Strongly tame modules for affine vertex algebras

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Admissible representations of vertex algebras

- $\mathfrak{g}$ a simple finite-dimensional Lie algebra
- κ a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$
- $\widehat{\mathfrak{g}}_\kappa = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ the affine Kac–Moody algebra with the commutation relations $[a \otimes f(t), b \otimes g(t)] =$

$$= [a, b] \otimes f(t)g(t) - \kappa(a, b) Res_{t=0}(f(t)dg(t))c$$

- $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}1 \oplus \mathbb{C}c$ Cartan subalgebra
- Let $\mathcal{V}_\kappa(\mathfrak{g}) = \mathbb{M}_{\kappa,\mathfrak{g}}(\mathbb{C})$ be the universal affine vertex algebra

  Simple quotient $\mathcal{L}_\kappa(\mathfrak{g}) = \mathbb{L}_{\kappa,\mathfrak{g}}(\mathbb{C})$ of $\mathbb{M}_{\kappa,\mathfrak{g}}(\mathbb{C})$ is the simple affine vertex algebra

Zhu's functor

- Let $\mathcal{V}$ be a $\mathbb{Z}$-graded vertex algebra. A graded $\mathcal{V}$-module M is a positive energy module if $M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$ and $M_\lambda \neq 0$, $\lambda \in \mathbb{C}$

  - The assignment $N \mapsto \mathbb{L}_{\kappa,\mathfrak{g}}(N)$ gives a one-to-one correspondence between the isomorphism classes of simple $\mathfrak{g}$-modules and simple positive energy $\mathcal{V}_\kappa(\mathfrak{g})$-modules
  - There is a one-to-one correspondence between isomorphism classes of simple modules over $U(\mathfrak{g})/I_\kappa$ and simple positive energy $\mathcal{L}_\kappa(\mathfrak{g})$-modules for some two-sided ideal I_κ of $U(\mathfrak{g})$
  - Relaxed highest weight modules: Feigin–Semikhatov–Tipunin 98, Adamovic–Milas 95, Adamovic 16, Arakawa–VF–Ramirez 17, Auger–Creutzig–Ridout 18, Kawasetsu–Ridout 19, VF–Ramirez–Morales 2020, VF–Krizka–Morales 21

• Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$:

$$k + h^\vee = \frac{p}{q} \text{ with } p, q \in \mathbb{N}, \ (p, q) = 1, \ p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) = r^\vee, \end{cases}$$

where $r^\vee$ is the lacing number of $\mathfrak{g}$

Let $\kappa = k\kappa_0$. We say that a $\mathfrak{g}$-module N is admissible of level k if $\mathbb{L}_{\kappa,\mathfrak{g}}(N)$ is an $\mathcal{L}_\kappa(\mathfrak{g})$-module (By Duflo and Joseph, a simple $\mathfrak{g}$-module N is admissible if $Ann_{U(\mathfrak{g})}N = Ann_{U(\mathfrak{g})}L(\lambda)$ for some admissible dominant λ)

• Admissible simple highest weight $\mathfrak{g}$-modules of level k were classified by Kac, Wakimoto (1989) and Arakawa (2016)

• For a primitive ideal J the associated variety is the closure of a coadjoint orbit $\mathbb{O}^*$. We say that a simple $\mathfrak{g}$-module N is in the nilpotent orbit $\mathbb{O}$ if the associated variety $\mathcal{V}(Ann_{U(\mathfrak{g})}N)$ is the closure of $\mathbb{O}^*$ (Borho, Brylinski, Joseph)

e.g. Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{g}$, $\mathbb{O}_{\mathfrak{p}} = (G.\mathfrak{p}^{\perp})^{\mathrm{reg}}$ the Richardson orbit of $\mathfrak{p}$, where $\mathfrak{p}^{\perp}$ is the orthogonal complement of $\mathfrak{p}$ with respect to the Killing form. If λ is a $\mathfrak{p}$-dominant weight then $M_{\mathfrak{p}}(\lambda) = Ind_{\mathfrak{p}}^{\mathfrak{g}}(L(\lambda))$ is in the orbit $\mathbb{O}_{\mathfrak{p}}$

• Nilpotent orbits of $\mathfrak{sl}_{n+1}$ are parameterized by partitions of $n+1$: if $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r]$ is a partition then

$$\mathfrak{p}_{\lambda} = \{x \in \mathfrak{g};\ x(V_i) \subset V_i \text{ for } i = 1, 2, \ldots, r\},$$

where $(V_0, V_1, V_2, \ldots, V_r)$ is a partial flag in $\mathbb{C}^{n+1}$, $V_0 = 0$, $\dim V_i = \sum_{j=1}^{i} \lambda_j$ for $i = 1, 2, \ldots, r$

Example

Let $\mathfrak{g} = \mathfrak{sl}_3$, $k = \frac{p}{q} - 3$ an admissible level, $p, q \in \mathbb{N}$, $(p, q) = 1$, $p \geq 3$. Then $\mathbb{O}_{[1^3]} = \mathbb{O}_{\text{zero}}$, $\mathbb{O}_{[2,1]} = \mathbb{O}_{\text{min}}$ $(dim = 4)$, $\mathbb{O}_{[3]} = \mathbb{O}_{\text{prin}}$ $(dim = 6)$

Simple admissible highest weight modules in these orbits are:

$$\begin{aligned}
\mathbb{O}_{\text{zero}} &: P = \{\lambda_1\omega_1 + \lambda_2\omega_2;\ \lambda_1, \lambda_2 \in \mathbb{N}_0,\ \lambda_1 + \lambda_2 \leq p - 3\}, \\
\mathbb{O}_{\text{min}} &: \Lambda_1 \cup s_1 \cdot \Lambda_1 \cup s_2 s_1 \cdot \Lambda_1, \\
\mathbb{O}_{\text{prin}} &: \Lambda_2 \cup s_1 \cdot \Lambda_2,
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 &= \left\{\lambda - \frac{p}{q} a\omega_1;\ \lambda \in P,\ a \in \mathbb{N},\ a \leq q - 1\right\}, \\
\Lambda_2 &= \left\{\lambda - \frac{p}{q}(a\omega_1 + b\omega_2);\ \lambda \in P,\ a, b \in \mathbb{N},\ a + b \leq q - 1\right\}
\end{aligned}$$

- Localization

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, α a root of $\mathfrak{g}$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ nonzero, $F_\alpha := \{f_\alpha^k \mid k \in \mathbb{Z}_{\geq 0}\} \subset U(\mathfrak{g})$

$D_\alpha U(\mathfrak{g})$ the localization of $U(\mathfrak{g})$ relative to F_α

For a $\mathfrak{g}$-module M define $D_\alpha M := D_\alpha U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$ (Deodhar 1982)

For $x \in \mathbb{C}$ consider the automorphism θ^x of $D_\alpha U(\mathfrak{g})$ such that

$$\theta^x(u) := \sum_{i \geq 0} \binom{x}{i} \operatorname{ad}(f_\alpha)^i(u)\, f_\alpha^{-i},$$

for $u \in D_\alpha U(\mathfrak{g})$, where $\binom{x}{i} := x(x-1)\ldots(x-i+1)/i!$ and $\binom{x}{0} := 1$. If M is a $D_\alpha U(\mathfrak{g})$-module, then $D_\alpha^x M$ is the twisted localization of M relative to α and x with the action:

$$u \cdot v := \theta^x(u) \cdot v, \quad u \in D_\alpha U(\mathfrak{g}), \quad v \in M$$

For a set $S = \{\alpha_1, \dots, \alpha_k\}$ of commuting roots,
$D_S M := D_{\alpha_1} \dots D_{\alpha_n} M$

Theorem (Mathieu, 2000)

Every simple weight representation M of $\mathfrak{g}$ with finite-dimensional weight spaces is a twisted localization of a highest weight representation: $M \simeq D_S^X L(\lambda)$, for some set S of commuting roots, $\lambda \in \mathfrak{h}^*$ and a subset $X \subset \mathbb{C}$

- If F is injective on M then

$$Ann_{U(\mathfrak{g})} M = Ann_{U(\mathfrak{g})} D_F M$$

- If $\lambda \in \mathfrak{h}^*$ is dominant, $\mathbf{x} \in \mathbb{C}^r$ and F is injective on $L(\lambda)$, then

$$Ann_{U(\mathfrak{g})} L(\lambda) = Ann_{U(\mathfrak{g})} D_F^{\mathbf{x}} L(\lambda) = Ann_{U(\mathfrak{g})} N$$

for any simple subquotient $N \neq 0$ of $D_F^{\mathbf{x}} L(\lambda)$

Example

If α is a simple root and $M_{s_\alpha(\pi)}(\lambda)$ is the Verma module of highest weight λ relative to $s_\alpha(\pi)$, then

$$0 \to M(\lambda) \to D_\alpha M(\lambda) \to M_{s_\alpha(\pi)}(\lambda + \alpha) \to 0$$

What if we localize in the direction of a non-simple root?

Example

If $n = 3$, $\alpha = \varepsilon_1 - \varepsilon_3$, $\lambda = -\varepsilon_1 + \varepsilon_3$, then $M(\lambda)$ is a simple Verma module, and $D_\alpha M(\lambda)$ has length 3:

$$0 \to M \to D_\alpha M(\lambda)/M(\lambda) \to M_{s_\alpha(\pi)}(\lambda + 2\alpha) \to 0,$$

where M is a weight module with infinite weight multiplicities

• The twisting functor on the category of $\mathfrak{g}$-modules:

$$T_\alpha(M) = (U(\mathfrak{g})_{(f_\alpha)}/U(\mathfrak{g})) \otimes_{U(\mathfrak{g})} M \simeq D_\alpha M/M$$

If α is a simple root then T_α preserves the category $\mathcal{O}(\mathfrak{g})$ up to a conjugation of the action of $\mathfrak{g}$. Not the case if α is not simple!

• Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$ with the nilradical $\mathfrak{u}$ and the Levi subalgebra $\mathfrak{l}$, Let $\Lambda^+(p)$ be the set of $\mathfrak{p}$-dominant weights and $\mathbb{F}_\lambda$ the simple finite-dimensional $\mathfrak{p}$-module with highest weight λ

• $M_\mathfrak{p}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{F}_\lambda$ (the generalized Verma $\mathfrak{g}$-module with highest weight λ)

• $W_\mathfrak{p}(\lambda, \alpha) = T_\alpha(M_\mathfrak{p}(\lambda))$

Theorem (VF, Křižka, 2019)

Let $\lambda \in \Lambda^+(\mathfrak{p})$, $\alpha \in \Delta_+$ such that $\mathfrak{g}_\alpha \subset \mathfrak{u}$, and assume that f_α is injective on $M_\mathfrak{p}(\lambda)$. Then $W_\mathfrak{p}(\lambda, \alpha)$ has finite Γ_α-multiplicities, where $\Gamma_\alpha \subset U(\mathfrak{g})$ is generated by the Cartan subalgebra and the Casimir element of the $\mathfrak{sl}_2$-subalgebra based on α. Moreover, $W_\mathfrak{p}(\lambda, \alpha)$ has finite Γ-multiplicities for any commutative subalgebra Γ of $U(\mathfrak{g})$ containing Γ_α (but infinite weight multiplicities)

Gelfand-Tsetlin modules

$\Bbbk$ an algebraically closed field of characteristic 0, $\mathfrak{g} = \mathfrak{gl}(n)$
$\pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ a set of simple roots

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n,$$

where $\mathfrak{gl}_k$ has simple roots $\{\alpha_1, \ldots, \alpha_{k-1}\}$, $k = 2, \ldots, n$, and

$$U_1 \subset U_2 \subset \ldots \subset U_n,$$

where $U_m = U(\mathfrak{gl}_m)$. The center Z_m of U_m is the polynomial algebra in m variables.

A Gelfand-Tsetlin subalgebra $\Gamma = \Gamma(\pi)$ of U is the polynomial algebra generated by $\{Z_m \mid m = 1, \ldots, n\}$

• A $\mathfrak{g}$-module M is a Γ-Gelfand-Tsetlin module if it is a locally Γ-finite module:

$$M = \bigoplus_{\chi \in \Gamma^*} M_\chi,$$

where

$$M_\chi = \{v \in M \mid \forall g \in \Gamma, \exists k > 0 \text{ such that } (g - \chi(g))^k v = 0\}$$

• A module M is a tame Γ-Gelfand-Tsetlin module if Γ has a simple spectrum on M

- Simple Gelfand-Tsetlin module need not to be tame
- Tame Γ-Gelfand-Tsetlin module doesn't need to be tame for $\Gamma' \neq \Gamma$:
  $\mathfrak{g} = \mathfrak{sl}_3$, $M(-\rho)$ is tame only for Γ_θ, where θ is the maximal root for π
- Finite-dimensional modules are tame for any Γ

Gelfand-Tsetlin tableaux

A Gelfand-Tsetlin tableau is an array $T(v)$ of $\binom{n+1}{2}$ complex numbers

$$
\begin{array}{ccccccccc}
v_{n,1} & & v_{n,2} & & \cdots & & v_{n,n-1} & & v_{n,n} \\
& v_{n-1,1} & & & \cdots & & & v_{n-1,n-1} & \\
& & \ddots & & \cdots & & \ddots & & \\
& & & v_{2,1} & & v_{2,2} & & & \\
& & & & v_{1,1} & & & &
\end{array}
$$

Standard: $v_{k,i} - v_{k-1,i} \in \mathbb{Z}_{\geq 0}$ and $v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{>0}$
Generic: $v_{k,i} - v_{k,j} \notin \mathbb{Z}, 1 \leq k < n$
Critical: $v_{k,i} = v_{k,j}$ for some i, j, k, $1 \leq k < n$

Theorem (Gelfand-Tsetlin)

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ *with* $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$, *and*

$$V(\lambda) = \mathrm{Span}_{\mathbb{C}} \{ T(v) \mid v_{n,i} = \lambda_i - i + 1 \text{ and } T(v) \text{ standard} \}$$

The space $V(\lambda)$ *is a finite-dimensional* $\mathfrak{gl}_n$*-module of highest weight* λ *with*

$$E_{k,k+1} T(v) = -\sum_{i=1}^{k} e_{k,i}^{+}(v) T(v + \delta^{k,i})$$

$$E_{k+1,k} T(v) = \sum_{i=1}^{k} e_{k,i}^{-}(v) T(v - \delta^{k,i})$$

$$E_{kk} T(v) = e_k(v) T(v)$$

where $e_{k,i}^{\pm}$ *are certain rational functions,* e_k *is a certain symmetric polynomial, and* $\delta^{k,i}$ *is the tableau with* $v_{k,i} = 1$ *and* $v_{r,s} = 0$

Classification of simple Gelfand-Tsetlin modules

- Generic tableau (Drozd, VF, Ovsienko, 1992): If $v \in \mathbb{C}^{\frac{n(n-1)}{2}}$ is generic, then the Gelfand-Tsetlin formulas define a $\mathfrak{gl}_n$-module structure on the universal generic Gelfand-Tsetlin module

$$V(T(v)) := \mathrm{Span}\left\{T(v+z) \mid z \in \mathbb{Z}^{\frac{n(n-1)}{2}}\right\}$$

  Module $V(T(v))$ is tame, the same holds for all simple subquotients

- Arbitrary tableau (Ramirez, Zadunaisky, 2017; Vishnyakova, 2017; Early, Mazorchuk, Vishnyakova, 2017; Webster, 2021)

• A Γ-Gelfand-Tsetlin $\mathfrak{gl}_n$-module V is a strongly tame (tableau) module if it satisfies the following conditions:

1) V is tame
2) V has a basis consisting of noncritical tableaux
3) The action of $\mathfrak{gl}_n$ on V is given by the Gelfand-Tsetlin formulas

(4) The action of Γ on a basis tableau is given by symmetric polynomials in the entries of the tableau)

• Strongly tame modules $\neq$ Tame modules
Strongly tame modules $\neq$ Tableau modules

Construction of strongly tame modules

- Finite-dimensional and generic modules
- Gelfand–Graev modules, 1965
- Lemire–Patera modules, 1979
- Mazorchuk modules, 1998
- VF, Ramirez, Zhang relation modules, 2018

Relation modules

Set $\mathfrak{V} := \{(i,j) \mid 1 \leq j \leq i \leq n\}$

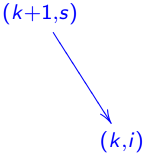
$$\mathcal{R} := \{((i,j),\ (k,s)) \mid |i-k| \leq 1\} \subset \mathfrak{V} \times \mathfrak{V}$$

$((k+1,s),\ (k,i)) \in \mathcal{R}$ $\qquad$ $((k,i),\ (k+1,t)) \in \mathcal{R}$

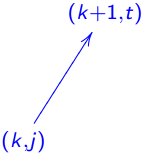
Let $\mathcal{C}$ be any subset of $\mathcal{R}$

We say that a Gelfand-Tsetlin tableau $T(v)$ is a $\mathcal{C}$-realization if $T(v)$ satisfies all the relations in $\mathcal{C}$, namely:

$v_{k+1,s} - v_{ki} \in \mathbb{Z}_{\geq 0}$ if



$v_{kj} - v_{k+1,t} \in \mathbb{Z}_{>0}$ if

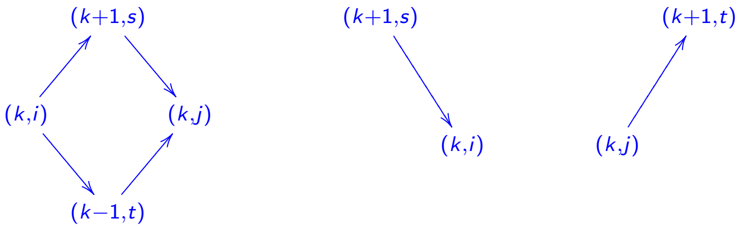


Example

A tableau $T(v)$ is standard if and only if $T(v)$ is the $\mathcal{S}$-realization, where

$$\mathcal{S} := \{((i+1,j),(i,j)),\ ((i,j),(i+1,j+1)) \mid 1 \leq j \leq i \leq n-1\}$$

• Denote by $\mathfrak{F}$ the set of all $\mathcal{C}$ satisfying the following condition: *for every adjoining pair (k,i) and (k,j) of $\mathcal{C}$, $1 \leq k \leq n-1$ there exist unique $s < t$ s.t. one of the following holds*



Fix any set of relations $\mathcal{C}$

Denote by $\mathcal{B}_{\mathcal{C}}(T(v))$ the set of all tableaux $T(v+z)$ satisfying $\mathcal{C}$, $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$

Let $V_{\mathcal{C}}(T(v))$ be the complex vector space spanned by $\mathcal{B}_{\mathcal{C}}(T(v))$

Theorem (V.F., Ramirez, Zhang, 2016)

$V_{\mathcal{C}}(T(v))$ is a Γ-Gelfand–Tsetlin $\mathfrak{gl}_n$-module if and only if $\mathcal{C} \in \mathfrak{F}$. Moreover,

- *$V_{\mathcal{C}}(T(v))$ is a strongly tame module*
- *$V_{\mathcal{C}}(T(v))$ is simple if and only if $\mathcal{C}$ is the maximal set of relations satisfied by $T(v)$*

• Highest weight relation modules (VF, Morales, Ramirez, Zhang, 2018-2020):

$L(\lambda)$ is a Γ-relation module if and only if one of the following conditions holds:

a) $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$, for all $\alpha \in \Delta_+ \setminus \{\alpha_{k,n} \mid k = 1, \ldots, n\}$

b) There exists a unique pair $i \leq j < n$ such that:

i) $\langle \lambda + \rho, \alpha_k^\vee \rangle \in \mathbb{Z}_{>0}$ for each $k > j$

ii) $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \Delta_+ \setminus \{\alpha_{i,k} \mid k \geq j\}$

iii) $\left\langle \lambda + \rho, \alpha_{i,n}^\vee \right\rangle \in \mathbb{Z}_{\leq 0}$ $(\alpha_{r,s} := \alpha_r + \ldots + \alpha_s)$

- Conjecture:
  - Any tableau module is isomorphic to a Γ-relation module for some Γ (known for $n \leq 4$)
  - Any strongly tame module is isomorphic to a Γ-relation module for some Γ (known for $n \leq 3$)

Theorem (VF, Morales, Ramirez, 2020)

Let $\mathfrak{g} = \mathfrak{sl}_n$, $\alpha = \alpha_1$ and $\Gamma = \Gamma(\pi)$. If M is a strongly tame (Γ-relation) Gelfand–Tsetlin module with an injective action of f_α, then $D_\alpha^x M$ is a strongly tame (Γ-relation) Gelfand–Tsetlin module

• Let M be a strongly tame Γ-Gelfand–Tsetlin module, $w \in W$. Then M^w is a strongly tame $w\Gamma$-Gelfand–Tsetlin module, with the action twisted by w (If $\Gamma = \Gamma(\pi)$ then $w\Gamma = \Gamma(w\pi)$)

Theorem (VF, Křižka, Morales, 2021)

Let $L_\pi(\lambda)$ be a simple admissible highest weight module

- *$L_\pi(\lambda)$ is a $\Gamma(\pi)$-relation Gelfand–Tsetlin module;*
- *If $w \in W$ then the module $L_{w\pi}(w\lambda)$ is a $w\Gamma(\pi)$-relation Gelfand–Tsetlin module;*
- *If $L_\pi(\lambda)$ is admissible in the principal orbit, then $L_\pi(\lambda)$ is a $w\Gamma(\pi)$-relation Gelfand–Tsetlin module for any $w \in W$*

Conjecture Let $\lambda \in \mathfrak{h}^*$ and $L_\pi(\lambda)$ an admissible highest weight module. If α is a root and $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$, then there exists $w \in W$ such that $D_\alpha^x L_\pi(\lambda)$ is a $\Gamma(w\pi)$-relation Gelfand–Tsetlin module, for any $x \in \mathbb{C}$

The conjecture holds in the following cases:

- Minimal orbit (VF, Morales, Ramirez, 2020). Moreover:
  - All simple admissible highest weight modules in the minimal nilpotent orbit are bounded Γ-relation Gelfand–Tsetlin modules
  - Classification of all admissible $sl(2)$-induced is known
- All admissible $sl(2)$-induced in the principal orbit (Arakawa–VF– Ramirez; VF– Křižka– Morales, 2017-2021);
- For representatives of the subregular and maximal parabolic orbits (VF, Křižka, Morales, 2021)

THANK YOU!