Double Poisson (vertex) algebras

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Joint work with L. A. Cónsul & D. Fernández

Physics	**Graded geometry**	**Geometry**	**Algebra**
Classical Mechanics	Symplectic $\mathbb{N}Q$-manifold of weight 1	Lie algebroids/ Lie–Rinehart algebras	Poisson
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?	Bisymplectic $\mathbb{N}Q$-algebras of weight 1	Double Lie algebroids/ Double Lie–Rinehart algebras	Double Poisson
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The arc algebra of a Poisson algebra

$$\{\text{Commutative algebras}\} \longleftrightarrow \left\{\begin{array}{c}\text{differential } \mathbb{Z}_+\text{- graded}\\ \text{commutative algebras}\end{array}\right\}$$

$$R \quad \mapsto \quad JR = R \oplus \Omega^1 R \oplus \cdots$$

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$$\{\text{Poisson Algebras}\} \longleftrightarrow \{\text{Poisson vertex algebras}\}$$

$$J_0 \qquad \mapsfrom \qquad J$$

Question: Why is this slide here?

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Lie algebroids

$P = \bigoplus_{k \geq 0} P_k$ a $\mathbb{Z}_+$-graded Poisson algebra. $a \in P_p$, $b \in P_q$.

$$a \cdot b \in P_{p+q}, \qquad \{a, b\} \in P_{p+q-1}.$$

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$\mathscr{O} := P_0$ is a commutative algebra.

$\mathscr{L} := P_1$ is a Lie algebra.

$\mathscr{L}$ is a $\mathscr{O}$-module.

$\mathscr{L}$ acts of $\mathscr{O}$ by derivations: $\mathscr{L} \ni \tau \mapsto \{\tau, \cdot\}$.

$[\tau, f \cdot \upsilon] = \tau(f) \cdot \upsilon + f[\tau, \upsilon]$. $\tau, \upsilon \in \mathscr{L}$, $f \in \mathscr{O}$.

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$X := \mathrm{Spec}\big(\mathrm{Sym}_{\mathscr{O}} \mathscr{L}[1]\big)$ is a Symplectic $\mathbb{N}Q$-manifold of weight 1.

[Kontsevich (?), Vaintrob (1997)]

Let $(\mathscr{O}, \mathscr{L})$ be a Lie algebroid, there exist a (unique up to isomorphism) graded Poisson algebra P with $P_0 = \mathscr{O}$ and $P_1 = \mathscr{L}$, generated in degrees 0 and 1, and universal with these properties.

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Example. Any lie algebra $\mathfrak{g}$ over k is a Lie algebroid $(k, \mathfrak{g})$

Example. Let $\mathscr{O}$ be a commutative algebra over k. $(\mathscr{O}, \mathrm{Der}_k \mathscr{O})$ is a Lie algebroid.

Example. Let X be a Poisson manifold

- $\mathscr{O} = \mathscr{O}_X$ is a Poisson algebra.
- $\mathscr{O}, \Omega^1_X$ is a Lie algebroid.

Courant-Dorfman algebroids

$P = \oplus_{k \geq 0} P_k$ a $\mathbb{Z}_+$-graded PVA. $a \in P_p$, $b \in P_q$.

$$Ta \in P_{p+1}, \qquad a \cdot b \in P_{p+q}, \qquad \{a_\lambda b\} = \sum_{j \geq 0} \lambda^j c_j, \quad c_j \in P_{p+q-j-1}.$$

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$\{a_\lambda b\} = [a,b] + \lambda\langle a,b\rangle$, $a, b \in \mathscr{L}$.

$$[\cdot,\cdot] : \mathscr{L} \otimes \mathscr{L} \to \mathscr{L}, \quad \langle\cdot,\cdot\rangle : \mathscr{L} \otimes \mathscr{L} \to \mathscr{O}.$$

$(\mathscr{L}, [,])$ is a Leibniz algebra. $\langle,\rangle$ is symmetric.

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$$[a, b] + [b, a] = 2T\langle a, b\rangle, \qquad \langle [a, b], c\rangle + \langle b, [a, c]\rangle = a \cdot \langle b, c\rangle.$$

Let $(\mathscr{O}, \mathscr{L})$ be a Courant-Dorfman algebroid, there exist a (unique up to isomorphism) graded Poisson vertex algebra P with $P_0 = \mathscr{O}$ and $P_1 = \mathscr{L}$, generated in degrees 0 and 1, and universal with these properties.

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Example. Any lie algebra $\mathfrak{g}$ with an invariant symmetric form $\langle \cdot, \cdot \rangle$ over k is a Courant-Dorfman algebroid $(k, \mathfrak{g})$

Example. Let $\mathscr{O}$ be a commutative algebra over k. $\mathscr{O}$ together with $\mathscr{L} := \mathrm{Der}_k \mathscr{O} \oplus \Omega^1_{\mathscr{O}}$ is a Courant-Dorfman algebroid.

[H. (2008), Roytenberg (2009), Ekstrand, Zabzine (2011)]

Non commutative versions

Let $(\mathscr{O}, \mathscr{L})$ be a Lie algebroid. There exists a unique associative algebra $U(\mathscr{L})$ together with embeddings

$$\imath : \mathscr{O} \hookrightarrow U(\mathscr{L}), \qquad j : \mathscr{L} \hookrightarrow U(\mathscr{L}),$$

such that

- $\imath$ is a morphism of algebras.
- j is a morphism of Lie algebras.
- $\imath(f \cdot a) = \imath(f) \cdot j(a)$ and $\imath(a \cdot f) = [j(a), \imath(f)]$ for $f \in \mathscr{O}$, $a \in \mathscr{L}$.
- $U(\mathscr{L})$ is universal with these properties.

$U(\mathscr{L})$ is filtered, and its associated graded is $\mathrm{Sym}_{\mathscr{O}} \mathscr{L}$.

When $(\mathscr{O}, \mathscr{L}) = (k, \mathfrak{g})$, $U(\mathscr{L}) = U(\mathfrak{g})$.

When $X = \mathrm{Spec}(\mathscr{O})$, $\mathscr{L} = \mathrm{Der}_k \mathscr{O}$, $U(\mathscr{L}) = \mathcal{D}_X$, differential operators on X.

[Beilinson, Schechtman (1988)]

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- $\imath$ is a morphism of algebras: $\imath(fg) = \imath(f) \cdot \imath(g)$.
- $[j(a)_\lambda j(b)] = j[a, b] + \lambda \imath\langle a, b\rangle$.
- $\imath(f \cdot a) = \imath(f) \cdot j(a)$, and $\imath(a \cdot f) = [j(a)_\lambda \imath(f)]$ for $f \in \mathscr{O}$, $a \in \mathscr{L}$.
- $jTf = T\imath f$.
- $V(\mathscr{L})$ is universal with these properties.

When $(\mathscr{O}, \mathscr{L}) = (k, \mathfrak{g})$, $V(\mathscr{L})$ is the affine Kac-Moody vertex algebra.

[Gorbounov, Malikov, Schechtman (2004), Beilinson-Drinfeld (2006)]

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Another flavor of non-commutativity

Definition. A an associative finitely generated R-algebra.

$$0 \to \Omega^1_{A/R} \hookrightarrow A \otimes_R A \to A.$$

A is smooth if $\Omega^1_{A/R}$ is is projective as an A-bimodule.

$$\Omega^\bullet_{A/R} := \bigoplus_{k \geq 0} \left(\Omega^1_{A/R}\right)^{\otimes_A k}, \quad d : \Omega^\bullet_{A/R} \to \Omega^{\bullet+1}_{A/R}.$$

$$DR^\bullet_{A/R} := \Omega^\bullet_{A/R}/[\Omega^\bullet_{A/R}, \Omega^\bullet_{A/R}].$$

$$\mathbb{D}er_{A/R} := \mathrm{Der}_{A/R}(A, A \otimes A) \simeq (\Omega^1_{A/R})^\vee := \mathrm{Hom}_{A-A}(\Omega^1_{A/R}, A \otimes A).$$

[Crawley-Boevey,Etingof,Ginzburg (2007)]

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Double Poisson Algebras

A an associative R algebra. A double Poisson algebra is

$$\{\!\{\cdot,\cdot\}\!\} : A \times A \to A^{\otimes 2}, \quad (a,b) \mapsto \{\!\{a,b\}\!\}.$$

$\{\!\{a,\cdot\}\!\} : A \to A^{\otimes 2}$ is an R-linear A-derivation.

$\{\!\{a,b\}\!\} = -\{\!\{b,a\}\!\}^{\sigma}$,

$$\{\!\{a,\{\!\{b,c\}\!\}\}\!\}_L + \tau_{(123)}\{\!\{b,\{\!\{c,a\}\!\}\}\!\}_L + \tau_{(132)}\{\!\{c,\{\!\{a,b\}\!\}\}\!\}_L = 0.$$

where:

$$(a \otimes b)^{\sigma} := b \otimes a, \qquad \tau_{(123)} a \otimes b \otimes c = c \otimes a \otimes b.$$

$$\left\{\!\!\left\{a, b_1 \otimes \cdots \otimes b_n\right\}\!\!\right\}_L := \left\{\!\!\left\{a, b_1\right\}\!\!\right\} \otimes b_2 \otimes \cdots \otimes b_n \in A^{\otimes(n+1)}$$

The double Schouten-Nijenhuis bracket

$$\mathbb{D}er_R A \otimes A \mapsto A \otimes A, \qquad X \otimes a \mapsto \{\!\{X, a\}\!\} := X(a).$$

$$\mathbb{D}er_{A/R} \otimes \mathbb{D}er_{A/R} \to \mathbb{D}er_{A/R} \otimes A \;\oplus\; A \otimes \mathbb{D}er_{A/R}.$$

$$\{\!\{X, Y\}\!\}_l^{\sim} := (X \otimes 1_A) \circ Y - (1_A \otimes Y) \circ X : A \to A^{\otimes 3}.$$

$$\{\!\{X, Y\}\!\} := \{\!\{X, Y\}\!\}_l^{\sim} + \{\!\{X, Y\}\!\}_r^{\sim}.$$

Using $\quad \mathbb{D}er_{A/R} \otimes A \simeq \mathrm{Der}_R(A, A^{\otimes 3}).$

This bracket can be extended to a double Poisson algebra structure on $T_A \mathbb{D}er_{A/R}$

[V. den Bergh(2008)]

Double Cartan calculus

For $X : A \to A^{\otimes l}$ a derivation (relative to R), we have a unique derivation of degree -1 defined by successively extending

$$\iota_X : \Omega^1_{A/R} \to A^{\otimes l}, \qquad X = \iota_X \circ d.$$

$$\iota_X : \Omega^\bullet_{A/R} \to \left(\Omega^\bullet_{A/R}\right)^{\otimes l} \qquad \text{Leibniz rule.}$$

$$\iota_X : T_k(\Omega^\bullet_{A/R}) \to T_k(\Omega^\bullet_{A/R}) \qquad \text{Leibniz rule.}$$

$$\iota_X \in \mathrm{Der}^{-1}_R\, T_k\left(\Omega^\bullet_{A/R}\right), \qquad d \in \mathrm{Der}^1_R\, T_k\left(\Omega^\bullet_{A/R}\right).$$

Define

$$L_X = \iota_X \circ d + d \circ \iota_X : T_k(\Omega^\bullet_{A/R}) \to T_k(\Omega^\bullet_{A/R}).$$

[V. den Bergh(2008), Crawley-Boevey,Etingof,Ginzburg (2007), A. Cónsul, Fernández, H. (2022)]

Double Cartan calculus

Theorem. Let A be an R-algebra that is finitely generated over R. Then for all $X, Y \in \mathbb{D}er_R A$, we have

$$\begin{aligned} d^2 &= 0, \\ L_X &= [d, i_X], \\ [d, L_X] &= 0, \\ \{\!\{ i_X, i_Y \}\!\} &= 0, \\ \{\!\{ i_X, L_Y \}\!\} &= \{\!\{ L_X, i_Y \}\!\} = i_{\{\!\{X,Y\}\!\}}, \\ \{\!\{ L_X, L_Y \}\!\} &= L_{\{\!\{X,Y\}\!\}}. \end{aligned}$$

Double Poisson vertex algebras

Definition A double Poisson vertex algebra is a differential associative algebra (V, ∂) with a double λ bracket

$$\{\!\{\cdot, \cdot\}\!\} : V \otimes V \to (V \otimes V)[\lambda],$$

Satisfying

$$\{\!\{\partial a_\lambda b\}\!\} = -\lambda \{\!\{a_\lambda b\}\!\}, \qquad \{\!\{a_\lambda \partial b\}\!\} = (\partial + \lambda) \{\!\{a_\lambda b\}\!\}.$$
$$\{\!\{a_\lambda b\}\!\} = - \{\!\{b_{-\lambda-\partial} a\}\!\}^\sigma.$$
$$\{\!\{a_\lambda bc\}\!\} = b \{\!\{a_\lambda c\}\!\} + \{\!\{a_\lambda b\}\!\} c.$$
$$\{\!\{a_\lambda \{\!\{b_\mu c\}\!\}\}\!\}_L = \{\!\{b_\mu \{\!\{a_\lambda c\}\!\}\}\!\}_R + \{\!\{ \{\!\{a_\lambda b\}\!\}_{\lambda+\mu} c\}\!\}_L.$$

$$\{\!\{a_\lambda (b \otimes c)\}\!\}_L := \{\!\{a_\lambda b\}\!\} \otimes c, \qquad \{\!\{(a \otimes b)_\lambda c\}\!\}_L = \{\!\{a_{\lambda+\partial} c\}\!\}_\rightarrow \otimes_1 b.$$

Where $(a \otimes b) \otimes_1 c = a \otimes c \otimes b$, and the arrow means apply ∂ to b.

[De Sole, Kac, Valeri (2015)]

Double Courant-Dorfman algebroids

V a $\mathbb{Z}_+$-graded double PVA, $a \in V_p$, $b \in V_q$,

$$Ta \in V_{p+1}, \quad a \cdot b \in V_{p+q}, \quad \{\!\{a_\lambda b\}\!\} = \sum_{j \geq 0} \lambda^j c_j, \quad c_j \in (V \otimes V)_{p+q-j-1}.$$

$\mathcal{O} := V_0$ is an associative unital algebra.

$\mathcal{L} := V_1$ is a $\mathcal{O}$-bimodule.

$\partial : \mathcal{O} \to \mathcal{L}$ is a derivation.

$\{\!\{a_\lambda b\}\!\} = [\![a, b]\!] + \lambda \langle\!\langle a, b \rangle\!\rangle$ where

$$[\![a, b]\!] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{O} \oplus \mathcal{O} \otimes \mathcal{L},$$
$$\langle\!\langle a, b \rangle\!\rangle : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O} \otimes \mathcal{O}.$$

Double Courant-Dorfman algebroids

These data satisfies the following axioms, for all $f, g \in \mathcal{O}$ and $a, b, c \in \mathscr{L}$:

$$
\begin{aligned}
[\![a,b]\!] + [\![b,a]\!]^{\sigma} &= \partial \langle\!\langle a,b \rangle\!\rangle, \\
[\![\partial f, a]\!] &= 0, \\
\langle\!\langle \partial f, \partial g \rangle\!\rangle &= 0, \\
[\![a, bf]\!] &= [\![a,b]\!] f + b \langle\!\langle a, \partial f \rangle\!\rangle, \\
[\![a, fb]\!] &= f [\![a,b]\!] + \langle\!\langle a, \partial f \rangle\!\rangle b, \\
[\![a, [\![b,c]\!]]\!]_L &= [\![[\![a,b]\!], c]\!]_L + [\![b, [\![a,c]\!]]\!]_R, \\
\langle\!\langle a, \partial \langle\!\langle b,c \rangle\!\rangle \rangle\!\rangle_L &= \langle\!\langle [\![a,b]\!], c \rangle\!\rangle_L + \langle\!\langle b, [\![a,c]\!] \rangle\!\rangle_R.
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[\![a, bf]\!] &= [\![a, b]\!] f + b \langle\!\langle a, \partial f \rangle\!\rangle, \\
[\![a, fb]\!] &= f [\![a, b]\!] + \langle\!\langle a, \partial f \rangle\!\rangle b, \\
[\![a, [\![b, c]\!]]\!]_L &= [\![[\![a, b]\!], c]\!]_L + [\![b, [\![a, c]\!]]\!]_R, \\
\langle\!\langle a, \partial \langle\!\langle b, c \rangle\!\rangle \rangle\!\rangle_L &= \langle\!\langle [\![a, b]\!], c \rangle\!\rangle_L + \langle\!\langle b, [\![a, c]\!] \rangle\!\rangle_R.
\end{aligned}
$$

$\mathscr{L}$ acts on $\mathscr{O}$ by double derivations: $\mathscr{L} \to \mathbb{D}er\mathscr{O}$, $a \mapsto \langle\!\langle a, \partial \cdot \rangle\!\rangle$.

$i_\partial : \Omega^1_{\mathscr{O}} \to \mathscr{L}$ by contraction by $\partial \in \mathrm{Der}(\mathscr{O}, \mathscr{L})$

Exact Double Courant-Dorfman algebroids

A double Courant-Dorfman algebroid is exact if the sequence

$$0 \to \Omega^1_{\mathscr{O}} \to \mathscr{L} \to \mathbb{D}er\mathscr{O} \to 0,$$

is exact.

If $\langle\langle \cdot, \cdot \rangle\rangle : \mathscr{L} \otimes \mathscr{L} \to \mathscr{O} \otimes \mathscr{O}$ is non-degenerate, it gives rise to a non-degenerate pairing

$$\Omega^1_{\mathscr{O}} \xrightarrow{\sim} \mathbb{D}er^{\vee}.$$

This happens if $\mathscr{O}$ is smooth.

$\mathscr{L} = \Omega^1_{\mathscr{O}} \oplus \mathbb{D}er\mathscr{O}$ is a Courant-Dorfman algebra.

$$[\![X + \alpha, Y + \beta]\!] = \{\!\{X, Y\}\!\} + L_X\beta - (\iota_Y\alpha)^{\sigma}.$$

Exact Double Courant-Dorfman algebroids

Let $\mathscr{O}$ be smooth, choose a isotropic splitting of

$$0 \to \Omega^1_{\mathscr{O}} \to \mathscr{L} \to \mathbb{D}er\mathscr{O} \to 0.$$

From the bracket we obtain a map

$$\mathbb{D}er\mathscr{O} \otimes \mathbb{D}er\mathscr{O} \to \Omega^1_{\mathscr{O}} \otimes \mathscr{O} \oplus \mathscr{O} \otimes \Omega^1_{\mathscr{O}}.$$

Since $\mathscr{O}$ is smooth we can dualize to

$$(\mathbb{D}er\mathscr{O})^{\otimes 3} \to \mathscr{O} \otimes \mathscr{O}.$$

Hence we obtain an element

$$H \in DR^3\mathscr{O} \twoheadleftarrow \left(\Omega^1_{\mathscr{O}}\right)^{\otimes 3}.$$

From the Jacobi condition it follows that H is is closed, and choosing another splitting we obtain $H + dB$ for some $B \in DR^2\mathscr{O}$.

Theorem Let $(\mathscr{O}, \mathscr{L})$ be a double Courant-Dorfman algebroid, there exists a unique $\mathbb{Z}_+$-graded double Poisson vertex algebra V of degree -1 such that $V_0 = \mathscr{O}$, $V_1 = \mathscr{L}$, it's generated in degrees 0 and 1, and it's universal with these properties.

[A. Cónsul, Fernández, H. (2022)]

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