_{\kappa,\mathfrak{g}}^\lambda(E)_{\rm top} \simeq E$ as $\mathfrak{g}$-modules provided $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{N}$ (antidominant)
- $\mathbb{M}_{\kappa,\mathfrak{g}}(E) \to \mathbb{W}_{\kappa,\mathfrak{g}}^\lambda(E)$ if $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{N}$ (antidominant)

Thank you for your attention!