VOAs arising from four dimensional superconformal filed theories

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Vertex algebras/Vertex operator algebra (VOA)

$\longleftrightarrow$ 2 dimensional chiral conformal field theories

Recent discovery (goes back to [Nakajima '94]):

VOAs appear in higher dimensional quantum field theories as well (in several ways)

$\rightsquigarrow$ new insights for representation theory of VOAs.

4D/2D duality

[Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees'15]

$$\mathbb{V} : \{\text{4D } \mathcal{N} = 2 \text{ SCFTs}\} \longrightarrow \{\text{VOAs}\} = \{\text{2D CFTs}\}$$

such that

$$\mathrm{Schur}(\mathcal{T}) = \chi_{\mathbb{V}(\mathcal{T})}(q) = \mathrm{Tr}_{\mathbb{V}(\mathcal{T})}(q^{-c/24+L_0})$$

for any 4D $\mathcal{N} = 2$ SCFT $\mathcal{T}$.

Some properties of $\mathbb{V}$

i) $c_{2D} = -12c_{4D}$
$\Rightarrow \mathbb{V}(T)$ is *never* unitary.
$\Rightarrow \mathbb{V}$ is not surjective.

ii) $\mathbb{V}$ is expected to be injective, i.e., $\mathbb{V}(\mathcal{T})$ is a *complete* invariant of 4D SCFT $\mathcal{T}$!

Beem-Rastelli conjecture

A 4D SCFT $\mathcal{T}$ has several interesting mathematical invariants (observables). One of them is the *Higgs branch* Higgs($\mathcal{T}$), which is a hyperKähler cone.

Conjecture (Beem-Rastelli '18)

$$\mathrm{Higgs}(\mathcal{T}) \cong X_{\mathbb{V}(\mathcal{T})}$$

for *any* 4D $\mathcal{N} = 2$ SCFT $\mathcal{T}$.

Here X_V is the *associated variety* of a VOA V ([A.'12]).

- The associated variety X_V of a VOA V is an affine Poisson variety that is introduced as an analogue of the associated variety of primitive ideals of $U(\mathfrak{g})$.

Remark

i) $\mathrm{Higgs}(\mathcal{T}) = \mathrm{Higgs}(\mathcal{T}_{3D}) \cong \mathrm{Coulomb}(\check{\mathcal{T}}_{3D})$ by the 3D mirror symmetry, where $\mathcal{T}_{3D}$ is the 3D theory obtained from $\mathcal{T}$ by the S^1-compactification.

ii) $X_V = \operatorname{Specm} R_V$, where R_V is Zhu's C_2-algebra of V. R_V is *not* reduced in general, while $\mathrm{Higgs}(\mathcal{T})$ is reduced.

$\tilde{X}_V = \operatorname{Spec} R_V$, the associated scheme of V.

Conjecture (Rastelli '22)

$\tilde{X}_{\mathbb{V}(\mathcal{T})}$ is a complete invariant of a 4D $\mathcal{N} = 2$ SCFT $\mathcal{T}$.

- $\mathrm{Higgs}(\mathcal{T}) = X_{\mathbb{V}(\mathcal{T})}$ is *not* a complete invariant.

What are the properties of VOAs coming from 4D theory?

Definition (A.-Kawasetsu'18)

A VOA V is called *quasi-lisse* if X_V has finitely many symplectic leaves.

- $\mathbb{V}(\mathcal{T})$ is expected to be quasi-lisse.

Theorem (A.-Kawasetsu'18)

Let V be a quasi-lisse VOA.

i) *There exists only finitely many simple ordinary representations of V;*

ii) *For an ordinary representation M, $\mathrm{tr}_M(q^{L_0 - c_{\chi(V)}/24})$ converges to a holomorphic function on the upper half place. Moreover, $\{\mathrm{tr}_M(q^{L_0 - c_{\chi(V)}/24}) \mid M \text{ ordinary }\}$ is a subspace of the space of the solutions of a modular linear differential equation (MLDE).*

Modularity of Schur index

- The space of the solutions of a MLDE is invariant under the action of $SL_2(\mathbb{Z})$.
- Together with Beem-Rastelli conjecture, the above theorem implies a certain modularity of the Schur index of a 4D $\mathcal{N} = 2$ SCFT.
- In particular, the notion of the *effective central charge* of $\mathbb{V}(\mathcal{T})$ is well-define, which gives another central charge of the 4D theory.
- It seems that the Schur index of a 4D $\mathcal{N} = 2$ SCFT ($=\chi_{\mathbb{V}(\mathcal{T})}(q)$) is a quasi-modular form.

Examples of quasi-lisse VOAs

$X = \mathbb{C}^2$, with coordinate (p, q).

X is symplectic with $\omega = dp \wedge dq$, $\{p, q\} = 1$.

$\underset{quantization}{\rightsquigarrow}$ the Weyl algebra $\mathcal{D}(\mathbb{C}^2) = \langle \frac{d}{dx}, x \mid [\frac{d}{dx}, x] = 1 \rangle$.

$\underset{\substack{chiralization \\ (affinization)}}{\rightsquigarrow}$ the $\beta\gamma$ system $\mathcal{D}^{ch}(\mathbb{C}^2)$, generated by

$$\beta(z) = \sum_{n \in \mathbb{Z}} \beta_n z^{-n}, \quad \gamma(z) = \sum_{n \in \mathbb{Z}} \gamma_n z^{-n-1},$$

with the OPE (operator product expantion)

$$\gamma(z)\beta(w) \sim \frac{1}{z - w} \left(\iff [\gamma_m, \beta_n] = \delta_{m,n} \right).$$

We have

$$X_{\mathcal{D}^{ch}(\mathbb{C}^2)} \cong \mathbb{C}^2 \text{ as Poisson varieties.}$$

$G = SL_2(\mathbb{C}) \curvearrowright \mathbb{C}^2$, Hamiltonian,

whose moment map is given by

$$\begin{array}{rccc} \mu : & \mathbb{C}^2 & \longrightarrow & \mathfrak{g}^*, \qquad\qquad \mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{sl}_2(\mathbb{C}) \\ & (p, q) & \mapsto & \frac{1}{2}\begin{pmatrix} pq & q^2 \\ -p^2 & -pq \end{pmatrix}. \end{array}$$

The comoment is a Lie algebra homomorphism given by

$$\begin{array}{rccc} \mu^* : & \mathfrak{g} & \to & \mathbb{C}[p, q] \\ & e & \mapsto & q^2/2, \\ & h & \mapsto & pq \\ & f & \mapsto & -p^2/2, \end{array}$$

where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$\rightsquigarrow$
chiralization

An action of affine Kac-Moody algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ on $\mathcal{D}^{ch}(\mathbb{C}^2)$ given by

$$e(z) = \sum_{n \in \mathbb{Z}} (et^n) z^{-n-1} \mapsto \frac{1}{2} : \beta(z)^2 :,$$

$$h(z) = \sum_{n \in \mathbb{Z}} (ht^n) z^{-n-1} \mapsto : \gamma(z)\beta(z) :,$$

$$f(z) = \sum_{n \in \mathbb{Z}} (ft^n) z^{-n-1} \mapsto -\frac{1}{2} : \gamma(z)^2 :,$$

$$K \mapsto -1/2.$$

The fields $e(z)$, $h(z)$, $f(z)$ generate a vertex subalgebra of $\mathcal{D}^{ch}(\mathbb{C}^2)$ isomorphic to the simple affine vertex algebra $L_{-1/2}(\mathfrak{sl}_2)$ associated with $\mathfrak{sl}_2$ at level $-1/2$. We have

$$X_{L_{-1/2}(\mathfrak{sl}_2)} \cong \mu(\mathbb{C}^2) = \mathcal{N} \subset \mathfrak{g}^*,$$

where $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g}$.

More generally, we have for $\mathfrak{g}$ simple, $k \in \mathbb{C}$,

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t,t^{-1}]\oplus \mathbb{C}K)} \mathbb{C}$$

the universal affine vertex algebra associated with g at level k generarted by $x(z)$, $x \in \mathfrak{g}$. The unique simple quotient $L_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$ is called the simple affine vertex algebra associated with $\mathfrak{g}$ at level k. We have

$$X_{V^k(\mathfrak{g})} = \mathfrak{g}^* \supset X_{L_k(\mathfrak{g})}, \quad G\text{-invariant, conic.}$$

- $k \notin \mathbb{Q} \Rightarrow L_k(\mathfrak{g}) = V^k(\mathfrak{g}) \Rightarrow X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$. (In fact, $L_k(\mathfrak{g}) = V^k(\mathfrak{g}) \iff X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$ [A.-Jiang-Moreau'21)
- $X_{L_k(\mathfrak{g})} = \{pt\} \iff L_k(\mathfrak{g})$ is integrable $\iff k \in \mathbb{Z}_{\geq 0}$.
- $L_k(\mathfrak{g})$ is quasi-lisse $\iff X_{L_k(\mathfrak{g})} \subset \mathcal{N}$.

(Kac-Wakimoto) admissible representations

$L_k(\mathfrak{g})$ is admissible

$\iff k + h^\vee = \frac{p}{q} \in \mathbb{Q}_{>0}$ with $p, q \in \mathbb{N}$, $(p, q) = 1$,

$p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1 \\ h & \text{if } (r^\vee, q) \neq 1, \end{cases}$ where $h^\vee$ is the dual Coxeter #, h is the Coxeter # of $\mathfrak{g}$, and $r^\vee$ is the lacing number of $\mathfrak{g}$.

Theorem (A.'15)

For an admissible $L_k(\mathfrak{g})$,

$$X_{L_k(g)} = \overline{\mathbb{O}}_q, \quad \exists \mathbb{O}_q \subset \mathcal{N}.$$

More precisely,

$$X_{L_k(\mathfrak{g})} = \begin{cases} \{x \in \mathfrak{g} \mid (\operatorname{ad} x)^{2q} = 0\} & \text{if } (r^\vee, q) = 1, \\ \{x \in \mathfrak{g} \mid \pi_{\theta_s}(x)^{2q/r^\vee} = 0\} & \text{if } (r^\vee, q) \neq 1. \end{cases}$$

- Using [Ginzburg-Kumar'93, Bendel-Nakano-Parshall-Pillen'14], one can show that

$$X_{L_k(\mathfrak{g})} \cong \operatorname{Spec} H^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C})$$

where $u_\zeta(\mathfrak{g})$ is the small quantum group at

$$\zeta = e^{2\pi i(k+h^\vee)}$$

if q is odd and not a bad prime for $\mathfrak{g}$ ([A.-van Ekeren-Moreau'22]).
- An admissible $L_k(\mathfrak{g})$ does appear from 4D theory if it is boundary admissible (i.e., p is as small as possible.) [Xie-Yan-Yau'16, Song-Xie-Yan'17, Wang-Xie'18].

F-theory and Deligne exceptional series

$$\{\text{isotrivial elliptic fibrations}\} \xrightarrow{\text{F-theory}} \{\text{4D } \mathcal{N} = 2 \text{ SCFTs}\}$$

By Kodaira, the singularities of isotrivial elliptic fibrations are labelled by simply-laced members of the *Deligne exceptional series* (DES)

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8.$$

Theorem (A.-Moreau'18)

For $\mathfrak{g} \in$ *DES*, $X_{L_{-h^\vee/6-1}(\mathfrak{g})} \cong \overline{\mathbb{O}}_{min}$.

- The above VOAs are the first examples of VOAs coming from 4D theory.

NIpotent Slodowy slices and W-algebras

For $f \in \mathcal{N}$, one can find an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}$ by Jacobson-Morozov theorem. The affine space

$$\mathcal{S}_f = f + \mathfrak{g}^e \subset \mathfrak{g}, \quad \mathfrak{g}^e = \{x \in \mathfrak{g} \mid [e, x] = 0\},$$

is transversal to G-orbits and is a Poisson variety called the *Slodowy slice* at f. For a nilpotent orbit $\mathbb{O}$,

$$S_{\mathbb{O},f} = \overline{\mathbb{O}} \cap \mathcal{S}_f$$

is a hyperKähler cone ([Kronheimer]) and called a *nilpotent Slodowy slice*.

- For a simply-laced $\mathfrak{g}$ and a subregular nilpotent element f_{subreg}, $S_{\mathbb{O}_{prin},f} = \mathcal{N} \cap S_{f_{subreg}}$ has the simple singularity as the same type as $\mathfrak{g}$ at f_{subreg} ([Brieskorn-Slodowy]).

$\mathbb{C}[\mathcal{S}_f]$

$\underset{quantization}{\rightsquigarrow}$ the finite W-algebra $U(\mathfrak{g}, f)$ ([Premet'02])

$\underset{chiralization}{\rightsquigarrow}$ the W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ ([Feigin-Frenkel'92, Kac-Roan-Wakimoto'03]):

$$\mathcal{W}^k(\mathfrak{g}, f) = H^0_{DS,f}(V^k(\mathfrak{g})),$$

where $H^\bullet_{DS,f}$ is the BRST cohomology associated with quantized Drinfeld-Sokolov reduction associated with $(\mathfrak{g}, f)$.

$$X_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathcal{S}_f \quad \text{([De-Sole-Kac'05]).}$$

Theorem (A.'15)

$\tilde{X}_{H^0_{DS,f}(L_k(\mathfrak{g}))} \cong \tilde{X}_{L_k(\mathfrak{g})} \times_{\mathfrak{g}^*} \mathcal{S}_f$ *for any* $\mathfrak{g}$, f *and* k.

In particuzlar, $X_{H^0_{DS,f}(L_k(\mathfrak{g}))} \cong X_{L_k(\mathfrak{g})} \cap \mathcal{S}_f$.

- Conjecturally, $H^0_{DS,f}(L_k(\mathfrak{g})) \cong \mathcal{W}_k(\mathfrak{g}, f)$, the simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$ ([Kac-Wakimoto]).
- For an admissible $L_k(\mathfrak{g})$, $X_{H^0_{DS,f}(L_k(\mathfrak{g}))} \cong S_{\mathbb{O}_q,f}$. The W-algebra $H^0_{DS,f}(L_k(\mathfrak{g}))$ appear from 4D SCFT if $L_k(\mathfrak{g})$ is boundary admissible [Xie-Yan- Yau'16, Song-Xie-Yan'17, Wang-Xie'18].
- $\mathcal{W}^k(\mathfrak{g}, f)$ has an interesting representation theory (cf. talks by Justine Fasquel, Naoki Genra, Anne Moreau, Sigenori Nakatsuka, Michael Penn at this conference).

Symmetric powers of $\mathbb{C}^2$

$\mathfrak{S}_n \curvearrowright \mathbb{C}^n$

$\rightsquigarrow$ symplectic action $\mathfrak{S}_n \curvearrowright T^*\mathbb{C}^n$

$\rightsquigarrow \mathfrak{S}^n\mathbb{C}^2 = \mathbb{C}^{2n}/\!/\mathfrak{S}_n = \mathrm{Spec}(\mathbb{C}[\mathbb{C}^2]^{\otimes n})^{\mathfrak{S}_n}$

A well-known resolution of singularity of $\mathfrak{S}^n\mathbb{C}^2$ is the Hilbert-Chow morphism

$$\mathrm{Hilb}^n\,\mathbb{C}^2 \to \mathfrak{S}^n\mathbb{C}^2,$$

where $\mathrm{Hilb}^n\,\mathbb{C}^2$ is the Hilbert scheme of n points in the place $\mathbb{C}^2$.

- [Kashiwara-Rouquier'08] constructed a sheaf of algebras on $\mathrm{Hilb}^n\,\mathbb{C}^2$ whose global section is the natural qunantizion of $\mathfrak{S}^n\mathbb{C}^2$ (the spherical rational Cherednik algebra).
- A naive chiralization of [Kashiwara-Rouquier'08] does not seem to work.

On the other hand, it seems that the 4D/2D duality expects the existence of a sheaf of $\mathcal{N}=4$ vertex *super*algebras on $\mathrm{Hilb}^n \mathbb{C}^2$:

"Theorem" (Ongoing work with Toshiro Kuwabara and Sven Möller])

There exists a natural sheaf $\mathcal{V}$ of $\mathcal{N}=4$ vertex *super*algebras on $\mathrm{Hilb}^n \mathbb{C}^2$ such that

$$X_{\Gamma(\mathrm{Hilb}^n \mathbb{C}^2, \mathcal{V})} \cong \mathfrak{S}^n \mathbb{C}^2.$$

- For $n=2$, $\Gamma(\mathrm{Hilb}^n \mathbb{C}^2, \mathcal{V}) \cong$ (the simple small $\mathcal{N}=4$ superconformal algebra at $c=-9) \otimes \mathcal{D}^{ch}(\mathbb{C}^2) \otimes (symplecticfermions)$. We recover Drazen's realization of the simple small $\mathcal{N}=4$ superconformal algebra at $c=-9$ ([Adamovic'15]).

Class $\mathcal{S}$ theory and Moore-Tachikawa varieties

The theory of class $\mathcal{S}(= \textit{six})$ ([Gaiotto'12])

$$\left\{ S_G(\Sigma) \mid \begin{array}{c} \Sigma\text{: a punctured Riemann surface,} \\ G\text{: complex semisimple group} \end{array} \right\}$$

- VOAs $\mathbb{V}(S_G(\Sigma))$ are called *chiral algebras of class* $\mathcal{S}$ [Beem-Peelaers-Rastellib-van Rees'15].
- Moore-Tachikawa'12 gave a mathematical description of the Higgs branch of $S_G(\Sigma)$ in terms of 2D TQFT, up to a conjecture.
- The Moore-Tachikawa conjecture was proved by Ginzburg-Kazhdan and Braverman-Finkelberg-Nakajima'19.
$\rightsquigarrow$ a new family of symplectic varieties.

For both chiral algebras of class $\mathcal{S}$ and Moore-Tachikawa vareities it is "enough" to describe them for genus zero Σ.

Theorem (A.'18)

For each semisimple group G, there exists a unique family of vertex algebras $\{\mathbf{V}_r \mid r \geq 1\}$ satisfying the desired properties of genus zero chiral algebras of class $\mathcal{S}$. Moreover, the associated variety of $\mathbf{V}_r$ is isomorphic to the Moore-Tachikawa variety.

- $\mathbf{V}_r$ admits a commuting action of r-copies of $\widehat{\mathfrak{g}}$ at the critical level.
- $\mathbf{V}_r$ satisfies the associativity that is compatible with the gluing of Riemann surfaces.
- The character of $\mathbf{V}_r$(= Schur index of class $\mathcal{S}$ theory) is closely related with multiple q-zeta values ([Milas'22]).

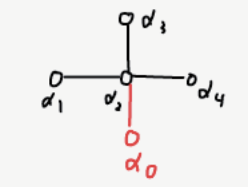
Exmaples

$G = SL_2$

$\mathrm{MT}_3 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, \quad \mathbb{C}^2 \curvearrowleft SL_2$

$\mathbf{V}_3 = \mathcal{D}^{ch}((\mathbb{C}^2)^{\otimes 3})$, $\beta\gamma$ system associated to the symplectic vector space $(\mathbb{C}^2)^{\otimes 3}$.

$\mathrm{MT}_4 = \overline{\mathbb{O}_{min}}$ in D_4



$\mathbf{V}_4 = L_{-2}(D_4)$, the simple affine vertex algebra associated with D_4 at level -2 (conjectured by [BL^{2}PRvR]).

Examples (contined)

The isomorphism $X_{L_{-2}(D_4)} \cong \overline{\mathbb{O}_{min}}$ reproves a previously stated result in [A.-Moreau'18].

The associativity gives:

- $((\mathbb{C}^2)^{\otimes 3} \times (\mathbb{C}^2)^{\otimes 3})/\Delta(SL_2) \cong \overline{\mathbb{O}_{min}}$,
  (ADHM construction of $\overline{\mathbb{O}_{min}}$)
- $H^{\infty/2+i}(\widehat{\mathfrak{sl}}_2, \mathfrak{sl}_2, \mathcal{D}^{ch}((\mathbb{C}^2)^{\otimes 3}) \otimes \mathcal{D}^{ch}((\mathbb{C}^2)^{\otimes 3})) \cong \delta_{i,0} L_{-2}(D_4)$.

Examples (continued)

$G = SL_3$

$\mathrm{MT}_3 = \overline{\mathbb{O}_{min}}$ in E_6.

$\mathbf{V}_3 = L_{-3}(E_6)$.

In general, neither MT_r nor $\mathbf{V}_r$ has a simple description.

- We have in general

$$\mathbb{V}_2 \cong \mathcal{D}^{ch}_{G,-h^\vee} \quad \text{(the cdo on } G \text{ at the critical level).}$$

By [Arlhipov-Gatsigory'02],

$$\mathcal{D}^{ch}_{G,-h^\vee}\text{-mod}^{G[[t]]} \cong \mathcal{D}_{\mathrm{Gr}_G}\text{-mod}_{-h^\vee},$$

where $\mathrm{Gr}_G = G((t))/G[[t]]$, the affine Grassmanian.

Thank you!