Schneider's p-adic continued fractions

Tomislav Pejković

Department of Mathematics
Faculty of Science
University of Zagreb, Croatia

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Continued fractions of real numbers

For (regular) continued fraction expansion of $\xi \in \mathbb{R}$ we have uniqueness and

1. $\xi \in \mathbb{Q} \Leftrightarrow$ continued fraction of ξ is finite
2. ξ quadratic irrationality $\Leftrightarrow$ continued fraction of ξ is periodic
3. Best rational approximations to ξ are convergents.

$$\xi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots}}} = [b_0, b_1, b_2, \ldots]$$

$$P_n/Q_n := [b_0, b_1, \ldots, b_n] \in \mathbb{Q}$$

$$|Q_0\xi - P_0| > |Q_1\xi - P_1| > |Q_2\xi - P_2| > \cdots$$

If $1 \leqslant Q \leqslant Q_n$ and $(P,Q) \neq (P_{n-1}, Q_{n-1}), (P_n, Q_n)$, then $|Q\xi - P| > |Q_{n-1}\xi - P_{n-1}|$.

Continued fractions of p-adic numbers

No continued fraction algorithm with all properties $1-3$.

Two main types of continued fractions in $\mathbb{Q}_p$:

Schneider (1968) and

Ruban (1970) modified by Browkin (1978).

Schneider's continued fraction

$\xi \in \mathbb{Q}_p, \quad |\xi|_p = 1$

$$\xi = b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{\ddots}}} = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, p^{a_3} \ldots]$$

a_j positive integers

$b_j \in \{1, 2, \ldots, p-1\}$

Algorithm:
b_0 such that $|\xi - b_0|_p < 1$
if $b_0 = \xi$, stop,
$p^{a_1} = |\xi - b_0|_p^{-1}$
$\xi_1 = p^{a_1}/(\xi - b_0)$, ...

Example

$$\alpha = 4+0\cdot 5+2\cdot 5^2+4\cdot 5^3+1\cdot 5^4+3\cdot 5^5+0\cdot 5^6+0\cdot 5^7+1\cdot 5^8+\cdots \in \mathbb{Q}_5$$

Schneider's continued fraction

$$\begin{aligned}
\alpha &= 4 + \frac{5^2}{3+1\cdot 5+0\cdot 5^2+4\cdot 5^3+1\cdot 5^4+1\cdot 5^5+\cdots} \\
&= 4 + \cfrac{5^2}{3+\cfrac{5}{1+0\cdot 5+1\cdot 5^2+3\cdot 5^3+\cdots}} = 4 + \cfrac{5^2}{3+\cfrac{5}{1+\cfrac{5^2}{\ddots}}}
\end{aligned}$$

Mahler's and Koksma's classification in $\mathbb{Q}_p$

For a p-adic number ξ, the value of Mahler's function w_2 (resp., of Koksma's function w_2^*) at ξ is the supremum of the real numbers w for which

$$0 < |P(\xi)|_p \leqslant \mathrm{H}(P)^{-w-1} \qquad \text{(resp.,} \quad 0 < |\xi - \alpha|_p \leqslant \mathrm{H}(\alpha)^{-w-1}\text{)}$$

is satisfied for infinitely many integer polynomials $P(X)$ (resp., algebraic numbers $\alpha \in \mathbb{Q}_p$) of degree at most two.

Mahler's and Koksma's classification in $\mathbb{Q}_p$

For every p-adic number ξ, we have

$$w_2^*(\xi) \leqslant w_2(\xi) \leqslant w_2^*(\xi) + 1$$

and every p-adic number ξ which is not rational or quadratic satisfies $w_2^*(\xi) \geqslant 2$.
The equality $w_2(\xi) = w_2^*(\xi)$ holds for almost all (with respect to the Haar measure) p-adic numbers ξ (including algebraic numbers).

Explicit examples of p-adic numbers for which the values of Mahler's and Koksma's functions differ by any prescribed value from the interval $[0, 1]$ can be constructed using Schneider's continued fractions.

Bugeaud and P. (2015)

Let $w > (5+\sqrt{17})/2$ be a real number, a a positive integer and $(\varepsilon_j)_{j\geqslant 0}$ a sequence taking its values in the set $\{0,1\}$. Define the sequence $(a_{n,w})_{n\geqslant 1}$ by

$$a_{n,w} = \begin{cases} a+3j+2, & \text{if } n = \lfloor w^j \rfloor \text{ for some } j \in \mathbb{Z}_{\geqslant 0}, \\ a+3j+\varepsilon_j, & \text{if } \lfloor w^j \rfloor < n < \lfloor w^{j+1} \rfloor \text{ for some } j \in \mathbb{Z}_{\geqslant 0}. \end{cases}$$

Set

$$\xi_w = [1, p^{a_{1,w}} : 1, p^{a_{2,w}} : 1, \ldots] \in \mathbb{Q}_p.$$

Then

$$w_2^*(\xi_w) = w - 1 \quad \text{and} \quad w_2(\xi_w) = w. \tag{1}$$

We introduce the sequence $(\varepsilon_j)_{j\geqslant 0}$ to show that our construction provides us with uncountably many explicitly given p-adic numbers ξ_w for which (1) is satisfied.

Bugeaud and P. (2015)

Let $w \geqslant 16$ be a real number, a a positive integer and $(\varepsilon_j)_{j\geqslant 0}$ a sequence taking its values in the set $\{0,1\}$. Let η be a positive real number with $\eta < \sqrt{w}/4$. Define the sequence $(a_{n,w,\eta})_{n\geqslant 1}$ by

$$a_{n,w,\eta} = \begin{cases} a+4j+3, & \text{if } n = \lfloor w^j \rfloor \text{ for some } j \in \mathbb{Z}_{\geqslant 0}, \\ a+4j+2, & \text{if } \lfloor w^j \rfloor < n < \lfloor w^{j+1} \rfloor \text{ for } j \in \mathbb{Z}_{\geqslant 0} \\ & \quad \text{and } (n - \lfloor w^j \rfloor)/\lfloor \eta w^j \rfloor \in \mathbb{Z}, \\ a+4j+\varepsilon_j, & \text{if } \lfloor w^j \rfloor < n < \lfloor w^{j+1} \rfloor \text{ for } j \in \mathbb{Z}_{\geqslant 0} \\ & \quad \text{and } (n - \lfloor w^j \rfloor)/\lfloor \eta w^j \rfloor \notin \mathbb{Z}. \end{cases}$$

Set $\xi_{w,\eta} = [1, p^{a_{1,w,\eta}} : 1, p^{a_{2,w,\eta}} : 1, \ldots] \in \mathbb{Q}_p$.
Then $w_2^*(\xi_{w,\eta}) = \dfrac{2w-2-\eta}{2+\eta}$ and $w_2(\xi_{w,\eta}) = \dfrac{2w-\eta}{2+\eta}$, hence

$$w_2(\xi_{w,\eta}) - w_2^*(\xi_{w,\eta}) = \frac{2}{2+\eta}.$$

$$\xi = b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{\ddots}}}, \qquad \frac{P_n}{Q_n} = b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{\ddots + \cfrac{p^{a_n}}{b_n}}}}$$

$$\begin{pmatrix} P_0 & P_{-1} \\ Q_0 & Q_{-1} \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{matrix} P_n = b_n P_{n-1} + p^{a_n} P_{n-2} \\ Q_n = b_n Q_{n-1} + p^{a_n} Q_{n-2} \end{matrix}, \text{ for } n \geqslant 1$$

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} b_n & 1 \\ p^{a_n} & 0 \end{pmatrix} = \prod_{i=0}^{n} \begin{pmatrix} b_i & 1 \\ p^{a_i} & 0 \end{pmatrix}$$

Taking determinant

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n+1} \prod_{i=1}^{n} p^{a_i} \Rightarrow \left| \frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} \right|_p = p^{-\sum_{i=1}^{n} a_i}.$$

Thus (P_n/Q_n) is Cauchy, $\xi = \lim_{n\to\infty} P_n/Q_n$ and

$$|Q_n \xi - P_n|_p = |\xi - P_n/Q_n|_p = p^{-\sum_{i=1}^{n+1} a_i}.$$

$\xi \in \mathbb{Q} \Leftrightarrow$ continued fraction of ξ is finite?

If the continued fraction is finite, then $\xi \in \mathbb{Q}$.

$\xi \in \mathbb{Q}$ has a finite or ultimately periodic continued fraction expansion (Bundschuh, 1977) ending in

$$p-1+\cfrac{p}{p-1+\cfrac{p}{p-1+\cfrac{p}{\ddots}}} = -1$$

e.g. $1+\frac{p}{1+\frac{p^2}{-1}} = \frac{p^2-p-1}{p^2-1} > 0$

No complete characterization of rational numbers with infinite Schneider's p-adic continued fraction expansion.
Hirsh and Washington (2011) handle some special cases.

We modify Bundschuh's result to obtain an upper bound on the required number of steps in the expansion before it terminates or reaches -1 as a complete quotient.

Expansion of rational numbers

Theorem 1

The expansion of a rational number ξ into a p-adic continued fraction either terminates or the complete quotient -1 is reached. The number of steps required, i.e. the number of complete quotients that need to be computed before either of the cases occur is $\mathcal{O}\big((\log \mathrm{H}(\xi))^2\big)$.

From Bundschuh's proof we had $\mathcal{O}(\mathrm{H}(\xi))$.

Expansion of rational numbers

Theorem 2

Let $\lambda = (1+\sqrt{1+4p})/2$. *If the* p*-adic continued fraction expansion of a rational number* ξ *is infinite, then within the first*

$$\left\lfloor \frac{\log \mathrm{H}(\xi)}{\log \lambda} \right\rfloor + 3$$

complete quotients, at least one has to be negative. This bound is in general asymptotically best possible.

Set

$$\nu_n = [1, (p:1)_{n-1}, p^2 : -1].$$

The complete quotients of ν_n, starting from the last one and moving backwards, are

$$-1 < 0, \quad [1, p^2 : -1] = 1 - p^2 < 0, \quad [1, p : 1, p^2 : -1] = \frac{1 + p - p^2}{1 - p^2} > 0.$$

This implies that in the continued fraction expansion of ν_n, the first negative complete quotient is obtained in the $(n-1)$-th step.

We also get

$$\left\lfloor \frac{\log \mathrm{H}(\nu_n)}{\log \lambda} \right\rfloor + 3 \sim n \sim n - 1.$$

Approximation by rational numbers

For rational approximation in $\mathbb{R}$ it is enough to bound the denominator of the rational number.

For approximation of $\xi \in \mathbb{Q}_p$, we have to bound both the numerator and the denominator of the rational approximation A/B since $|B\xi - A|_p$ can be as small as we like if we only bound the size of B (e.g. set $B = 1$ and $A \equiv \xi \pmod{p^k}$ for k as large as wanted).

Even with this restriction, the convergents in Schneider's continued fraction expansion of $\xi \in \mathbb{Q}_p$ are not necessarily the best rational approximations (e.g. for $\xi = [1, (p^a : 1)_\infty]$).

Quality of approximation and finiteness of expansion

Let $\xi \in \mathbb{Z}_p^{\times}$ be a p-adic unit and A/B a rational number written as a reduced fraction. Slightly changing the terminology, we say that the rational number with the reduced fraction u/v is a *better rational approximation* of ξ than A/B if

$$|v\xi - u|_p \leqslant |B\xi - A|_p \quad \text{while} \quad |u| \leqslant |A|, \ |v| \leqslant |B|,$$

with at least one of the bounds on $|u|, |v|$ being strict.

Theorem 3

A rational number with the reduced fraction $u/v \in \mathbb{Z}_p$ has an infinite p-adic Schneider's continued fraction expansion if and only if it is a better rational approximation of some $\xi \in \mathbb{Z}_p$ than some convergent of ξ.

We proved that a better rational approximation (in our terminology) than P_{k-1}/Q_{k-1} does not exist if

$$p^{\sum_{i=1}^{k} a_i} > 2P_{k-1}Q_{k-1}. \tag{2}$$

Thus the sequence $(a_n)_{n\geqslant 1}$ has to grow quite rapidly.

One sufficient condition for the existence of such an approximation is

$$p^{\sum_{i=1}^{k} a_i} \leqslant P_{k-1}Q_{k-1} \quad \text{and} \quad p^{a_k}P_{k-2}/P_{k-1} \in (1, p-1). \tag{3}$$

This shows that it is of interest to find good estimates of size for numerators and denominators of convergents given sequences $(a_n)_n$ with different rates of growth.

Bounds on height of convergents for some examples

We are interested in the lower and upper bounds of the sequence

$$X_0 = X_1 = 1, \quad X_n = cX_{n-1} + p^{a_n} X_{n-2} \text{ for } n \geqslant 2, \qquad (4)$$

where $c \in \{1, \ldots, p-1\}$ is fixed.

Comparing the initial values, we get $P_n \gg X_n$ and $Q_n \gg X_n$ if $c = 1$ is chosen, while $P_n \ll p^{a_1} X_n$, $Q_n \ll X_n$ if $c = p - 1$ is taken. In these bounds, the implied constants depend only on p. For all our examples the choice of sequence $(b_n)_{n \geqslant 0}$ in $\{1, \ldots, p-1\}$ will be irrelevant in the estimates of size and we can disregard the value of c.

Define

$$T_n = \prod_{k=1}^{n} p^{a_k} = p^{\sum_{k=1}^{n} a_k} \quad \text{and} \quad Y_n = \frac{X_n}{\sqrt{T_{n+1}}} \qquad (n \geqslant 0), \quad (5)$$

so that (4) becomes

$$Y_n = cp^{-a_{n+1}/2} Y_{n-1} + p^{(a_n - a_{n+1})/2} Y_{n-2}. \qquad (6)$$

Let $g(n) = p^{(a_n - a_{n+1})/2} + cp^{-a_{n+1}/2}$ for $n \geqslant 1$. Then

$$Y_n \geqslant \min\{Y_0, Y_1, Y_2\} \prod_{\substack{3 \leqslant k \leqslant n \\ 2|(n-k)}} \min\{g(k), g(k-1), g(k)g(k-1)\}.$$

$$Y_n \leqslant \max\{Y_0, Y_1, Y_2\} \prod_{\substack{3 \leqslant k \leqslant n \\ 2|(n-k)}} \max\{g(k), g(k-1), g(k)g(k-1)\}.$$

Then $p^{a_n} \asymp n^\alpha$ and we want to bound T_n and Y_n in this case. We obtain

$$T_n = e^{\alpha(n \log n + \mathcal{O}(n))}.$$

$$\begin{aligned} Y_n &\gg_\alpha e^{\frac{1}{2(2-\alpha)} n^{1-\frac{\alpha}{2}} - \frac{1}{2}\alpha \log n}, \\ Y_n &\ll_\alpha e^{p^{\frac{3}{2}} \frac{2}{2-\alpha} n^{1-\frac{\alpha}{2}}}. \end{aligned} \tag{7}$$

It follows that

$$\begin{aligned} \lim_{n\to\infty} Y_n &= +\infty \quad \text{for } \alpha < 2, \\ Y_n &= \mathcal{O}_\alpha(1) \quad \text{for } \alpha > 2. \end{aligned}$$

Unfortunately, for $\alpha > 2$ we cannot in general conclude that $\lim_{n\to\infty} Y_n = 0$. The difference between the upper and lower bound in (7) comes from the fact that the decrease in the sequence $(Y_n)_n$ happens at $n = \lceil p^{k/\alpha} \rceil - 1$ for positive integers k. Let D be the infinite set of all these indices n where the descent occurs. The sequence of Y_n with odd (even) indices will tend to 0 if and only if there are infinitely many odd (even) numbers in D.

The claim in one direction is obvious. If, for example, there are only finitely many odd numbers in D, then (6) shows that for odd n which are large enough, we have $Y_n > Y_{n-2}$ so that $Y_n \gg 1$ for odd n.

Note that by choosing $p^{1/\alpha}$ to be an even (odd) integer, we get that $\lceil p^{k/\alpha} \rceil - 1$ is odd (even) for every positive integer k, so that D contains only odd (even) numbers.

On the other hand, since $\lceil p^{k/\alpha} \rceil$ is odd if and only if the fractional part $\{\frac{1}{2}p^{k/\alpha}\}$ is in $(0, \frac{1}{2}]$, we see that this question is closely related to the problem of distribution modulo 1 of powers of a real number. From a result by Koksma, we know that for almost all real numbers $r > 1$ (in the sense of Lebesgue measure), the sequence $(\{\frac{1}{2}r^k\})_{k \geqslant 0}$ is uniformly distributed in the interval $[0, 1)$. This shows that for almost all $\alpha > 2$ the sequence $(\{\frac{1}{2}p^{k/\alpha}\})_{k \geqslant 0}$ is uniformly distributed in $[0, 1)$ and thus the set D contains infinitely many odd and infinitely many even numbers, so that $\lim_{n \to \infty} Y_n = 0$ really holds.

Quality of approximation by convergents for examples

Theorem 4

For a positive real number α*, let* $a_n = \lfloor \alpha \log_p n \rfloor$*,* $n \geqslant 1$*. If* $\alpha > 2$*, better rational approximations do not exist for all convergents with large enough odd indices or for all convergents with large enough even indices. For almost all* $\alpha > 2$*, better rational approximations exist for at most finitely many convergents.*

If $a_n = \lfloor n^{1/r} \rfloor$ *(*$n \geqslant 1$*) or* $a_n = n^r$ *(*$n \geqslant 1$*) or* $a_n = \lfloor \beta^n \rfloor$ *(*$n \geqslant 1$*), where* r *is any positive integer and* $\beta > 1$ *any real number, then for all but finitely many convergents there are no better rational approximations.*

Sketch of proof

$$X_0 = X_1 = 1, \quad X_n = cX_{n-1} + p^{a_n} X_{n-2} \text{ for } n \geqslant 2 \tag{4}$$

$$T_n = \prod_{k=1}^{n} p^{a_k} = p^{\sum_{k=1}^{n} a_k} \quad \text{and} \quad Y_n = \frac{X_n}{\sqrt{T_{n+1}}} \qquad (n \geqslant 0) \tag{5}$$

$$p^{\sum_{i=1}^{k} a_i} > 2P_{k-1}Q_{k-1} \tag{2}$$

Taking into account (4), (5) we see that

$$P_n Q_n / p^{\sum_{i=1}^{n+1} a_i}$$

satisfies the same lower and upper bounds as those obtained for Y_n^2. Now the conclusion follows from (2) and the bounds on $(Y_n)_n$.

The irrationality exponent $\mu(\xi)$ of an irrational p-adic number ξ is the supremum of the real numbers μ such that

$$\left|\xi - \frac{a}{b}\right|_p < \mathrm{H}(a/b)^{-\mu}$$

has infinitely many solutions in rational numbers a/b.

It is easily seen that this inequality can be replaced by

$$|b\xi - a|_p < \mathrm{H}(a/b)^{-\mu}.$$

The lower bound $\mu(\xi) \geqslant 2$ always holds. In order to determine the irrationality exponent of numbers introduced previously, we use the following Lemma.

Lemma 5

For $\xi \in \mathbb{Q}_p$*, let* $(\vartheta_k)_{k \geqslant 0}$ *be a sequence of real numbers such that* $\liminf_{k\to\infty} \vartheta_k > 1$ *and let* $(P_k/Q_k)_{k \geqslant 0}$ *be a sequence of distinct rational numbers such that*

$$\left| \xi - \frac{P_k}{Q_k} \right|_p = \mathrm{H}(P_k/Q_k)^{-\vartheta_k}$$

holds for $k \geqslant 0$*. If*

$$\limsup_{k\to\infty} \vartheta_k \geqslant 1 + \limsup_{k\to\infty} \frac{\log \mathrm{H}(P_{k+1}/Q_{k+1})}{(\vartheta_k - 1)\log \mathrm{H}(P_k/Q_k)},$$

then $\mu(\xi) = \limsup_{k\to\infty} \vartheta_k$*.*

Using the convergents of the p-adic continued fraction as a sequence of rational approximations $(P_k/Q_k)_{k\geqslant 0}$ from the previous Lemma, we obtain the following result.

Theorem 6

Let $\alpha > 0$ and $\beta > 1$ be real numbers and r a positive integer.

If $a_n = \lfloor \alpha \log_p n \rfloor$ $(n \geqslant 1)$ or $a_n = \lfloor n^{1/r} \rfloor$ $(n \geqslant 1)$ or $a_n = n^r$ $(n \geqslant 1)$, the irrationality exponent of $\xi = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, p^{a_3} \dots]$ is 2.

If $a_n = \lfloor \beta^n \rfloor$ $(n \geqslant 1)$, we have $\mu(\xi) = \beta + 1$.

Convergence in the reals

The sequence of convergents $(P_n/Q_n)_n$ of a p-adic number ξ has a limit in $\mathbb{R}$ if and only if the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{Q_{k-1}Q_k} p^{\sum_{i=1}^{k} a_i} \tag{8}$$

converges in $\mathbb{R}$.

If in an example $\sum_{k=1}^{\infty} Y_k^2$ converges or diverges regardless of $c \in \{1, \dots, p-1\}$, then (8) converges in the reals if and only if

$$\sum_{k=1}^{\infty} Y_k^2 = +\infty. \tag{9}$$

The case $a_n = \lfloor \alpha \log_p n \rfloor$ and $\alpha > 2$

Here the situation is not straightforward. As discussed before, the rate of decrease of the associated sequence $(Y_n)_n$ depends on the distribution of $(\{p^{k/\alpha}/2\})_k$ in $[0, 1)$. As was shown, for some $\alpha > 2$, the subsequences of $(Y_n)_n$ obtained by choosing only even n or only odd n are non-decreasing and thus (9) certainly holds in those cases.

However, for almost all $\alpha > 2$, the sequence $(\{p^{k/\alpha}/2\})_{k\geqslant 0}$ is uniformly distributed in $[0,1)$. Thus, for any $\varepsilon \in (0,1/2)$, for all K large enough, there are between $(\frac{1}{2}-\varepsilon)K$ and $(\frac{1}{2}+\varepsilon)K$ integers k in $(0,K)$ satisfying $\{p^{k/\alpha}/2\} \in (0,\frac{1}{2}]$, or equivalently, $\lceil p^{k/\alpha} \rceil - 1$ is even.

This shows that, starting with a large enough number, the sequence $(n_i)_i$ of even integers in D satisfies

$$p^{\frac{2i}{(1+2\varepsilon)\alpha}} < n_i < p^{\frac{2i}{(1-2\varepsilon)\alpha}} \tag{10}$$

or, equivalently,

$$\left(\frac{1}{2}-\varepsilon\right)\alpha \log_p n_i < i < \left(\frac{1}{2}+\varepsilon\right)\alpha \log_p n_i \tag{11}$$

for i large enough.

For any large enough even integer n, we have $n_i \leqslant n < n_{i+1}$ for some i and now a more precise bound on Y_n and (10) give

$$\begin{aligned} Y_n &\ll p^{(\frac{1}{\alpha}-\frac{1}{2})i} \ll p^{(\frac{1}{\alpha}-\frac{1}{2})(i+1)} \ll p^{(\frac{1}{\alpha}-\frac{1}{2})(\frac{1}{2}-\varepsilon)\alpha \log_p n_{i+1}} \\ &\ll n_{i+1}^{(1-\frac{\alpha}{2})(\frac{1}{2}-\varepsilon)} \ll n^{(1-\frac{\alpha}{2})(\frac{1}{2}-\varepsilon)}. \end{aligned} \tag{12}$$

For $\alpha > 4$, choose $\varepsilon > 0$ such that

$$\varepsilon < \frac{\alpha - 4}{2\alpha}.$$

Then (12) implies

$$Y_n \ll n^{(1-\frac{\alpha}{2})(\frac{1}{2}-\varepsilon)} \ll n^{-\frac{1}{2}-\varepsilon}.$$

Analogously, we prove that the same bound holds for all odd numbers n which are large enough. This shows that (9) does not hold for such α.

Theorem 7

First, let $a_n = \lfloor \alpha \log_p n \rfloor$ *for some positive real number* α *and all positive integers* n. *If* $\alpha < 2$, *the continued fraction* $[b_0, p^{a_1} : b_1, p^{a_2} : b_2, p^{a_3} \dots]$ *converges in the field of real numbers. For* $p > 2$, *there exist* $\alpha > 2$ *such that this continued fraction converges in* $\mathbb{R}$. *For almost all real numbers* $\alpha > 4$, *this continued fraction does not converge in* $\mathbb{R}$.

If $a_n = \lfloor n^{1/r} \rfloor$ $(n \geqslant 1)$ *or* $a_n = n^r$ $(n \geqslant 1)$ *or* $a_n = \lfloor \beta^n \rfloor$ $(n \geqslant 1)$, *where* r *is any positive integer and* $\beta > 1$ *real number, then* $[b_0, p^{a_1} : b_1, p^{a_2} : b_2, p^{a_3} \dots]$ *does not converge in* $\mathbb{R}$.