Asymptotics, strange identities and the Habiro ring

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Main goals

- D. Zagier, "Vassiliev invariants and a strange identity related to the Dedekind eta-function", Topology **40** (2001), no. 5, 945–960.

- K. Habiro, "Cyclotomic completions of polynomial rings", Publ. Res. Inst. Math. Sci. **40** (2004), no. 4, 1127–1146.

- A. Goswami, A. Jha, B. Kim, –, "Asymptotics and sign patterns for coefficients in expansions of Habiro elements", `https://arxiv.org/abs/2204.02628`

Fishburn numbers

- Consider the usual q-Pochhammer symbol

$$(a)_n = (a;q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}),$$

valid for $n \in \mathbb{N}_0 \cup \{\infty\}$ and the generating function

$$\sum_{n\geq 0} (1-q;1-q)_n = \sum_{n\geq 0} \xi(n)\, q^n = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \cdots .$$

- A *Fishburn matrix* is an upper-triangular matrix with non-negative integer entries and without zero rows or columns such that the sum of all entries is n. So, $\xi(3) = 5$ since

$$(3)\,,\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- For other interesting combinatorial interpretations of $\xi(n)$, see A022493.

Arithmetic and asymptotic properties of $\xi(n)$

- Andrews and Sellers (2016), Guerzhoy, Kent and Rolen (2014), Garvan (2015), Ahlgren and Kim (2015), Straub (2015) studied prime power congruences for $\xi(n)$.

- For example, we have for all $r, m \in \mathbb{N}$

$$\xi(5^r m - 1) \equiv \xi(5^r m - 2) \equiv 0 \pmod{5^r},$$

$$\xi(7^r m - 1) \equiv 0 \pmod{7^r}$$

and

$$\xi(11^r m - 1) \equiv \xi(11^r m - 2) \equiv \xi(11^r m - 3) \equiv 0 \pmod{11^r}.$$

- In 2001, Zagier proved that as $n \to \infty$

$$\xi(n) \sim \left(\frac{6}{\pi^2}\right)^n n! \sqrt{n} \frac{12\sqrt{3}}{\pi^{\frac{5}{2}}} e^{\frac{\pi^2}{12}}.$$

Strange series

- On October 14, 1997, Kontsevich introduced the expression

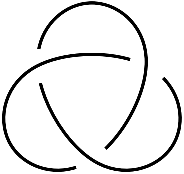
$$F(q) := \sum_{n\geq 0} (q)_n,$$

which does not converge on any open subset of $\mathbb{C}$, but is well-defined when q is a root of unity and q is replaced by $1 - q$.

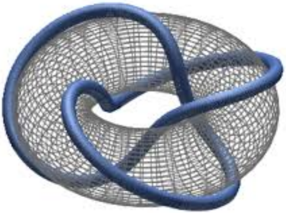
- There are other examples of such strange series arising in knot theory ...

Knots

- Let K be a knot in $\mathbb{R}^3$. For example, the right-handed trefoil knot is given by

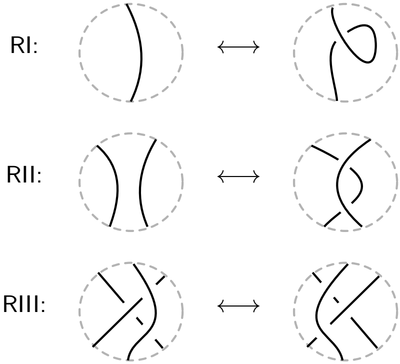


- We will consider the family of *torus knots* $T(p,q)$. For example, $T(3,2)$ is given by



Knots

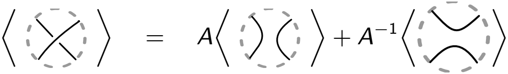
- (Reidemeister, 1927) Let K and K' be two knots with diagrams D and D'. Then K is isotopic to K' in $\mathbb{R}^3$ if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the moves *RI*, *RII* and *RIII* given by the following:



The Kauffman bracket

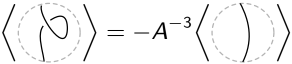
- The *Kauffman bracket* $\langle D \rangle$ of D is defined by

$$\left\langle D \sqcup \bigcirc \right\rangle = (-A^2 - A^{-2})\langle D \rangle$$



$$\left\langle \text{empty diagram} \right\rangle = 1.$$

- $\langle D \rangle$ is invariant under RII and RIII, but not RI as



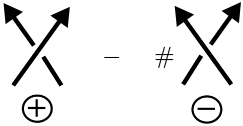
The Jones polynomial

- The Jones polynomial $V(K) = V(K; q)$ is given by

$$V(K) = \frac{1}{(-A^2 - A^{-2})}(-A)^{-3w(D)}\langle D\rangle\Big|_{A^2=q^{-1/2}}$$

where

$$w(D) = \quad \# \qquad - \quad \#$$



is the "writhe" of D.

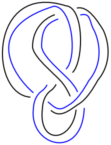
- $V(K)$ is invariant under RI, RII and RIII.

The colored Jones polynomial

- The colored Jones polynomial $J_N(K;q)$ is a linear combination of cablings of D using Chebyshev polynomials: $S_1(x) = 1$, $S_2(x) = x$, $S_N(x) = xS_{N-1}(x) - S_{N-2}(x)$.

- For example, $S_3(x) = x^2 - 1$. So, we have

$$J_3(4_1;q) \quad = \quad \star \Big\langle$$



$$\Big\rangle - 1$$

- The $N = 2$ case recovers the Jones polynomial.

Strange series and knot theory

- Habiro (2000), T. Lê (2003) proved the expansion

$$J_N(\text{trefoil}; q) = q^{1-N} \sum_{n \geq 0} q^{-nN} (q^{1-N})_n.$$

- Observe that

$$F(\zeta_N) = \zeta_N^{-1} J_N(\text{trefoil}; \zeta_N)$$

and so

$$F(q) := \sum_{n \geq 0} (q)_n$$

is "extracted" from the colored Jones polynomial of the trefoil.

The Habiro ring

- Habiro (2004): Consider the projective system

$$\left(\left(\mathbb{Z}[q] / \langle (q)_i \rangle \right)_{i \in \mathbb{N}}, \ (f_{ij})_{i \leq j \in \mathbb{N}} \right)$$

where

$$f_{ij} : \mathbb{Z}[q] / \langle (q)_j \rangle \longrightarrow \mathbb{Z}[q] / \langle (q)_i \rangle$$

$$m(q) = \sum_{k=0}^{j-1} \underbrace{m_k(q)}_{\deg \leq k} (q)_k \mapsto m(q) \pmod{(q)_i}.$$

Then, $\widehat{\mathbb{Z}[q]} := \varprojlim_n \mathbb{Z}[q] / \langle (q)_n \rangle$.

- Every $m \in \widehat{\mathbb{Z}[q]}$ can be expressed as

$$m(q) = \sum_{n \geq 0} m_n(q) \, (q)_n$$

where $m_n(q) \in \mathbb{Z}[q]$, $\deg(m_n) \leq n$ for all $n \in \mathbb{N}$.

Strange identities

- The key to Zagier's asymptotic result for the Fishburn numbers $\xi(n)$ is the "strange identity"

$$F(q) \text{``}=\text{''} -\frac{1}{2}\sum_{n\geq 1} n\left(\frac{12}{n}\right) q^{\frac{n^2-1}{24}}.$$

- Behind every strange identity is *actual* q-series identity. For example, we have

$$2\sum_{n\geq 0}\Big[(q)_n-(q)_\infty\Big]+(q)_\infty\left(-1+2\sum_{n\geq 1}\frac{q^n}{1-q^n}\right)=-\sum_{n\geq 1} n\left(\frac{12}{n}\right) q^{\frac{n^2-1}{24}}$$

using Bailey pairs (Lovejoy, 2022).

- Express $\xi(n)$ in terms of the Taylor series coefficients of $F(e^{-t})$, then use the strange identity to obtain estimates for these coefficients.

Other strange identities

- Hikami (2006): For $m \in \mathbb{N}$ and $0 \leq \ell \leq m-1$, consider

$$X_m^{(\ell)}(q) := \sum_{k_1, k_2, \ldots, k_m = 0}^{\infty} (q)_{k_m} q^{k_1^2 + \cdots + k_{m-1}^2 + k_{\ell+1} + \cdots + k_{m-1}} \prod_{i=1}^{m-1} \underbrace{\begin{bmatrix} k_{i+1} + \delta_{i,\ell} \\ k_i \end{bmatrix}}_{q\text{-binomial coefficient}} \in \widehat{\mathbb{Z}[q]}.$$

- We have that

$$X_m^{(0)}(\zeta_N) = \zeta_N^{-m} J_N(T(2, 2m+1); \zeta_N)$$

  and $X_1^{(0)}(q) = F(q)$.

- $X_m^{(\ell)}(q)$ satisfies a strange identity.

Other strange identities

- Consider the family of torus knots $T(3,2^t)$, $t \geq 1$. In 2016, Konan proved

$$J_N(T(3,2^t);q) = (-1)^{h''(t)} q^{2^t-1-h'(t)-N} \sum_{n\geq 0} (q^{1-N})_n q^{-Nnm(t)}$$
$$\times \sum_{3\sum_{\ell=1}^{m(t)-1} j_\ell \ell \equiv 1 \pmod{m(t)}} (-q^{-N})^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t)+\sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}}$$
$$\times \sum_{k=0}^{m(t)-1} q^{-kN} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}.$$

- Let $\mathcal{F}_t(q) := (-1)^{h''(t)} q^{-h'(t)} \sum_{n\geq 0} (q)_n \sum_{j_\ell}{}' (-1)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^v \sum_{k=0}^{m(t)-1} \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}$.

  Then
$$\mathcal{F}_t(\zeta_N) = \zeta_N^{1-2^t} J_N(T(3,2^t);\zeta_N),$$
  $\mathcal{F}_1(q) = F(q)$ and $\mathcal{F}_t(q)$ satisfies a strange identity.

Main result

- Let $f : \mathbb{Z} \to \mathbb{C}$ be a function of period $M \geq 2$. For integers $a \geq 0$ and $b > 0$, consider the partial theta series

$$\theta^{(\nu)}_{a,b,f}(q) := \sum_{n \geq 0} n^{\nu} f(n)\, q^{\frac{n^2 - a}{b}}$$

where $\nu \in \{0, 1\}$.

- Suppose we have

$$F_f(q) \text{``}=\text{''} \theta^{(\nu)}_{a,b,f}(q) \qquad (\star)$$

where $F_f(q) := \sum_{n \geq 0} A_{n,f}(q)\,(q)_n$ and $A_{n,f}(q) \in \mathbb{Z}[q]$.

- Write

$$F_f(1-q) =: \sum_{n \geq 0} \xi_f(n) q^n.$$

Theorem (Goswami, Jha, Kim, –)

Assume ($\star$) is true. Then as $n \to \infty$, we have

$$\xi_f(n) \sim (-1)^{\nu} \left(\frac{M}{2\pi k_{\nu}}\right)^{2n+\nu+1} \frac{C_{f,\nu} 2^{2n+\nu} n! n^{\nu-\frac{1}{2}}}{b^n \sqrt{\pi M}} e^{\frac{bk_{\nu}^2 \pi^2}{2M^2}}.$$

- As a consequence, write

  $$X_m^{(\ell)}(1-q) := \sum_{n \geq 0} \xi_{\ell,m}(n)\, q^n$$

  and

  $$\mathcal{F}_t(1-q) := \sum_{n \geq 0} \xi_t(n)\, q^n.$$

  Then as $n \to \infty$, we have

  $$\xi_{\ell,m}(n) \sim \sin\left(\frac{\pi(\ell+1)}{2m+1}\right) \left(\frac{2m+1}{\pi^2}\right)^{n+1} \frac{2^{n+3} n! \sqrt{n}}{\sqrt{\pi}} e^{\frac{\pi^2}{8m+4}}$$

  and

  $$\xi_t(n) \sim \frac{\sin(\frac{\pi}{2^t})}{2^t\sqrt{3\pi}} \left(\frac{3 \cdot 2^t}{\pi}\right)^{2n+2} \frac{2^{2n+1} n! \sqrt{n}}{(3 \cdot 2^{t+2})^n} e^{\frac{\pi^2}{3 \cdot 2^{t+1}}}.$$

Sketch of proof

- Express $\xi_f(n)$ in terms of Stirling numbers of the first kind and the coefficients $B_{n,f}$ of

$$F_f(e^{-t}) := \sum_{n \geq 0} \frac{B_{n,f}}{n!} t^n.$$

- Use the strange identity $(\star)$ to relate $B_{n,f}$ to $L(-2n-\nu, f)$.

- Compute estimates for $B_{n,f}$. This yields the estimates for $\xi_f(n)$.

Future work

- There are other expansions for $F(q)$. We have (see A138265)

$$F\left(\frac{1}{1+q}\right) = 1 + q + q^2 + 2q^3 + 5q^4 + 16q^5 + 61q^6 + 271q^7 + 1372q^8 + \cdots$$

  and (see A289312)

$$F\left(\frac{1-q}{1+q}\right) = 1 + 2q + 6q^2 + 26q^3 + 142q^4 + 946q^5 + 7446q^6 + 67658q^7 + \cdots .$$

- Recall that $\mathcal{F}_1(q) = F(q)$ and $X_1^{(0)}(q) = F(q)$.

Future work

- Coefficients for $\mathcal{F}_t(\frac{1}{1+q})$:

$$
\begin{aligned}
t &= 1 \quad 1 + q + q^2 + 2q^3 + 5q^4 + 16q^5 + 61q^6 + 271q^7 + 1372q^8 + 7795q^9 + \dots \\
t &= 2 \quad 1 + 3q + 8q^2 + 31q^3 + 160q^4 + 1029q^5 + 7910q^6 + 70658q^7 + 718687q^8 + \dots \\
t &= 3 \quad 1 + 7q + 42q^2 + 329q^3 + 3395q^4 + 43638q^5 + 670663q^6 + 11980513q^7 + \dots \\
t &= 4 \quad 1 + 15q + 190q^2 + 3005q^3 + 61885q^4 + 1587420q^5 + 48722721q^6 + \dots \\
t &= 5 \quad 1 + 31q + 806q^2 + 25637q^3 + 1054465q^4 + 54008696q^5 + 3311724885q^6 + \dots
\end{aligned}
$$

- Coefficients for $\mathcal{F}_t(\frac{1-q}{1+q})$:

$$
\begin{aligned}
t &= 1 \quad 1 + 2q + 6q^2 + 26q^3 + 142q^4 + 946q^5 + 7446q^6 + 67658q^7 + 697118q^8 + \dots \\
t &= 2 \quad 1 + 6q + 38q^2 + 318q^3 + 3406q^4 + 44790q^5 + 699126q^6 + 12630702q^7 + \dots \\
t &= 3 \quad 1 + 14q + 182q^2 + 2982q^3 + 62734q^4 + 1630174q^5 + 50474886q^6 + \dots \\
t &= 4 \quad 1 + 30q + 790q^2 + 25590q^3 + 1064590q^4 + 54905390q^5 + 3382387174q^6 + \dots \\
t &= 5 \quad 1 + 62q + 3286q^2 + 211606q^3 + 17496462q^4 + 1797007566q^5 + \dots
\end{aligned}
$$

- Are the coefficients of $\mathcal{F}_t(1-q)$, $\mathcal{F}_t(\frac{1}{1+q})$ and $\mathcal{F}_t(\frac{1-q}{1+q})$ positive for all $t \geq 1$?

Future work

- Coefficients for $X_5^{(\ell)}(\frac{1}{1+q})$:

$$\begin{aligned}
\ell = 0 \quad & 1 + 5q + 25q^2 + 180q^3 + 1725q^4 + 20538q^5 + 291571q^6 + 4801844q^7 + \dots \\
\ell = 1 \quad & 2 + 9q + 45q^2 + 330q^3 + 3195q^4 + 38286q^5 + 545949q^6 + 9020385q^7 + \dots \\
\ell = 2 \quad & 3 + 12q + 60q^2 + 446q^3 + 4350q^4 + 52374q^5 + 749294q^6 + 12410001q^7 + \dots \\
\ell = 3 \quad & 4 + 14q + 70q^2 + 525q^3 + 5145q^4 + 62139q^5 + 890925q^6 + 14779290q^7 + \dots \\
\ell = 4 \quad & 5 + 15q + 75q^2 + 565q^3 + 5550q^4 + 67134q^5 + 963578q^6 + 15997212q^7 + \dots
\end{aligned}$$

- Coefficients for $X_5^{(\ell)}(\frac{1-q}{1+q})$:

$$\begin{aligned}
\ell = 0 \quad & 1 + 10q + 110q^2 + 1650q^3 + 32230q^4 + 776666q^5 + 22237534q^6 + \dots \\
\ell = 1 \quad & 2 + 18q + 198q^2 + 3018q^3 + 59598q^4 + 1446210q^5 + 41605014q^6 + \dots \\
\ell = 2 \quad & 3 + 24q + 264q^2 + 4072q^3 + 81048q^4 + 1976760q^5 + 57067560q^6 + \dots \\
\ell = 3 \quad & 4 + 28q + 308q^2 + 4788q^3 + 95788q^4 + 2344076q^5 + 67828068q^6 + \dots \\
\ell = 4 \quad & 5 + 30q + 330q^2 + 5150q^3 + 103290q^4 + 2531838q^5 + 73345162q^6 + \dots
\end{aligned}$$

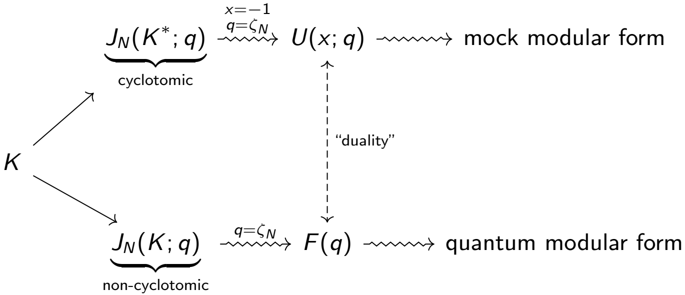
- Are the coefficients of $X_m^{(\ell)}(1-q)$, $X_m^{(\ell)}(\frac{1}{1+q})$ and $X_m^{(\ell)}(\frac{1-q}{1+q})$ positive for all $m \in \mathbb{N}$ and $0 \le \ell \le m-1$?

Future work

- (Habiro, 2008) For *any* knot K, we have the "cyclotomic expansion"

$$J_N(K;q) = \sum_{n\geq 0} \underbrace{C_n(K;q)}_{\in\mathbb{Z}[q^{\pm 1}]} (q^{1+N})_n (q^{1-N})_n.$$

- Consider the picture:



- $T(3,2)$: Zagier $\rightsquigarrow F\checkmark$, Hikami, Lovejoy $\rightsquigarrow U\checkmark$
  $T(2,2m+1)$: Hikami $\rightsquigarrow F\checkmark$, Mortenson, Zwegers $\rightsquigarrow U\checkmark$
  $T(3,2^t)$: Goswami, – $\rightsquigarrow F\checkmark$, NO U *yet*!!
  Satellite knots? Hyperbolic knots?

Thank you!