"MODULE TENSOR CATEGORIES AND THE LANDAU-GINZBURG/CFT CORRESPONDENCE"

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(submitted!)

joint with Thomas Wasserman
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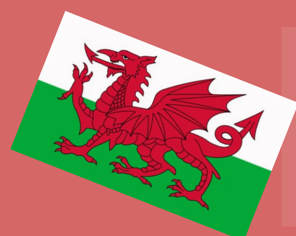
Representation Theory XVII
Dubrovnik, Hrvatska



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0 Motivation

[Vafa-Warner, Martinec, Witten, Howe-West]: physics literature, late 80s-early 90s.

$\Rightarrow$ **LANDAU-GINZBURG / CFT CORRESPONDENCE**

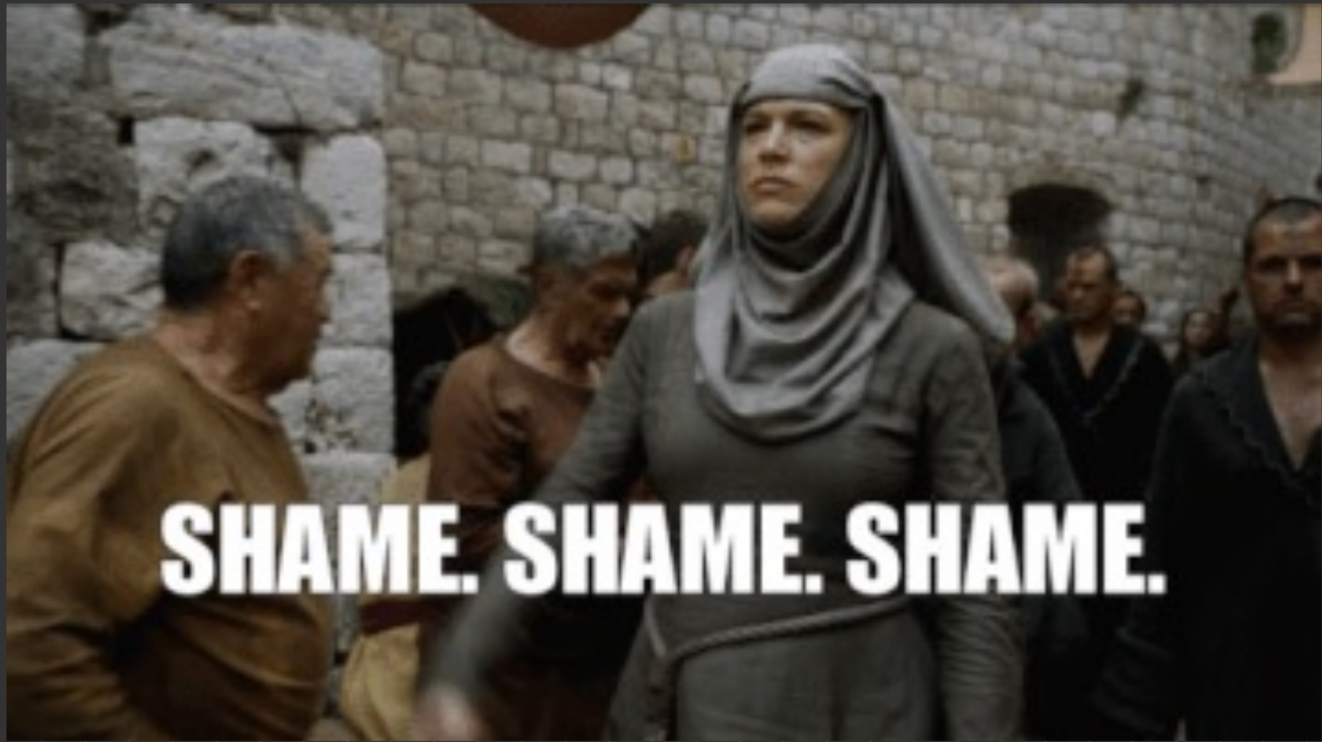
Predicting mathematically:

$$\boxed{\text{Categories of representations of VOAs}} \overset{?}{\longleftarrow\!-\!-\!-\!\longrightarrow} \text{Categories of matrix factorizations}$$

As:
- $\mathbb{C}$-linear cats. ("boundary conds")
- $\otimes$-cats. ("defects")

<u>Current state of affairs</u>

- $\mathbb{C}$-linear equivalences: several examples available [Carqueville-Runkel-RC, Newton-RC, ...]
- $\otimes$-equivalences: 1! example available [Davydov-Runkel-RC] =: [DRCR]



TODAY generalize [DRCR]!

Plan of action:

1. Brief intro to matrix factorizations.
2. [DRCR]'s main theorem.
3. Module tensor categories and how to generalize 2.
4. Outro.

① Brief intro to matrix factorizations

For our purposes: $k = \mathbb{C}$, $S = \mathbb{C}[x_1, \dots, x_m]$, $W \in S$.

Defn: • W potential $:\Leftrightarrow \dim_k \left(S / \langle \partial_{x_1} W, \dots, \partial_{x_m} W \rangle \right) < \infty$

• A matrix factorization of a potential W consists of a pair (M, d^M):

- M free, $\mathbb{Z}_2$-graded $(= M_0 \oplus M_1)$ finite rank S-module,
- $d^M : M \to M$ degree 1 $\left(= \begin{pmatrix} 0 & d_1^M \\ d_0^M & 0 \end{pmatrix}\right)$ S-linear morphism

s.t. $$d^M \circ d^M = W \cdot \mathrm{id}_M \qquad (\text{"twisted differential"})$$

• A morphism of matrix factorizations $f : (M, d^M) \to (N, d^N)$ is an S-linear map from $M \to N$

With these, construct a category:

$$HMF(W) := \begin{cases} \text{ob: same as } MF(W), \\ \text{mo: } \dfrac{\{ f : (M, d^M) \to (N, d^N) \mid |f| = 0 \text{ and } f \circ d^M = d^N \circ f \}}{\{ g : (M, d^M) \to (N, d^N) \mid |g| = 1 \text{ and can be written down as compositions of } d\text{'s and morph. of degree } 0 \}} \end{cases}$$

Some facts:

- Use bimodules instead of S-modules $\rightsquigarrow$ "matrix bifactorization"

  $\Rightarrow \mathrm{HMF}_{bi}(\omega)$

Example: (important for us!)

$$(S_1, \omega_1) = (\mathbb{C}[x], x^d) \qquad \rightsquigarrow \quad M = \mathbb{C}[x,y]^{\oplus 2}$$
$$(S_2, \omega_2) = (\mathbb{C}[y], y^d)$$

$$d^M := \begin{pmatrix} 0 & \prod_{j\in\{0,\dots,e\}} (x - \eta^{j+m} y) \\ \dfrac{x^d - y^d}{\prod_{j\in\{0,\dots,e\}} (x - \eta^{j+m} y)} & 0 \end{pmatrix}$$

for $\eta^d = 1$, $e \in \{0, \dots, d-1\}$, $m \in \mathbb{Z}_d$.

Notation: $P_{m:e}$, "permutation-type matrix factorization".

- With bimodules we can $\otimes$!

Given (S_1, W_1), (S_2, ω_2), (S_3, ω_3), (M, d^M) mat. fact. of $W_1 - W_2$

(N, d^N) mat. fact. of $\omega_2 - \omega_3$,

the tensor product matrix factorization $(M \otimes_{S_2} N, d^{M\otimes N})$ is the matrix factorization of $\omega_1 - \omega_3$ with:

- $M \otimes_{S_2} N$ S_1-S_3-bimodule,
- $d^{M\otimes N} = d^M \otimes \mathrm{id}_N + \mathrm{id}_M \otimes d^N$

Some results:

Lemma: for $w_1 = w_2 = w$, $\mathrm{HMF}_{bi}(w)$ is monoidal.
(possible to upgrade to more general settings).

Also, there are duals that can be described very explicitly:

Example: $(P_{m:e})^{\vee} \simeq P_{-m:e}$

Set: $\mathcal{P}_d$ full subcategory of $\mathrm{HMF}_{bi}(w)$ with objects isomorphic to finite $\oplus$'s of $P_{m:e}$'s.

Thm: [Brunner-Roggenkamp, DRCR] $\mathcal{P}_d$ is closed under $\otimes$.
Explicitly, for $e, e' \in \{0, \dots, d-2\}$,

$$P_{m:e} \otimes P_{m':e'} \simeq \bigoplus_{\substack{\nu = |e-e'| \\ \text{step } 2}}^{\min(e+e',\, 2d-4-e-e')} P_{m+m'+\frac{1}{2}(e+e'-\nu):\nu}$$

And that's it
for matrix factorizations!

② [DRCR]'s main theorem.

Denote: $\mathcal{V}$ VOA corresponding to the $N=2$ unitary minimal model with central charge $c = 3(1-2/d)$, $d \in \mathbb{Z}_{>2}$. $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ here.
[Adamović, Eholzer-Gaberdiel, DiVecchia-Petersen-Yu-Zheng ...]

Prop: [Fröhlich-Fuchs-Runkel-Schweigert] $\mathrm{Rep}(\mathcal{V}_0)$ can be realized as the subcategory (of local modules over a commutative algebra A) in the product:

$$\mathcal{C}(d) := \mathrm{Rep}(\widehat{su}(2)_{d-2}) \boxtimes \overline{\mathrm{Rep}}(\hat{u}(1)_{2d}) \boxtimes \mathrm{Rep}(\hat{u}(1)_4).$$

integrable highest weight representations of affine $su(2)$ at $d-2$.

reps of VOA for $u(1)$ extended by two fields of weight d; opposite braiding + twisting

dito

they are pointed fusion categories!

with simples $[\ell, m, s]$, $\ell \in \{0, \dots, d-2\}$, $m \in \mathbb{Z}_{2d}$, $s \in \mathbb{Z}_4$ satisfying $\ell+m+s$ even.

Two interesting subcategories:

- $\mathcal{C}_{NS}(d) := \{$ subcategory of $\mathcal{C}(d)$ with $\ell+m$ even, s even $\}$
- $\mathcal{C}_R(d) := \{$ " " " " " odd, " odd $\}$.

Thm: • [Brunner-Roggenkamp, '07]: $\mathcal{P}_d \simeq \mathcal{E}_{NS}(d)$ as $\mathbb{C}$-linear.

• [DRCR, '14]: $\mathcal{P}_d \simeq_{\otimes} \mathcal{E}_{NS}(d)$ for d odd.

Rmk: - fusion rules work for any value of d.

- no physical obstructions to the result being true $\forall d$.
- a "quantum" (blessing / curse) hint in the proof:

Prop: [DRCR, Step 1 of proof] for d odd, there exists an equivalence of braided fusion categories

$$\mathcal{E}_{NS}(d) \simeq \frac{TL_{2\cos(\pi/d)}}{\langle p_{d-1} \rangle} \boxtimes \mathrm{Vec}_{\mathbb{Z}_d}$$

Temperley-Lieb category, generated by a self-dual object of dimension $2\cos(\pi/d)$ mod the ideal of morphisms generated by the so-called Wenzl-Jones idempotent

category of $\mathbb{Z}_d$-graded vector spaces (with trivial associators)

quite remarkable, let's explore it!

③ Module tensor categories and how to generalize ②.

Turns out that to generalize the main theorem, we needed a fresh new perspective...

① Introduce a braiding for $\mathrm{Vec}_{\mathbb{Z}_d}$, given by a quadratic form:

$$q_d : \mathbb{Z}_d \longrightarrow \mathbb{C}^\times$$
$$r \longmapsto e^{\pi i r^2/d}$$

Denote: $\mathcal{U}_d := (\mathrm{Vec}_{\mathbb{Z}_d}, q_d)$

② Regard $\mathcal{E}_{NS}(d)$ and $\mathcal{P}_d$ as $\mathcal{U}_d$-module tensor categories ([Henriques-Penneys-Tener, '15]: let $\mathcal{M}$ tensor category, $\mathcal{C}$ braided category. $\mathcal{M}$ is called a ==$\mathcal{C}$-module tensor category== if it is equipped with a functor $\Phi : \mathcal{C} \to \mathcal{M}$ that can be factorized as $\mathcal{C} \longrightarrow Z(\mathcal{M}) \longrightarrow \mathcal{M}$)

Prop: [Wasserman-RC] $\mathcal{E}_{NS}(d)$ and $\mathcal{P}_d$ indeed have this structure.

In particular, $\mathcal{E}_{NS}(d)$ is generated by one object with dimension $2\cos(\pi/d)$.

③ Use this to prove: **Thm:** [WRC] $\mathcal{C}_{NS}(d) \simeq \mathcal{P}d$ as (spherical, free) $\mathcal{D}d$-module tensor categories.

A direct corollary:

Thm: [WRC] $\mathcal{C}_{NS}(d) \simeq_{\otimes} \mathcal{P}d \quad \forall d.$

Rmk:
- Ours is one of the first concrete examples of these categories.
- [W-RC, current work in progress]: great key to prove more general results (beyond x^d).
- Never seen before in the area of matrix factorizations, nor (I believe?) in VOAs.

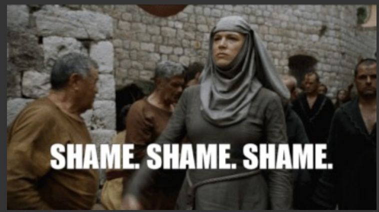
My question to the audience!

(4) Outro

Plenty of open questions here...

[WRC, work in progress]

- Further $\otimes$-equivalences beyond x^d? (E.g. $x^d y$)

  (because... 

we still couldn't go beyond).

  (Possibly extracted from the $\mathbb{C}$-linear equivalences we have?)

- Within x^d: replicating some classification results by Vafa-Warner?

- Physical evidence of more equivalences beyond min. models ($N=2$ Kazama-Suzuki): can we exploit this new structure to prove results here?

- Higher categorical statement for LG/CFT: dito?

More soon! :)

DIOLCH!

HVALA!

THANK YOU!

¡MUCHAS GRACIAS!

Any questions? :)

Maybe later?

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