Feigin - Semikhatov duality

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arXiv : 2005.10713 / 2104.00942.

§1. W - algebras.

$\mathfrak{g}$: simple Lie (super) alg.

$V^k(\mathfrak{g})$: affine vertex (super) alg asso. to $\mathfrak{g}$ at level k.

$f \in \mathfrak{g}$: nilpotent element in $\mathfrak{g}_{\bar{0}}$

$$W^k(\mathfrak{g}, f) := H^0_{DS,f}(V^k(\mathfrak{g}))$$

↑ Drinfeld - Sokolov reduction for f.

: W - algebras asso. to $\mathfrak{g}, f$ at level k.

ex. • $f = 0$: $W^k(\mathfrak{g}, 0) = V^k(\mathfrak{g})$

• $f = f_{prin}$: principal nilpotent element
(or regular)

$W^k(\mathfrak{g}) = W^k(\mathfrak{g}, f_{prin})$: principal W-alg

$\mathfrak{g} = \mathfrak{sl}_2$: $W^k(\mathfrak{sl}_2)$ = Virasoro vertex alg
of central charge $1 - 6\frac{(k+1)^2}{k+2}$

$\mathfrak{g} = \mathfrak{sl}_3$: $W^k(\mathfrak{sl}_3)$ = Zamolodchikov's W_3-alg

$\mathfrak{g} = \mathfrak{sl}_n$: $W^k(\mathfrak{sl}_n)$ = Fateev-Lukyanov's W_n-alg.

$\mathfrak{g} = \mathfrak{osp}(1|2)$: $W^k(\mathfrak{osp}(1|2))$ = $N=1$ Super Conformal Alg

$\mathfrak{g} = \mathfrak{sl}(2|1)$: $W^k(\mathfrak{sl}(2|1))$ = $N=2$ SCA. ... etc.

§2. Duality

- Feigin – Frenkel duality

  $\mathfrak{g}$: simple Lie alg

  ${}^L\mathfrak{g}$: Langlands dual of $\mathfrak{g}$

$$\Rightarrow \quad W^k(\mathfrak{g}) \simeq W^\ell({}^L\mathfrak{g}),$$

where $r^\vee(k+h^\vee)(\ell + {}^Lh^\vee) = 1$

$$\left\{\begin{array}{l} r^\vee = \text{lacing number of } \mathfrak{g} = \left\{\begin{array}{ll} 1 & \text{ADE} \\ 2 & \text{BCF} \\ 3 & \text{G} \end{array}\right. \\ h^\vee = \text{dual Coxeter number of } \mathfrak{g} \\ {}^Lh^\vee = \qquad \text{〃} \qquad \text{of } {}^L\mathfrak{g} \end{array}\right.$$

cf. $k = -h^\vee$: Drinfeld's conj

$$Z(\hat{\mathfrak{g}}) \cong \operatorname{Fun} \operatorname{Oper}_{{}^LG}(D^\times)$$

Motivation : Generalizations of FF dualities.

cf. G. '17

$$\mathcal{W}^k(\mathfrak{osp}(1|2n)) \simeq \mathcal{W}^\ell(\mathfrak{osp}(1|2n))$$

$$4(k+n+\tfrac{1}{2})(\ell+n+\tfrac{1}{2}) = 1$$

Today : $\mathcal{W}^k(\mathfrak{sl}_n, f_{sub}) \longleftrightarrow \mathcal{W}^\ell(\mathfrak{sl}(n|1))$

↑ subregular nilpotent element

- Coset

V : vertex superalg $\supset$ W : vertex subalg

$$\mathrm{Com}(W, V) := \{ A \in V \mid A(z)B(w) \sim 0 \quad \forall B \in W \}.$$

coset of W in V.

- $$\begin{array}{ccccc} \mathcal{W}^k(\mathfrak{sl}_n, f_{sub}) & & \longleftrightarrow & & \mathcal{W}^\ell(\mathfrak{sl}(n|1)) \\ \cup & & & & \cup \\ \pi^+ & \leftarrow & \text{Heis. VA of rk 1} & \rightarrow & \pi^- \\ \| & & & & \| \\ \langle H^+ \rangle & & & & \langle H^- \rangle \end{array}$$

Thm 1 (Creutzig – G. – Nakatsuka)

Feigin – Semikhatov duality

(1) $\forall k, \ell \in \mathbb{C}$ s.t. $(k+n)(\ell+n-1) = 1$

and $k+n \neq \frac{n-1}{n}$. Then

$$\mathrm{Com}(\pi^+, W^k(\mathfrak{sl}_n, f_{sub})) \simeq \mathrm{Com}(\pi^-, W^\ell(\mathfrak{sl}(n|1)))$$

(2) $\forall k, \ell \in \mathbb{C}$ s.t. $(k+n)(\ell+n-1) = 1$

Then

$$\mathrm{Com}(\tilde{\pi}^+, W^k(\mathfrak{sl}_n, f_{sub}) \otimes V_{\mathbb{Z}})$$
$$\simeq W^\ell(\mathfrak{sl}(n|1)).$$

$$\mathrm{Com}(\tilde{\pi}^-, W^\ell(\mathfrak{sl}(n|1)) \otimes V_{\sqrt{-1}\mathbb{Z}})$$
$$\simeq W^k(\mathfrak{sl}_n, f_{sub}).$$

where $\tilde{\pi}^\pm \subset \pi^\pm \otimes V_{\sqrt{\pm 1}\mathbb{Z}}$: Heis. VA of rk 1.

$\quad\; = \langle H^\pm \otimes 1 + 1 \otimes \phi^\pm \rangle$

Idea of proof

Feigin - Frenkel :

Free Field Realizations of $W^k(\mathfrak{g})$
(FFR)

$$\begin{tikzcd}[column sep=small] & \text{Heis. VA of } \mathrm{rk}\,\mathfrak{g} & \\ W^k(\mathfrak{g}) \arrow[ur, hook] & \simeq & W^\ell({}^L\mathfrak{g}) \arrow[ul, hook'] \end{tikzcd}$$

Embedings are characterized by screening op.

if k, ℓ are generic. and then

we can show the isomorphisms.

k, ℓ : non-generic. [Aganagic - Frenkel - Okounkov]

Consider each conformal wt sp.

$$\begin{tikzcd}[column sep=small] & (\text{Heis. VA})_\Delta & \\ W^k(\mathfrak{g})_\Delta \arrow[ur, hook] & \simeq & W^\ell({}^L\mathfrak{g})_\Delta \arrow[ul, hook'] \end{tikzcd}$$

k, ℓ : continuous

isom holds for dense open set $k \in U \subset \mathbb{C}$

$\Rightarrow$ for all $k \in \mathbb{C} \setminus \{-h^\vee\}$

Feigin - Semikhatov : FFR of $W^k(\mathfrak{sl}_n, f_{sub})$

Our contributions : FFR of $W^\ell(\mathfrak{sl}(n|1))$ //

- $W_k(\mathfrak{g}, f)$ = simple quotient of $W^k(\mathfrak{g}, f)$.

Thm 2 (CGN)

Thm 1 also holds for $W_k(\mathfrak{sl}_n, f_{sub})$ and $W_\ell(\mathfrak{sl}(n|1))$.

ex. $n = 2$:

affine $\mathfrak{sl}_2$ $\longleftrightarrow$ $N=2$ SCA

: Kazama - Suzuki coset construction

[Adamovic, Creutzig - Linshaw,
Feigin - Semikatov - Tipunin, Sato etc.]

§3. Representation Theory

V : vertex (super) alg.

- V is C_2-cofinite (or lisse.)

$\Leftrightarrow$ $\dim V/C_2(V) < \infty$.

$C_2(V) = \mathrm{Span}_{\mathbb{C}}\{A_{(-2)}B \mid A, B \in V\}$

$\Rightarrow$ (+ suitable conditions on V)

The cat. V-mod of generalized modules of V has a fusion product $\boxtimes$ induced from $P(z)$-tensor and is $\mathbb{C}$-lin. additive monoidal braided.

(V : not super $\Rightarrow$ tensor)

- V : rational

$\Leftrightarrow$ V-mod is semisimple

(w/ finitely many simple modules)
(up to isoms)

<u>Rational cases</u> : Fix $k \in \mathbb{Q}$ s.t.

$$k+n = \frac{n+r}{n-1}, \quad (n+r, n-1) = 1, \quad n, r \geq 2.$$

$$\mathrm{Com}(\pi^+, W_k(\mathfrak{sl}_n, f_{sub})) \simeq \mathrm{Com}(\pi^-, W_\ell(\mathfrak{sl}(n|1)))$$

$\cong$ Creutzig - Linshaw

$W_{k'}(\mathfrak{sl}_r)$

Level - rank duality. (Arakawa - Lam - Yamada,

Arakawa - Creutzig - Linshaw

$$k'+r = \frac{r+n}{r-1}.$$

... for small n)

$\Rightarrow$ $W_k(\mathfrak{sl}_n, f_{sub})$ is a simple current ext

of $W_{k'}(\mathfrak{sl}_r)$ and $\exists$ lattice VA

$\uparrow$ lisse, rational (Arakawa) $\qquad$ $\uparrow$ lisse, rational (Dong.)

$\Rightarrow$ $W_k(\mathfrak{sl}_n, f_{sub})$ is lisse and rational

$\Rightarrow$ $W_\ell(\mathfrak{sl}(n|1))$ is $\quad$ "

Thm 2.

- $L_k(\mathfrak{g}) :=$ simple quot of $V^k(\mathfrak{g})$.

<u>Thm 3</u> (Creutzig - G. - Nakatsuka - Sato).

k, l, r, k' are as above.

(1) There exists one-to-one correspondence between isom classes of simple modules of $L_{k'}(\mathfrak{sl}_r)$ and 〃 of $W_k(\mathfrak{sl}_n, f_{sub})$ that induces an isom of the fusion rings.

(2) We have an explicit description of the fusion ring of $W_l(\mathfrak{sl}(n|1))$, a parametrization of the simple modules and the character formulae.

cf. Arakawa - van Ekeren : (1).

General levels :

$$(W^+, W^-) = \left(W^k(\mathfrak{sl}_n, f_{sub}),\ W^\ell(\mathfrak{sl}(n|1)) \right)$$

$$\text{or} \quad \left(W_k(\mathfrak{sl}_n, f_{sub}),\ W_\ell(\mathfrak{sl}(n|1)) \right)$$

M : $W^\pm$ - module.

M is a weight module

$$\Leftrightarrow \quad M = \bigoplus_{\lambda, \Delta} M_{\lambda,\Delta} \qquad \begin{array}{l} \lambda : \text{Heis. weight} \\ \Delta : \text{conformal weight} \end{array}$$

w/ $\dim M_{\lambda,\Delta} < \infty$.

$\mathcal{C}^\pm$ = category of weight $W^\pm$- modules

$$\cong \bigoplus_{[\lambda] \in \mathbb{C}/\mathbb{Z}} (\mathcal{C}^\pm)_{[\lambda]} . \quad \text{block decomp.}$$

$V^\pm = V_{\sqrt{\pm 1}\mathbb{Z}}$ $\qquad$ Fock $\tilde{\pi}^\pm$-mod.

$$\begin{array}{c} W^\pm \otimes V^\pm \\ \cup \\ \tilde{\pi}^\pm \end{array} = \bigoplus_\lambda \underset{\underset{W^\mp\text{- mod.}}{\uparrow}}{\Omega_\lambda(W^\pm \otimes V^\pm)} \otimes \tilde{\pi}^\pm_\lambda$$

$$F_\lambda^+ : (\mathcal{C}^+)_{[\lambda]} \to (\mathcal{C}^-)_{[\lambda^\vee]}$$
$$M \mapsto \Omega_{-\lambda}(M \otimes V^+)$$

$$F_{\lambda^\vee}^- : (\mathcal{C}^-)_{[\lambda^\vee]} \to (\mathcal{C}^+)_{[\lambda]}$$
$$N \mapsto \Omega_{-\lambda^{\vee\prime}}(N \otimes V^-)$$

<u>Thm 4</u> (CGNS)

F_λ^+ gives an equivalence of $(\mathcal{C}^+)_{[\lambda]}$ and $(\mathcal{C}^-)_{[\lambda^\vee]}$, and the inverse is $F_{\lambda^\vee}^-$ //

☆ To study $F_\lambda^\pm$, we reconstruct $F_\lambda^\pm$ by using relative semi-infinite cohomology

I. Frenkel – Garland – Zuckerman :

$$H^{\frac{\infty}{2}+i}(\hat{\mathfrak{gl}}_1, \mathfrak{gl}_1, \pi_\lambda \otimes \pi_\mu)$$

$$\simeq \delta_{i0}\, \delta_{\lambda+\mu,0}\, \mathbb{C}$$

spanned by $|\lambda\rangle \otimes |\mu\rangle$

$$\tilde{V}^{\pm}_{\lambda} := V^{\pm} \otimes \tilde{\pi}^{\pm}_{\lambda}$$

$$H^{rel,\pm}(W^{\pm}) := H^{\frac{\infty}{2}+0}(\hat{gl}_1, gl_1, W^{\pm} \otimes \tilde{V}^{\pm})$$

$$\simeq W^{\mp} \quad \text{as} \quad \text{VSA}$$

For $M \in (\mathcal{C}^{\pm})_{[\lambda]}$.

$$H^{rel,\pm}_{\lambda}(M) := H^{\frac{\infty}{2}+0}(\hat{gl}_1, gl_1, M \otimes \tilde{V}^{\pm}_{\lambda})$$

$$: W^{\mp}\text{-module}.$$

<u>Thm 5</u> (CGNS)

(1) $F^{\pm}_{\lambda} \simeq H^{rel,\pm}_{\lambda}(?)$. for all $\lambda \in \mathbb{C}$.

(2) Let $M_i \in (\mathcal{C}^{\pm})_{[\lambda_i]}$ $(i = 1, 2, 3)$.

s.t. $[\lambda_1] + [\lambda_2] = [\lambda_3]$. Then

$$I\begin{pmatrix} & M_3 & \\ M_1, & & M_2 \end{pmatrix} \simeq I\begin{pmatrix} & H^{rel,\pm}_{\lambda_3}(M_3) & \\ H^{rel,\pm}_{\lambda_1}(M_1), & & H^{rel,\pm}_{\lambda_2}(M_2) \end{pmatrix}$$

(3) If $\boxtimes$ is well-defined,

$$H^{rel,\pm}_{\lambda_1}(M_1) \boxtimes H^{rel,\pm}_{\lambda_2}(M_2)$$

$$\simeq H^{rel,\pm}_{\lambda_1+\lambda_2}(M_1 \boxtimes M_2) \quad //$$

Idea of proof for (2) ((3) is similar)

$C_\lambda^{rel,\pm}(M)$: cpx defining $H_\lambda^{rel,\pm}(M)$

$$f \in I\binom{M_3}{M_1,\ M_2}$$

$$\rightsquigarrow \tilde{f} \in I\binom{C_{\lambda_3}^{rel,+}(M_3)}{C_{\lambda_1}^{rel,+}(M_1),\ C_{\lambda_2}^{rel,+}(M_2)}$$

$$\rightsquigarrow [\tilde{f}] \in I\binom{H_{\lambda_3}^{rel,+}(M_3)}{H_{\lambda_1}^{rel,+}(M_1),\ H_{\lambda_2}^{rel,+}(M_2)}$$

$$[\tilde{f}] =: H^{rel,+}(f)$$

Then we can show that

$$(H^{rel,-} \circ H^{rel,+})(f) = f$$

Similarly, $H^{rel,+} \circ H^{rel,-} = \mathrm{id}$.

$\therefore$ $H^{rel,+}$ is an isom.