Combinatorial bases of modules of affine Lie algebras

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B., S. Kožić, M. Primc, *Parafermionic bases of standard modules for affine Lie algebras*, Mathematische Zeitschrift 298 (2021), 1003-1032., arXiv:2002.00435

Motivation

- J. Lepowsky, M. Primc, *Structure of the standard modules for the affine Lie algebra* $A_1^{(1)}$, Contemp. Math. **46**, Amer. Math. Soc., Providence, 1985.
- D. Gepner, *New conformal field theories associated with Lie algebras and their partition functions*, Nuclear Phys. B **290** (1987), 10–24.
- A. Kuniba, T. Nakanishi, J. Suzuki, *Characters in Conformal Field Theories from Thermodynamic Bethe Ansatz*, Modern Phys. Lett. A **08** (1993), 1649–1659.
- G. Georgiev, *Combinatorial constructions of modules for infinite dimensional Lie algebras, II. Parafermionic space*, preprint arXiv:q-alg/9504024.

Construction of bases of parafermionic spaces

Principal subspaces

$\downarrow$

Standard modules

$\downarrow$

Parafermionic spaces

Affine Kac-Moody Lie algebra

- $\widetilde{\mathfrak{g}}$ - affine Kac-Moody Lie algebra associated to the simple Lie algebra $\mathfrak{g}$ of type X_l

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

  with Lie bracket:

$$[x(j_1), y(j_2)] = [x, y]\,(j_1 + j_2) + \langle x, y \rangle\, j_1 \delta_{j_1+j_2,0} c$$

$$[c, x(j)] = 0, \;\; [d, x(j)] = jx(j),$$

  where $x(j) = x \otimes t^j$ for $x \in \mathfrak{g}$ and $j \in \mathbb{Z}$ and $\langle \cdot, \cdot \rangle$ nondegenerate form on $\mathfrak{g}$.

- $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ - Cartan subalgebra
- $Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_l$ - root lattice
- $Q^\vee = \mathbb{Z}\alpha_0^\vee \oplus \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_l^\vee$ - coroot lattice
- $\Lambda_0, \Lambda_1, \ldots, \Lambda_l$ - fundamental weights

Standard modules of affine Lie algebra

- $L(\Lambda)$ - standard $\widetilde{\mathfrak{g}}$-module, i.e. the integrable highest weight $\widetilde{\mathfrak{g}}$-module of level k,
- Λ - rectangular weight, i.e. the highest weight of the form

$$\Lambda = k_0\Lambda_0 + k_j\Lambda_j,$$

where $k_0, k_j \in \mathbb{Z}_+$, and Λ_j is the fundamental weight such that $\langle \Lambda_j, c\rangle = 1$.

$L(\Lambda)$ is the unique irreducible quotient of the generalized Verma $\widetilde{\mathfrak{g}}$-module,

$$N(\Lambda) = U(\widetilde{\mathfrak{g}}) \otimes_{U(\widetilde{\mathfrak{g}}^{\geqslant 0})} U, \qquad \text{where}$$

- U - finite-dimensional, irreducible $\mathfrak{g}$-module of highest weight $k_j\Lambda_j$,
- $\widetilde{\mathfrak{g}}^{\geqslant 0} = \oplus_{n\geqslant 0}(\mathfrak{g} \otimes t^n) \oplus \mathbb{C}c \oplus \mathbb{C}d$ acts on U by

$$\mathfrak{g} \otimes t^n \cdot u = 0 \text{ for } n > 0, \quad c \cdot u = ku, \quad d \cdot u = 0 \qquad \text{for all } u \in U.$$

Principal subspace

- $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, $\mathfrak{n}_+ = \oplus_{\alpha \in R_+} \mathbb{C} x_\alpha$
- $\widetilde{\mathfrak{n}}_+ = \mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}]$

Principal subspace

$$W_{L(\Lambda)} = U(\widetilde{\mathfrak{n}}_+) v_\Lambda$$

Character of the principal subspace

$$\mathit{ch}\, W_{L(\Lambda)} = \sum_{m, r_1, \ldots, r_l \geq 0} \mathit{dim}\, (W_{L(\Lambda)})_{-m\delta + r_1\alpha_1 + \ldots + r_l\alpha_l} q^m \prod_{i=1}^{l} y_i^{r_i}$$

- $(W_{L(\Lambda)})_{-m\delta + r_1\alpha_1 + \ldots + r_l\alpha_l}$ the weight subspaces of $W_{L(\Lambda)}$ with respect to $\widetilde{\mathfrak{h}}$

Quasi-particles

- $L(\Lambda)$ is a module of VOA $L(k\Lambda_0)$, where

$$x_{\alpha_i}(z) = \sum_{m\in\mathbb{Z}} x_{\alpha_i}(m) z^{-m-1} \in (\mathrm{End}\, L(\Lambda))[[z, z^{-1}]].$$

Quasi-particle of color i, charge r and energy -m

$$x_{r\alpha_i}(z) = x_{\alpha_i}(z)^r = \sum_{m\in\mathbb{Z}} x_{r\alpha_i}(m) z^{-r-m} \in (\mathrm{End}\, L(\Lambda))[[z, z^{-1}]].$$

For fixed $m \in \mathbb{Z}$ quasi-particle $x_{r\alpha_i}(m)$ is

$$x_{r\alpha_i}(m) = \sum_{\substack{m_1,\ldots,m_r\in\mathbb{Z} \\ m_1+\cdots+m_r=m}} x_{\alpha_i}(m_r)\cdots x_{\alpha_i}(m_1).$$

Quasi-particle bases of principal subspaces $W_{L(\Lambda)}$

Theorem (B. Feigin and A. Stoyanovsky 1994, G. Georgiev 1995, M. B., S. Kožić, M. Primc 2014–2021)

For any rectangular highest weight Λ the set

$$\mathfrak{B}_{W_{L(\Lambda)}} = \left\{ b \cdot v_\Lambda \, : \, b \in B_{W_{L(\Lambda)}} \right\}$$

forms a basis for the principal subspace $W_{L(\Lambda)}$ of untwisted affine Lie algebras.

- $B_{W_{L(\Lambda)}}$ the set of monomials of the form

$$b(\alpha_l) \cdots b(\alpha_1),$$

where

$$b(\alpha_i) = x_{n_{r_i^{(1)},i}\alpha_i}(m_{r_i^{(1)},i}) \cdots x_{n_{2,i}\alpha_i}(m_{2,i}) x_{n_{1,i}\alpha_i}(m_{1,i})$$

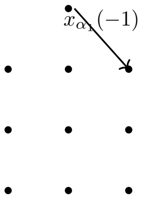
- charges of quasi-particles in color $i = 1, \ldots, l$ decrease from right to left,
- maximal charge for color $i = 1, \ldots, l$ is $k_{\alpha_i} = 2k / \langle \alpha_i, \alpha_i \rangle \in \{k, 2k, 3k\}$,
- energies of quasi-particles satisfy certain conditions.

Quasi-particle bases-example

- $V = L(\Lambda_0)$, $L(\Lambda_1)$, $L(\Lambda_2)$ the standard $B_2^{(1)}$ - modules of level 1
- $x_{1\alpha_2}(m_{3,2})x_{2\alpha_2}(m_{2,2})x_{2\alpha_2}(m_{1,2})x_{\alpha_1}(m_{3,1})x_{\alpha_1}(m_{2,1})x_{\alpha_1}(m_{1,1})v_V$
- conditions on energies of color $i = 1$:

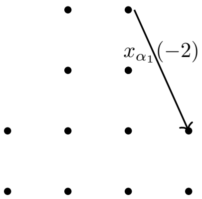
- $W_{L(\Lambda_0)}$:

$$m_{1,1} \leq -1$$



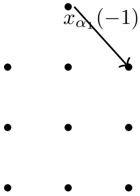
- $W_{L(\Lambda_1)}$:

$$m_{1,1} \leq -2$$



- $W_{L(\Lambda_2)}$:

$$m_{1,1} \leq -1$$



Quasi-particle bases-example

- $V = L(\Lambda_0)$, $L(\Lambda_1)$, $L(\Lambda_2)$ the standard $B_2^{(1)}$ - modules of level 1
- $x_{1\alpha_2}(m_{3,2})x_{2\alpha_2}(m_{2,2})x_{2\alpha_2}(m_{1,2})x_{\alpha_1}(m_{3,1})x_{\alpha_1}(m_{2,1})x_{\alpha_1}(m_{1,1})v_V$
- conditions on energies of color $i = 1$ follow from relation on level 1 module V:

$$x_{2\alpha_1}(z) = 0 \quad \Rightarrow \quad \sum_{m \in \mathbb{Z}} \left(\sum_{\substack{m_1, m_2 \in \mathbb{Z} \\ m_1 + m_2 = m}} x_\alpha(m_1) x_\alpha(m_2) \right) z^{-m-2} v_V = 0 \quad \Rightarrow \quad m_2 \leq m_1 - 2$$

- $W_{L(\Lambda_0)}$:

$$m_{1,1} \leq -1$$
$$m_{2,1} \leq -3$$
$$m_{3,1} \leq -5$$

- $W_{L(\Lambda_1)}$:

$$m_{1,1} \leq -2$$
$$m_{2,1} \leq -4$$
$$m_{3,1} \leq -6$$

- $W_{L(\Lambda_2)}$:

$$m_{1,1} \leq -1$$
$$m_{2,1} \leq -3$$
$$m_{3,1} \leq -5$$

Quasi-particle bases-example

- $x_{1\alpha_2}(m_{3,2})x_{2\alpha_2}(m_{2,2})x_{2\alpha_2}(m_{1,2})v_V$

- conditions on energies of color $i = 2$:

- $W_{L(\Lambda_0)}$:

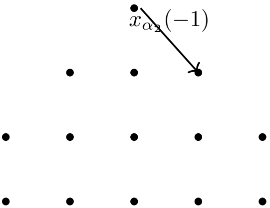
  $$m_{1,2} \leq -2$$

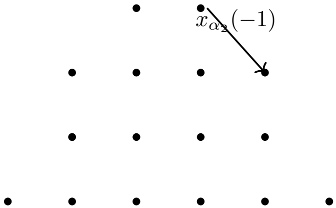
- $W_{L(\Lambda_1)}$:

  $$m_{1,2} \leq -2$$

- $W_{L(\Lambda_2)}$:

  $$m_{1,2} \leq -3$$





Quasi-particle bases-example

- $x_{1\alpha_2}(m_{3,2})x_{2\alpha_2}(m_{2,2})x_{2\alpha_2}(m_{1,2})v_V$

- conditions on energies of color $i = 2$:

- $W_{L(\Lambda_0)}$:

$$m_{1,2} \leq -2$$
$$m_{2,2} \leq -6$$
$$m_{3,2} \leq -8$$

- $W_{L(\Lambda_1)}$:

$$m_{1,2} \leq -2$$
$$m_{2,2} \leq -6$$
$$m_{3,2} \leq -8$$

- $W_{L(\Lambda_2)}$:

$$m_{1,2} \leq -3$$
$$m_{2,2} \leq -7$$
$$m_{3,2} \leq -9$$

- conditions on energies of color $i = 2$ follow from the commutativity of vertex operators:

$$\left.\begin{array}{c} x_{\alpha_2}(z)x_{2\alpha_2}(z) \\ x_{2\alpha_2}(z)x_{2\alpha_2}(z) \end{array}\right\} \Longrightarrow \begin{array}{ccc} x_{\alpha_2}(m_2)x_{2\alpha_2}(m_1) & \Rightarrow & m_2 \leq m_1 - 2 \\ x_{2\alpha_2}(m_2)x_{2\alpha_2}(m_1) & \Rightarrow & m_2 \leq m_1 - 4 \end{array}$$

Quasi-particle bases-example

- $x_{1\alpha_2}(m_{3,2})x_{2\alpha_2}(m_{2,2})x_{2\alpha_2}(m_{1,2})x_{\alpha_1}(m_{3,1})x_{\alpha_1}(m_{2,1})x_{\alpha_1}(m_{1,1})v_V$
- conditions on energies of color $i = 2$:

- $W_{L(\Lambda_0)}$:

$$m_{1,2} \leq 1$$
$$m_{2,2} \leq -3$$
$$m_{3,2} \leq -5$$

- $W_{L(\Lambda_1)}$:

$$m_{1,2} \leq 1$$
$$m_{2,2} \leq -3$$
$$m_{3,2} \leq -5$$

- $W_{L(\Lambda_2)}$:

$$m_{1,2} \leq 0$$
$$m_{2,2} \leq -4$$
$$m_{3,2} \leq -6$$

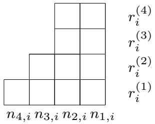
- conditions on energies of color $i = 1$ follow from relation on level 1 module V:

$$(z_1 - z_2)x_{\alpha_1}(z_1)x_{n\alpha_2}(z_2) = (z_1 - z_2)x_{n\alpha_2}(z_2)x_{\alpha_1}(z_1),\ n \leq 2$$

$$\Rightarrow \quad x_{n\alpha_2}(m_2)x_{\alpha_1}(m_1) \quad \Rightarrow \quad m_2 \leq m_1 + 1$$

Character of $W_{L(\Lambda)}$

Write conditions on energies in terms of dual partitions of $\sum_{t=1}^{r_i^{(1)},} n_{t,i}$ with parts $r_i^{(s)}$, $s = 1, \dots k_{\alpha_i}$



- in the case of level one $B_2^{(1)}$ -modules we have:

$$\text{ch } W_{L(\Lambda_0)} = \sum_{\substack{r_1^{(1)} \geqslant 0 \\ r_2^{(1)} \geqslant r_2^{(2)} \geqslant 0}} \frac{q^{r_1^{(1)2} + r_2^{(1)2} + r_2^{(2)2} - r_1^{(1)}(r_2^{(1)} + r_2^{(2)})}}{(q;q)_{r_1^{(1)}} (q;q)_{r_2^{(1)} - r_2^{(2)}} (q;q)_{r_2^{(2)}}} y_1^{r_1^{(1)}} y_2^{r_2^{(1)} + r_2^{(2)}},$$

$$\text{ch } W_{L(\Lambda_1)} = \sum_{\substack{r_1^{(1)} \geqslant 0 \\ r_2^{(1)} \geqslant r_2^{(2)} \geqslant 0}} \frac{q^{r_1^{(1)2} + r_2^{(1)2} + r_2^{(2)2} - r_1^{(1)}(r_2^{(1)} + r_2^{(2)}) + r_1^{(1)}}}{(q;q)_{r_1^{(1)}} (q;q)_{r_2^{(1)} - r_2^{(2)}} (q;q)_{r_2^{(2)}}} y_1^{r_1^{(1)}} y_2^{r_2^{(1)} + r_2^{(2)}},$$

$$\text{ch } W_{L(\Lambda_2)} = \sum_{\substack{r_1^{(1)} \geqslant 0 \\ r_2^{(1)} \geqslant r_2^{(2)} \geqslant 0}} \frac{q^{r_1^{(1)2} + r_2^{(1)2} + r_2^{(2)2} - r_1^{(1)}(r_2^{(1)} + r_2^{(2)}) + r_2^{(2)}}}{(q;q)_{r_1^{(1)}} (q;q)_{r_2^{(1)} - r_2^{(2)}} (q;q)_{r_2^{(2)}}} y_1^{r_1^{(1)}} y_2^{r_2^{(1)} + r_2^{(2)}},$$

where

$$(a;q)_r = \prod_{i=1}^{r} (1 - aq^{i-1}).$$

Character of $W_{L(\Lambda)}$

Theorem

For any rectangular weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ *of level* $k = k_0 + k_j$ *in types* B, C, F *and* G *the character* $\operatorname{ch} W_{L(\Lambda)}$ *equals*

$$\sum_{\substack{r_1^{(1)} \geqslant \cdots \geqslant r_1^{(k_{\alpha_1})} \geqslant 0 \\ \vdots \\ r_l^{(1)} \geqslant \cdots \geqslant r_l^{(k_{\alpha_l})} \geqslant 0}} \frac{q^{\sum_{i=1}^{l} \sum_{t=1}^{k_{\alpha_i}} {r_i^{(t)}}^2 - \sum_{i=2}^{l} \sum_{t=1}^{k} \sum_{p=0}^{\nu_i - 1} r_{i-1}^{(t)} r_i^{(\nu_i t - p)} + \sum_{t=\nu_j k_0 + (\nu_j - 1)k_j + 1}^{k_{\alpha_j}} r_j^{(t)}}}{\prod_{i=1}^{l} (q;q)_{r_i^{(1)} - r_i^{(2)}} \cdots (q;q)_{r_i^{(k_{\alpha_i})}}} \prod_{i=1}^{l} y_i^{n_i},$$

where $\nu_i = k_{\alpha_i}/k_{\alpha_{i'}}$ *for* $i = 2, \ldots, l$, *and*

$$i' = \begin{cases} l-2, & \text{if } i = l \text{ and } \mathfrak{g} = D_l, \\ 3, & \text{if } i = l \text{ and } \mathfrak{g} = E_6, E_7, \\ 5, & \text{if } i = l \text{ and } \mathfrak{g} = E_8, \\ i-1, & \text{otherwise.} \end{cases}$$

Character of $W_{N(\Lambda)}$

- principal subspace of GVM is defined as $W_{N(\Lambda)} = U(\widetilde{\mathfrak{n}}_+)v_\Lambda$
- $W_{N(\Lambda)} \cong U(\widetilde{\mathfrak{n}}_+^{<0})$, $\widetilde{\mathfrak{n}}_+^{<0} = \mathfrak{n}_+ \otimes t^{-1}\mathbb{C}\left[t^{-1}\right]$

Theorem (M. B., S. Kožić, M. Primc)

For any rectangular weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ of level $k = k_0 + k_j$, and for any untwisted affine Lie algebra $\widetilde{\mathfrak{g}}$ we have

$$\sum \frac{q^{\sum_{i=1}^{l}\sum_{t\geqslant 1} {r_i^{(t)}}^2 - \sum_{i=2}^{l}\sum_{t\geqslant 1}\sum_{p=0}^{\nu_i-1} r_{i'}^{(t)} r_i^{(\nu_i t - p)}}}{\prod_{i=1}^{l}\prod_{j\geqslant 1}(q;q)_{r_i^{(j)}-r_i^{(j+1)}}} \prod_{i=1}^{l} y_i^{n_i} = \frac{1}{\prod_{\alpha\in R_+}(\alpha;q)_\infty},$$

where $\nu_i = \nu_i/\nu_{i'}$, $n_i = \sum_{t\geqslant 1} r_i^{(t)}$ for $i = 1,\ldots,l$ and the sum on the left hand side goes over all descending infinite sequences

$$r_1^{(1)} \geqslant \cdots \geqslant r_1^{(m)} \geqslant \cdots \geqslant 0$$
$$\vdots$$
$$r_l^{(1)} \geqslant \cdots \geqslant r_l^{(m)} \geqslant \cdots \geqslant 0$$

of integers with finite support. ($r_i^{(j)}$=number of q-p's of color i and charge $\geqslant j$).

Idea of the proof of linear independence

$$\sum_{a \in A} c_a b v_\Lambda \quad \Longrightarrow \quad \sum_{a \in A} c_a b^+ v_\Lambda$$

$$b < b^+$$

- *In the case of* $B_2^{(1)}$

$$bv_{\Lambda_0} \xmapsto{A_{\lambda_1}} be_{\lambda_1} v_{\Lambda_0} \xrightarrow{e_{\lambda_1}^{-1}} b^+ v_{\Lambda_0}$$

$$bv_{\Lambda_2} \xmapsto{A_{\lambda_1}} be_{\lambda_1} v_{\Lambda_2} \xrightarrow{e_{\lambda_1}^{-1}} b^+ v_{\Lambda_2}$$

$$A_{\lambda_1} \in I \begin{pmatrix} & L(\Lambda_1) & \\ L(\Lambda_1) & & L(\Lambda_0) \end{pmatrix}$$

$$A_{\lambda_1} \in I \begin{pmatrix} & L(\Lambda_2) & \\ L(\Lambda_1) & & L(\Lambda_2) \end{pmatrix}$$

Construction of bases of parafermionic spaces

Principal subspaces

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Standard modules

↓

Parafermionic spaces

Standard modules

- we use relations

$$\frac{1}{p!}(zx_\alpha(z))^p = \frac{1}{q!}E^-(-\alpha^\vee, z)(-zx_{-\alpha}(z))^q E^+(-\alpha^\vee, z)e_{\alpha^\vee} z^{c_\alpha + \alpha^\vee},$$

  where $k_\alpha = p + q$, $p, q \geqslant 0$.

- The coroot lattice $Q^\vee = \sum_{i=0}^{l} \mathbb{Z}\alpha_i^\vee$ acts on the standard module $L(\Lambda)$ via Weyl translations:

$$Q^\vee \ni \alpha^\vee \mapsto e_{\alpha^\vee} = e^{x_{-\alpha}(1)} e^{-x_\alpha(-1)} e^{x_{-\alpha}(1)} e^{x_\alpha(0)} e^{-x_{-\alpha}(0)} e^{x_\alpha(0)} \in \operatorname{End} L(\Lambda).$$

- $B_{U(\widehat{\mathfrak{h}}^-)}$ ... Poincaré–Birkhoff–Witt-type basis of the universal enveloping algebra of

$$\widehat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}],$$

- $B'_{W_{L(\Lambda)}} = \left\{ b \in B_{W_{L(\Lambda)}} \,:\, b \text{ does not contain q-p's of max. charge } k_{\alpha_i} \right\}$.

Standard modules

Theorem (G. Georgiev 1995, M. B., S. Kožić, M. Primc 2021)

For any rectangular highest weight Λ the set

$$\mathfrak{B}_{L(\Lambda)} = \left\{ e_{\alpha^\vee} \cdot h \cdot b \cdot v_\Lambda \,:\, \alpha^\vee \in Q^\vee,\, h \in B_{U(\widehat{\mathfrak{h}}^-)},\, b \in B'_{W_{L(\Lambda)}} \right\}$$

forms a basis for the standard module $L(\Lambda)$.

Theorem (M. B., S. Kožić, M. Primc)

For any rectangular highest weight Λ the character of the standard module $L(\Lambda)$ is

$$\frac{1}{\prod_{i=1}^{l}(q;q)_\infty^i} \sum_{a_1,\ldots,a_l \in \mathbb{Z}} \sum_{\substack{r_1^{(1)} \geqslant \cdots \geqslant r_1^{(k_{\alpha_1})} \geqslant 0 \\ \vdots \\ r_l^{(1)} \geqslant \cdots \geqslant r_l^{(k_{\alpha_l})} \geqslant 0}} \frac{q^{\sum_{i=1}^{l}\sum_{t=1}^{k_{\alpha_i}} {r_i^{(t)}}^2 - \sum_{i=2}^{l}\sum_{t=1}^{k}\sum_{p=0}^{\nu_i-1} r_{i-1}^{(t)} r_i^{(\nu_i t - p)} + \sum_{t=\nu_j k_0 + (\nu_j-1)k_j+1}^{k_{\alpha_j}} r_j^{(t)}}}{\prod_{i=1}^{l}(q;q)_{r_i^{(1)}-r_i^{(2)}} \cdots (q;q)_{r_i^{(k_{\alpha_i}-1)}}}$$

$$q^{\frac{1}{2}\left(\sum_{i=1}^{l} k_i^2 a_i^2 \langle\alpha_i,\alpha_i\rangle + \sum_{j=1}^{l-1} k_i k_j a_i a_j \langle\alpha_i,\alpha_j\rangle\right) + \sum_{i=1}^{l}\sum_{s=1}^{k_{\alpha_i}-1}\left(k_i a_i s p_i^{(s)} \langle\alpha_i,\alpha_i\rangle + \sum_{j=1}^{l} k_i a_i s p_{j-1}^{(s)}\right)} \prod_{i=1}^{l} y_i^{n_i + k_{\alpha_i} a_i},$$

where $\nu_i = k_{\alpha_i}/k_{\alpha_{i'}}$ for $i = 2,\ldots,l$

Construction of bases of parafermionic spaces

Principal subspaces

$\downarrow$

Standard modules

$\downarrow$

Parafermionic spaces

Vacuum subspace of the standard module $L(\Lambda)$

- $L(\Lambda)^{\widehat{\mathfrak{h}}^+}$ - vacuum space of the standard module $L(\Lambda)$

$$L(\Lambda)^{\widehat{\mathfrak{h}}^+} = \left\{ v \in L(\Lambda) \, : \, \widehat{\mathfrak{h}}^+ \cdot v = 0 \right\}, \qquad \text{where} \qquad \widehat{\mathfrak{h}}^+ = \mathfrak{h} \otimes t\mathbb{C}[t].$$

- $\mathcal{Z}$-operators for quasi-particles of higher charge

$$\mathcal{Z}_{n\alpha}(z) = E^-(\alpha, z)^{n/k} x_{n\alpha}(z) E^+(\alpha, z)^{n/k}$$

- $\mathcal{Z}$-operators for quasi-particle monomials of charge type $\mathcal{R}' = (n_{r_l^{(1)},l}, \ldots, n_{1,1})$

$$\begin{aligned} \mathcal{Z}_{\mathcal{R}'}(z_{r_l^{(1)},l}, \ldots, z_{1,1}) = \\ = E^-(\alpha_l, z_{r_l^{(1)},l})^{n_{r_l^{(1)},l}/k} \ldots E^-(\alpha_1, z_{1,1})^{n_{1,1}/k} x_{\mathcal{R}'}(z_{r_l^{(1)},l}, \ldots, z_{1,1}) \\ \times E^+(\alpha_l, z_{r_l^{(1)},l})^{n_{r_l^{(1)},l}/k} \ldots E^+(\alpha_1, z_{1,1})^{n_{1,1}/k}, \end{aligned}$$

where

$$x_{\mathcal{R}'}(z_{r_l^{(1)},l}, \ldots, z_{1,1}) = x_{n_{r_l^{(1)},l}\alpha_l}(z_{r_l^{(1)},l}) \ldots x_{n_{1,1}\alpha_1}(z_{1,1}).$$

$$\mathcal{Z}_{\mathcal{R}'}(z_{r_l^{(1)},l}, \ldots, z_{1,1}) = \sum_{m_{r_l^{(1)},l}, \ldots, m_{1,1} \in \mathbb{Z}} \mathcal{Z}_{\mathcal{R}'}(m_{r_l^{(1)},l}, \ldots, m_{1,1}) z_{r_l^{(1)},l}^{-m_{r_l^{(1)},l} - n_{r_l^{(1)},l}} \ldots z_{1,1}^{-m_{11} - n_{11}}$$

- the coefficients act on the vacuum space

$$\mathcal{Z}_{\mathcal{R}'}(m_{r_l^{(1)},l}, \ldots, m_{1,1}) \colon L(\Lambda)^{\widehat{\mathfrak{h}}^+} \to L(\Lambda)^{\widehat{\mathfrak{h}}^+}.$$

Vacuum subspace of the standard module $L(\Lambda)$

- The direct sum decomposition of the standard module,

$$L(\Lambda) = L(\Lambda)^{\widehat{\mathfrak{h}}^+} \oplus \widehat{\mathfrak{h}}^- U(\widehat{\mathfrak{h}}^-) \cdot L(\Lambda)^{\widehat{\mathfrak{h}}^+},$$

  defines the projection

$$\pi^{\widehat{\mathfrak{h}}^+} : L(\Lambda) \to L(\Lambda)^{\widehat{\mathfrak{h}}^+}.$$

- We have Weyl translations $e_{\alpha^\vee} : L(\Lambda)^{\widehat{\mathfrak{h}}^+} \to L(\Lambda)^{\widehat{\mathfrak{h}}^+}$ for $\alpha^\vee \in Q^\vee$.

Theorem (G. Georgiev 1995, M. B., S. Kožić, M. Primc 2021)

For any rectangular highest weight Λ the set of vectors

$$e_\mu \, \mathcal{Z}_{\mathcal{R}'}(m_{r_l^{(1)},l}, \ldots, m_{1,1}) v_{L(\Lambda)},$$

such that $\mu \in Q^\vee$ and the charge-type $\mathcal{R}'$ and the energy-type $(m_{r_l^{(1)},l}, \ldots, m_{1,1})$ satisfy difference and initial conditions for $B'_{W_{L(\Lambda)}}$, is a basis of the vacuum space $L(\Lambda)^{\widehat{\mathfrak{h}}^+}$.

Parafermionic space of the standard module $L(\Lambda)$

- The action of $e_{\alpha_i^\vee}$ produces isomorphisms of the $\mathfrak{h}$-weight subspaces

$$e_{\alpha_i^\vee} : L(\Lambda)_{\nu}^{\widehat{\mathfrak{h}}^+} \to L(\Lambda)_{\nu+k_{\alpha_i}\alpha_i}^{\widehat{\mathfrak{h}}^+}.$$

- $Q(k) = \coprod_{i=1}^{l} \mathbb{Z}k_{\alpha_i}\alpha_i \subset Q,$ where $k_{\alpha_i} = 2k/\langle \alpha_i, \alpha_i \rangle$.

Parafermionic subspace of $L(\Lambda)$

$$L(\Lambda)_{Q(k)}^{\widehat{\mathfrak{h}}^+} = \coprod_{\substack{0 \leqslant m_1 \leqslant k_{\alpha_1}-1 \\ \cdots \\ 0 \leqslant m_l \leqslant k_{\alpha_l}-1}} L(\Lambda)_{\Lambda+m_1\alpha_1+\cdots+m_l\alpha_l}^{\widehat{\mathfrak{h}}^+} \subset L(\Lambda)^{\widehat{\mathfrak{h}}^+}$$

Parafermionic space of the standard module $L(\Lambda)$

- Georgiev defines the *parafermionic space of highest weight Λ* as the space of kQ-coinvariants in the kQ-module $L(\Lambda)^{\widehat{\mathfrak{h}}^+}$:

$$L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{kQ} = L(\Lambda)^{\widehat{\mathfrak{h}}^+} / \operatorname{span}\left\{ (\rho(k\alpha) - 1) \cdot v \;\middle|\; \alpha \in Q,\, v \in L(\Lambda)^{\widehat{\mathfrak{h}}^+} \right\},$$

where $\rho(k\alpha) = e^{\alpha} \otimes \ldots \otimes e^{\alpha} \colon L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{\nu} \to L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{\nu + k\alpha}$

- parafermionic projection

$$\pi^{\widehat{\mathfrak{h}}^+}_{kQ} \colon L(\Lambda)^{\widehat{\mathfrak{h}}^+} \to L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{kQ}.$$

Parafermionic space of the standard module $L(\Lambda)$

- For any $\mu = (\Lambda + m_1\alpha_1 + \ldots + m_l\alpha_l)|_{\mathfrak{h}}$ there is a unique $e^{p_1}_{\alpha_1^\vee} \ldots e^{p_l}_{\alpha_l^\vee}$ such that

$$e^{p_1}_{\alpha_1^\vee} \ldots e^{p_l}_{\alpha_l^\vee} \colon L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{\mu} \xrightarrow{\cong} L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{\Lambda+(m_1+p_1k_{\alpha_1})\alpha_1+\ldots+(m_l+p_lk_{\alpha_l})\alpha_l} \subset L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{Q(k)},$$

so that we can identify the $\mathfrak{h}$-weight subspaces $L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{\Lambda+\mu}$ and $L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{\Lambda+\mu'}$ with $\mathfrak{h}$-weights μ and μ' in the same class $\mu + Q(k) \in Q/Q(k)$.

- parafermionic projection

$$\pi^{\widehat{\mathfrak{h}}^+}_{Q(k)} \colon L(\Lambda)^{\widehat{\mathfrak{h}}^+} \to L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{Q(k)}$$

Parafermionic space of the standard module $L(\Lambda)$

- parafermionic current

$$\Psi_\alpha(z) = \mathcal{Z}_\alpha(z) z^{-\alpha/k}, \qquad \Psi_\alpha(z) = \sum_{m \in \frac{1}{k_\alpha} + \mathbb{Z}} \psi_\alpha(m) z^{-m-1},$$

- parafermionic currents of charge n:

$$\Psi_{n\alpha}(z) = \mathcal{Z}_{n\alpha}(z) z^{-n\alpha/k}, \qquad \Psi_{n\alpha}(z) = \sum_{m \in \frac{n}{k_\alpha} + \mathbb{Z}} \psi_{n\alpha}(m) z^{-m-n}.$$

Lemma

For a simple root β and a positive integer n we have

$$\Psi_{n\beta}(z) = \left(\prod_{1 \leqslant p < s \leqslant n} \left(1 - \frac{z_p}{z_s}\right)^{\langle \beta, \beta \rangle / k} z_s^{\langle \beta, \beta \rangle / k} \right) \Psi_\beta(z_n) \ldots \Psi_\beta(z_1) \Big|_{z_n = \ldots = z_1 = z}.$$

Parafermionic space basis in types B, C, F, G

- For monomials of quasi-particles of charge type $\mathcal{R}' = (n_{r_l^{(1)},l}, \ldots, n_{1,1})$ we define the corresponding Ψ-operators

$$\Psi_{\mathcal{R}'}(z_{r_l^{(1)}\,l}, \ldots, z_{1,1}) = \mathcal{Z}_{\mathcal{R}'}(z_{r_l^{(1)}\,l}, \ldots, z_{1,1}) z_{r_l^{(1)}\,l}^{-n_{r_l^{(1)}}\alpha_l/k} \ldots z_{1,1}^{-n_{1,1}\alpha_1/k},$$

$$\Psi_{\mathcal{R}'}(z_{r_l^{(1)},l}, \ldots, z_{1,1}) = \sum_{m_{r_l^{(1)},l},\ldots,m_{1,1}} \psi_{\mathcal{R}'}(m_{r_l^{(1)},l}, \ldots, m_{1,1}) \prod_{i=1}^{l} \prod_{p=1}^{r_i^{(1)}} z_{p,i}^{-m_{p,i}-n_{p,i}}$$

where the summation in the second equality is over all sequences $(m_{r_l^{(1)},l}, \ldots, m_{1,1})$ such that $m_{i,r} \in \frac{n_{i,r}}{k_{\alpha_r}} + \mathbb{Z}$.

Lemma

For any simple roots $\beta_r, \ldots, \beta_1$ and charges $n_r, \ldots, n_1$ we have

$$\Psi_{n_r\beta_r,\ldots,n_1\beta_1}(z_r, \ldots, z_1) = \left(\prod_{1 \leqslant p < s \leqslant r} \left(1 - \frac{z_p}{z_s}\right)^{\langle n_l\beta_s, n_p\beta_p\rangle/k} z_s^{\langle n_s\beta_s, n_p\beta_p\rangle/k} \right) \times \Psi_{n_r\beta_r}(z_r) \ldots \Psi_{n_1\beta_1}(z_1).$$

Parafermionic space of the standard module $L(\Lambda)$

Theorem (M. B., S. Kožić, M. Primc 2021)

For any highest weight Λ the set of vectors

$$\pi_{Q(k)}^{\widehat{\mathfrak{h}}^+} \mathcal{Z}_{\mathcal{R}'}(m_{r_l^{(1)},l}, \ldots, m_{1,1})v_\Lambda$$
$$= \psi_{\mathcal{R}'}(m_{r_l^{(1)},l} + \langle n_{r_l^{(1)},l}\alpha_l, \Lambda\rangle/k, \ldots, m_{1,1} + \langle n_{1,1}\alpha_1, \Lambda\rangle/k)v_\Lambda,$$

such that the charge-type $\mathcal{R}'$ and the energy-type $(m_{r_l^{(1)},l}, \ldots, m_{1,1})$ satisfy difference and initial conditions for $B'_{W_{L(\Lambda)}}$, is a basis of the parafermionic space $L(\Lambda)_{Q(k)}^{\widehat{\mathfrak{h}}^+}$ of type B, C, F, G.

Parafermionic character formulas of Kuniba, Nakanishi, Suzuki in types B, C, F, G

Theorem (M. B., S. Kožić, M. Primc 2021)

For any rectangular highest weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ *character of parafermionic space is equal to*

$$\operatorname{ch} L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{Q(k)} = q^{-c_\Lambda} \operatorname{tr} q^{L_\Omega(0)} = \sum_{\mathcal{P}} D'_{\mathcal{P}}(q)\, G'_{\mathcal{P}}(q)\, B'_{\mathcal{P}}(q),$$

where the sum goes over all finite sequences $\mathcal{P} = (\mathcal{P}_l, \ldots, \mathcal{P}_1)$ *of nonnegative integers such that* $\mathcal{P}_i = (p_i^{(1)}, \ldots, p_i^{(k_{\alpha_i}-1)})$ *and*

$$D'_{\mathcal{P}}(q) = \frac{1}{\prod_{i=1}^{l}\prod_{r=1}^{k_{\alpha_i}-1} (q;q)_{p_i^{(r)}}}, \qquad \nu_i = k_{\alpha_i}/k = 2/\langle \alpha_i, \alpha_i\rangle\,,$$

$$G'_{\mathcal{P}}(q) = q^{\frac{1}{2k}\sum_{i,r=1}^{l}\sum_{m=1}^{k_{\alpha_i}-1}\sum_{n=1}^{k_{\alpha_r}-1}\langle\alpha_i,\alpha_r\rangle p_i^{(m)} p_r^{(n)}\left(\min\left\{k_{\alpha_r}m, k_{\alpha_i}n\right\}-mn\right)},$$

$$B'_{\mathcal{P}}(q) = q^{\sum_{t=\nu_j k_0-(\nu_j-1)k_j+1}^{k_{\alpha_j}-1}(t-\nu_j k_0+(\nu_j-1)k_j)p_j^{(t)}}\, q^{-\frac{k_j}{k_{\alpha_j}}\sum_{t=1}^{k_{\alpha_j}-1} t p_j^{(t)}}.$$

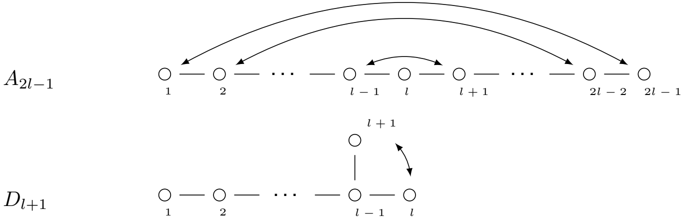
Parafermionic spaces associated to twisted affine Lie algebras

M. Okado, R. Takenaka, *Parafermionic Bases of Standard Modules for Twisted Affine Lie Algebras of Type* $A^{(2)}_{2l-1}, D^{(2)}_{l+1}, E^{(2)}_{6}$ *and* $D^{(3)}_{4}$, Algebras and Representation Theory (2022)

R. Takenaka. *Vertex algebraic construction of modules for twisted affine Lie algebras of type* $A^{(2)}_{2l}$, arXiv:2205.05271 [math.RT]

Standard modules of twisted affine Kac-Moody Lie algebra

- Let $\mathfrak{g}$ be a complex simple Lie algebra of type A_{2l-1} or D_{l+1}



- ν induces an the decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)},$$

where

$$\mathfrak{g}_{(m)} = \{x \in \mathfrak{g} \mid \nu(x) = (-1)^m x\}.$$

Standard modules of twisted affine Kac-Moody Lie algebra

- $\widetilde{\mathfrak{g}}[\nu]$ - twisted affine Kac-Moody Lie algebra associated to the simple Lie algebra $\mathfrak{g}$

$$\widetilde{\mathfrak{g}}[\nu] = \coprod_{m \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_{(2m)} \otimes t^m \oplus \mathbb{C}c \oplus \mathbb{C}d$$

- $L(\Lambda)$ - standard $\widetilde{\mathfrak{g}}[\nu]$-module of level k with the highest weight of the form

$$\Lambda = k_0\Lambda_0 + k_j\Lambda_j$$

- $L(\Lambda) \cong U(\tilde{\mathfrak{g}}[\hat{\nu}]) \cdot v_\Lambda$ is realized as a submodule of $V_Q^{T\otimes k}$ with a highest weight vector

$$v_\Lambda = v_{\Lambda_{j_k}} \otimes \cdots \otimes v_{\Lambda_{j_1}},$$

where

$$j_s = \begin{cases} 0 & \text{for} \quad 1 \leq s \leq k_0 \\ j & \text{for} \quad k_0 + 1 \leq s \leq k \end{cases}.$$

Principal subspace

- $\widetilde{\mathfrak{n}}_+[\nu] = \coprod_{m\in\frac{1}{2}\mathbb{Z}} \mathbb{C}x_{\alpha(2m)} \otimes t^m \oplus \mathbb{C}c$

Principal subspace

$$W_{L(\Lambda)} = U(\widetilde{\mathfrak{n}}_+[\nu])v_\Lambda$$

Twisted quasi-particle of color i, charge r and energy -m

$$x^\nu_{r\alpha_i}(z) = Y^\nu(x_{\alpha_i}(-1)^r\mathbf{1}, z) = \sum_{m\in\frac{1}{2}\mathbb{Z}} x^\nu_{r\alpha_i}(m)z^{-m-r} \in (\operatorname{End} V)[[z, z^{-1}]].$$

Quasi-particle bases of principal subspaces $W_{L(\Lambda)}$

Theorem

For any rectangular highest weight Λ the set

$$\mathfrak{B}_{W_{L(\Lambda)}} = \left\{ b \cdot v_\Lambda \;:\; b \in B_{W_{L(\Lambda)}} \right\}$$

forms a basis for the principal subspace $W_{L(\Lambda)}$ of twisted affine Lie algebras of type $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$.

- $B_{W_{L(\Lambda)}}$ the set of monomials of the form

$$b^\nu(\alpha_l) \cdots b^\nu(\alpha_1),$$

  where

$$b^\nu(\alpha_i) = x^\nu_{n_{r_i^{(1)},i}\alpha_i}(m_{r_i^{(1)},i}) \cdots x^\nu_{n_{2,i}\alpha_i}(m_{2,i}) x^\nu_{n_{1,i}\alpha_i}(m_{1,i})$$

- charges of quasi-particles in color $i = 1, \ldots, l$ decrease from right to left,
- maximal charge for color $i = 1, \ldots, l$ is k.

Quasi-particle bases of principal subspaces $W_{L(\Lambda)}$

$$b^{\nu}(\alpha_l)\cdots b^{\nu}(\alpha_1),$$

where

$$b^{\nu}(\alpha_i)=x^{\nu}_{n_{r_i^{(1)},i}\alpha_i}(m_{r_i^{(1)},i})\cdots x^{\nu}_{n_{2,i}\alpha_i}(m_{2,i})x^{\nu}_{n_{1,i}\alpha_i}(m_{1,i})$$

- Energies of quasi-particles satisfy certain conditions:

$$m_{p,i}\leq -\mu_i n_{p,i}-\langle \alpha_{i(0)},\alpha_{i-1(0)}\rangle \sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{i-1},n_i\}-2(p-1)\mu_i n_{p,i}-\sum_{s=1}^{n_{p,i}}\mu_i\delta_{i,j_s},$$

for $1\leq p\leq r_i^{(1)}$,

$$m_{p+1,i}\leq m_{p,i}-2\mu_i n_{p,i},\ \text{for } n_{p+1,i}=n_{p,i},\ \ 1\leq p\leq r_i^{(1)}-1,$$

where $r_0^{(1)}=0$ and $\mu_i=\frac{\langle \alpha_{i(0)},\alpha_{i(0)}\rangle}{2}$.

Character of principal subspaces $W_{L(\Lambda)}$

Theorem (M. B., S. Kožić, M. Primc)

For affine Lie algebras $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$, the principal subspace $W_{L(\Lambda)}$ has character given by:

$$\mathrm{ch}\ W_{L(\Lambda)} = \sum_{\mathcal{P}} \frac{q^{\frac{1}{2}\sum_{i,j=1}^{l}\sum_{m,n}^{t}\langle \alpha_{i\,(0)},\alpha_{j\,(0)}\rangle \min\{m,n\} p_i^{(m)} p_j^{(n)} + \tilde{p}_j}}{\prod_{i=1}^{l}\prod_{s=1}^{t}(q^{\mu_i};q^{\mu_i})_{p_i^{(s)}}} \prod_{i=1}^{l} y_i^{n_i},$$

where $\tilde{p}_j = \sum_{s=k_0+1}^{k}(s-k_0)\mu_j p_j^{(s)}$ and $n_i = \sum_{s=1}^{t} r_i^{(s)}$. The sum goes over all finite sequences $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l)$ of lt nonnegative integers, where $t = k$ for every color i.

Standard modules

- operators e_α on the standard module $L(\Lambda)$ correspond to the translation operator of the affine Weyl group
- $B_{U(\widetilde{\mathfrak{h}}[\nu]^-)}$ ... Poincaré–Birkhoff–Witt-type basis of the universal enveloping algebra of

$$\widetilde{\mathfrak{h}}[\nu]^- = \mathfrak{h}_{(0)} \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{h}_{(1)} \otimes t^{-1/2}\mathbb{C}[t^{-1}],$$

- $B'_{W_{L(\Lambda)}} = \left\{ b \in B_{W_{L(\Lambda)}} \,:\, b \text{ does not contain q-p's of max. charge } k \right\}$.

Theorem

For any rectangular highest weight Λ the set

$$\mathfrak{B}_{L(\Lambda)} = \left\{ e_\alpha \cdot h \cdot b \cdot v_\Lambda \,:\, \alpha \in Q_{(0)},\, h \in B_{U(\widetilde{\mathfrak{h}}[\nu]^-)},\, b \in B'_{W_{L(\Lambda)}} \right\}$$

forms a basis for the standard module $L(\Lambda)$.

Character of standard modules

Theorem (M. B., S. Kožić, M. Primc)

For any rectangular highest weight Λ the character of the standard module $L(\Lambda)$ is

$$\frac{1}{\prod_{i=1}^{l}(q^{\mu_i};q^{\mu_i})_\infty^i}\sum_{a_1,\ldots,a_l\in\mathbb{Z}}\sum_{\mathcal{P}}\frac{q^{\frac{1}{2}\sum_{i,j=1}^{l}\sum_{m,n}^{t}\langle\alpha_{i\,(0)},\alpha_{j\,(0)}\rangle\min\{m,n\}p_i^{(m)}p_j^{(n)}+\tilde{p}_j}}{\prod_{i=1}^{l}\prod_{s=1}^{t}(q^{\mu_i};q^{\mu_i})_{p_i^{(s)}}}$$

$$q^{\frac{1}{2}(\sum_{i=1}^{l}k^2a_i^2\langle\alpha_{i\,(0)},\alpha_{i\,(0)}\rangle+\sum_{j=1}^{l-1}k^2a_ia_j\langle\alpha_{i\,(0)},\alpha_{j\,(0)}\rangle)+\sum_{i=1}^{l}\sum_{s=1}^{k-1}(ka_isp_i^{(s)}\langle\alpha_{i\,(0)},\alpha_{i\,(0)}\rangle+\sum_{j=1}^{l}ka_isp_{j-1}^{(s)})}\prod_{i=1}^{l}y_i^{n_i+a_i},$$

where $\tilde{p}_j=\sum_{s=k_0+1}^{k}(s-k_0)\mu_jp_j^{(s)}$ *and* $n_i=\sum_{s=1}^{t}r_i^{(s)}$.

Vacuum subspace of the standard module $L(\Lambda)$

- $L(\Lambda)^{\widetilde{\mathfrak{h}}[\nu]^+}$ - vacuum space of the standard module $L(\Lambda)$

$$L(\Lambda)^{\widetilde{\mathfrak{h}}[\nu]^+} = \left\{ v \in L(\Lambda) \,\middle|\, \widetilde{\mathfrak{h}}[\nu]^+ \cdot v = 0 \right\},$$

- The direct sum decomposition of the standard module,

$$L(\Lambda) = L(\Lambda)^{\widetilde{\mathfrak{h}}[\nu]^+} \oplus \widetilde{\mathfrak{h}}[\nu]^- U(\widetilde{\mathfrak{h}}[\nu]^-) \cdot L(\Lambda)^{\widetilde{\mathfrak{h}}[\nu]^+},$$

defines the projection

$$\pi^{[\nu]^+} : L(\Lambda) \to L(\Lambda)^{\widetilde{\mathfrak{h}}[\nu]^+}$$

Theorem

For any rectangular highest weight Λ the vectors

$$e_\alpha \cdot \pi^{\widehat{\mathfrak{h}}^+} (b \cdot v_\Lambda), \quad \textit{where} \quad \alpha \in Q,\, b \in B'_{W_{L(\Lambda)}},$$

form a basis for the vacuum space $L(\Lambda)^{\widetilde{\mathfrak{h}}[\nu]^+}$.

Parafermionic space of the standard module $L(\Lambda)$

Parafermionic subspace of $L(\Lambda)$

$$L(\Lambda)_{kQ}^{\widetilde{\mathfrak{h}}[\nu]^+} = L(\Lambda)^{\widetilde{\mathfrak{h}}[\nu]^+} / \textit{span}\left\{(\rho(k\alpha)-1)\cdot v \mid \alpha \in Q,\, v \in L(\Lambda)^{\widetilde{\mathfrak{h}}[\nu]^+}\right\}$$

Character of the parafermionic subspace

$$\operatorname{ch} L(\Lambda)_{kQ}^{\widetilde{\mathfrak{h}}[\nu]^+} = \sum_{m, r_1, \ldots, r_l \geq 0} \textit{dim}\, (L(\Lambda)_{kQ}^{\widetilde{\mathfrak{h}}[\nu]^+})_{(m, r_1, \ldots r_l)} q^m \prod_{i=1}^{2} y_i^{r_i},$$

- $(L(\Lambda)_{kQ}^{\widetilde{\mathfrak{h}}[\nu]^+})_{(m, r_1, \ldots r_l)}$ the weight subspaces of $L(\Lambda)_{kQ}^{\widetilde{\mathfrak{h}}[\nu]^+}$ with respect to the degree operator

$$-d - D^{\widetilde{\mathfrak{h}}[\nu]^+}, \quad D^{\widetilde{\mathfrak{h}}[\nu]^+}\Big|_{L(\Lambda)_\mu^{\widetilde{\mathfrak{h}}[\nu]^+}} = \frac{\langle \mu_{(0)}, \mu_{(0)} \rangle}{2k}$$

Parafermionic space basis

- parafermionic projection

$$\pi_{kQ}^{[\nu]^+} : L(\Lambda)^{\widetilde{\mathfrak{h}}[\nu]^+} \to L(\Lambda)_{kQ}^{\widetilde{\mathfrak{h}}[\nu]^+}$$

Theorem (M. B., S. Kožić, M. Primc)

For any rectangular highest weight Λ the vectors

$$\pi_{kQ}^{[\nu]^+}\left(\pi^{[\nu]^+}(b \cdot v_\Lambda)\right), \quad \textit{where} \quad b \in B'_{W_{L(\Lambda)}},$$

form a basis for the parafermionic space $L(\Lambda)_{kQ}^{\widetilde{\mathfrak{h}}[\nu]^+}$ of type $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$.

Parafermionic character formula

Theorem (M. B., S. Kožić, M. Primc)

For any rectangular highest weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$ character of parafermionic space is equal to

$$\mathrm{ch}\, L(\Lambda)^{\widehat{\mathfrak{h}}^+}_{Q(k)} = \sum_{\mathcal{P}} D'_{\mathcal{P}}(q)\, G'_{\mathcal{P}}(q)\, B'_{\mathcal{P}}(q),$$

where the sum goes over all finite sequences $\mathcal{P} = (\mathcal{P}_l, \ldots, \mathcal{P}_1)$ of nonnegative integers such that $\mathcal{P}_i = (p_i^{(1)}, \ldots, p_i^{(k-1)})$ and

$$D'_{\mathcal{P}}(q) = \frac{1}{\prod_{i=1}^{l} \prod_{s=1}^{k-1} (q^{\mu_i}; q^{\mu_i})_{p_i^{(s)}}},$$

$$G'_{\mathcal{P}}(q) = q^{\frac{1}{2}\sum_{i,j=1}^{l} \sum_{m,n}^{k-1} (\langle \alpha_{i\,(0)}, \alpha_{j\,(0)} \rangle \min\{m,n\} p_i^{(m)} - \frac{mn}{2}},$$

$$B'_{\mathcal{P}}(q) = q^{\sum_{t=k_0+1}^{k-1} (t-k_0) p_j^{(t)}}\, q^{-\frac{k_j}{k} \sum_{t=1}^{k-1} t p_j^{(t)}}.$$

Thank you