Bernstein–Zelevinsky classification of irreducible representations of coverings groups of $\mathrm{GL}_r(F)$, F non-archimedean local field

Erez Lapid
Joint work with Eyal Kaplan and Jiandi Zou

October 2022

Covering groups over local and global fields have been studied extensively since the 1960s.

- Weil (1964): the Weil representation on the metaplectic group (covering of the symplectic group)
- Kubota (1967): coverings of GL_2 and automorphic forms on them.
- Steinberg (1968), Moore (1968), Matsumoto (1969): covering of local groups.
- Serre (1967): connection to the congruence subgroup problem.
- Shimura (1972) (and a later perspective): relation between representations of $\mathrm{GL}_2(F)$ and its double cover.
- Flicker (1980): ditto for higher order covers of $\mathrm{GL}_2(F)$.
- Kazhdan–Patterson (1980s): relation between representations of central coverings of $\mathrm{GL}_r(F)$, $r \geq 1$ and $\mathrm{GL}_r(F)$.

- Bump et al. (1980s–): extensive study of Eisenstein series of covering groups with deep connections to physics.
- Brylinski–Deligne (2000): algebraic and functorial approach for central extensions.
- Weissman (2015): L-groups for Brylinski–Deligne covering groups and formulation of local Langlands correspondence.
  Wee Teck Gan and his students worked out many special cases.

Special representations of double covers (such as Weil representation) play an important role in the global theory, even for algebraic groups.
The jury is still out on the role of higher covering groups in number theory.

Henceforth let F be a local non-archimedean field and let $\underline{G}$ be a reductive group over F.
By a covering group of $\underline{G}(F)$ we mean a short exact sequence

$$1 \to A \to G \to \underline{G}(F) \to 1$$

where A is finite and central in G.
From now on all groups are implicitly ℓ-groups, i.e., Hausdorff, separable, topological groups having a system of neighborhoods of the identity consisting of open subgroups.

- $\mathcal{M}(G)$ – the category of complex, smooth representations of G. (The stabilizer in G of any vector is open.)
- (For later) If χ is a character of a central subgroup B of G, we denote by $\mathcal{M}_\chi(G)$ the subcategory where B acts by χ.

Fact (Deligne)
Let $\underline{P}$ be a parabolic subgroup of $\underline{G}$ defined over F. Let $\underline{U}$ be its unipotent radical. Then, there is a unique lifting

$$\underline{U}(F) \to G$$

that is $\operatorname{Ad} \underline{P}(F)$-equivariant.
(The statement is non-trivial only if char F divides the size of A. Otherwise, the exact sequence splits over $\underline{U}(F)$ uniquely.)

We therefore have $P = M \ltimes \underline{U}$ and we can define parabolic induction as a functor

$$\mathcal{M}(M) \to \mathcal{M}(G)$$

where M is the induced covering group of the Levi quotient of $\underline{M}(F)$.

The basic elements of Bernstein–Zelevinsky theory stay put for covering groups.
(Standard properties of parabolic induction and Jacquet functor; Jacquet subrepresentation theorem; cuspidal support; geometric lemma; description of Bernstein center; Casselman's pairing; second adjointness; etc.)

Throughout fix $n \geq 1$ and assume that F^* contains n n-th roots of unity. (In particular, char $F \nmid n$.)
Consider the n-th order Hilbert symbol

$$(\cdot,\cdot)_n : F^* \times F^* \to \mu_n$$

It is a bi-multiplicative, antisymmetric pairing that descends to a non-degenerate pairing

$$F^*/(F^*)^n \times F^*/(F^*)^n \to \mu_n.$$

Using $(\cdot,\cdot)_n$ Matsumoto defined a central extension of $\mathrm{SL}_r(F)$, $r \geq 1$ (and in fact for any simply connected, split group over F).
This defines a central extension G_r of $\mathrm{GL}_r(F)$, $r \geq 1$ by embedding

$$\mathrm{GL}_r \to \mathrm{SL}_{r+1}, \quad g \mapsto \begin{pmatrix} \det(g)^{-1} & \\ & g \end{pmatrix}$$

This central extension (together with its twists) was considered by Kazhdan and Patterson in the 1980s.
They conjectured that the genuine irreducible representation of G_r are in natural correspondence with the irreducible representations of $\mathrm{GL}_r(F)$ whose central character is trivial on μ_n.
(The case $r = 2$ had been considered by Flicker.)

The goal of this talk is to give a classification of the irreducible representations of G_r that is parallel to that of Bernstein–Zelevinsky in the linear case.

Note

An important tool in the Bernstein–Zelevinsky theory is the notion of derivative (a functor from $\mathcal{M}(\mathrm{GL}_r(F))$ to $\mathcal{M}(\mathrm{GL}_{r-1}(F))$).
If we consider inner forms of GL_r, we no longer have this functor.
However, using ideas of Tadić, Mínguez, Sécherre and others it is possible to circumvent this functor and give a uniform proof of Zelevinsky's classification for inner forms of GL_r.

The takeaway from this talk is that essentially the same method works for Kazhdan–Patterson coverings of GL_r.
However, it is a little bit more than just dotting the i's and crossing the t's.

What is the issue?

Parabolic induction defines a biadditive, biexact bifunctor

$$\mathcal{M}(\mathrm{GL}_{r_1}(F)) \times \mathcal{M}(\mathrm{GL}_{r_2}(F)) \to \mathcal{M}(\mathrm{GL}_r(F)), \quad r = r_1 + r_2.$$

(Bernstein–Zelevinsky product).
The problem is that the preimage of $\mathrm{GL}_{r_1}(F) \times \mathrm{GL}_{r_2}(F)$ in G_r is **not** the fibered product $G_{r_1} \underset{\mu_n}{\times} G_{r_2}$.
In other words, the two blocks G_{r_1} and G_{r_2} (the inverse images of $\{\begin{pmatrix} g & \\ & I_{r_2} \end{pmatrix} \mid g \in \mathrm{GL}_{r_1}(F)\}$ and $\{\begin{pmatrix} I_{r_1} & \\ & g \end{pmatrix} \mid g \in \mathrm{GL}_{r_2}(F)\}$) do not commute in G_r.
This phenomenon was highlighted and explicated by Banks–Levy–Sepanski (1999).
Thus, there is no naive way to define the Bernstein–Zelevinsky product in the covering case.
We need a way around it.

A key case

Suppose that N is of Heisenberg type, that is

1. The derived group $[N, N]$ is finite, cyclic and central in N.
2. $N/Z(N)$ is finite (and abelian).

Let ψ be a genuine character of $Z(N)$, that is $\psi|_{[N,N]}$ is faithful. Then, we have an equivalence of categories

$$\mathcal{M}_\psi(N) \simeq \mathcal{M}_\psi(Z(N)).$$

More precisely, choosing a maximal abelian subgroup L of N and an extension χ of ψ to L we obtain equivalences of categories (Stone, von Neumann)

$$\mathcal{LI}_{(L,\chi)} : \mathcal{M}_\psi(Z(N)) \to \mathcal{M}_\chi(L) \xrightarrow{\mathrm{Ind}_L^N} \mathcal{M}_\psi(N).$$

Moreover, for any two choices of data (L_i, χ_i), $i = 1, 2$ we have a natural equivalence of functors

$$T_{L_1,\chi_1}^{L_2,\chi_2} : \mathcal{LI}_{(L_1,\chi_1)} \to \mathcal{LI}_{(L_2,\chi_2)}.$$

A sting in the tail

This does not mean that we have a canonical functor !
This is because generally, for three choices (L_i, χ_i), $i = 1, 2, 3$

$$T_{L_1,\chi_1}^{L_3,\chi_3} \neq T_{L_2,\chi_2}^{L_3,\chi_3} \circ T_{L_1,\chi_1}^{L_2,\chi_2}.$$

To rectify the situation we would need to normalize $T_{L_1,\chi_1}^{L_2,\chi_2}$.
Such a problem was considered by S. Gurevich–Hadani and Kamgampour–Thomas for finite groups of Heisenberg type.
(It is related to defining the Weil representation as a representation of the symplectic group, rather than a projective representation.)
Unfortunately, they exclude the case of even order.
We will avoid this splitting hairs issue..

Consider a central extension

$$1 \to A \to G \xrightarrow{p} \underline{G} \to 1$$

where A is a finite cyclic group.
Let $\underline{H}$ be a finite index subgroup of $\underline{G}$ and let $H = p^{-1}(\underline{H})$.

We say that H is a special subgroup of G if

1. $p(Z_G(H)) \le Z(\underline{G})$.
2. $Z_G(H \cap Z_G(H)) = H \cdot Z_G(H)$.

For instance, if N is of Heisenberg type, then $Z(N)$ is a special subgroup of N.
In general, if H is special, then $Z_G(H)$ is of Heisenberg type.

Basic example
$G = G_r$, $\underline{G} = \mathrm{GL}_r(F)$, $\underline{H} = \{g \in \mathrm{GL}_r(F) \mid \det g \in (F^*)^n\}$.
In this case

$$Z_G(H) = Z_r = p^{-1}(\{\lambda I_r \mid \lambda \in F^*\}).$$

Proposition

Suppose that H is a special subgroup of G.
Let ψ be a character of $H \cap Z_G(H)$ and χ a character of $Z(G)$.
Suppose that ψ and χ are genuine (i.e., their restriction to A is faithful) and they agree on $H \cap Z(G)$.
Then, we have an equivalence of categories (Lagrangian induction)

$$\mathcal{LI} : \mathcal{M}_\psi(H) \to \mathcal{M}_\chi(G)$$

Moreover, $\mathrm{Ind}_H^G \sigma \simeq d \cdot \mathcal{LI}\sigma$ where $[Z_G(H) : Z(Z_G(H))] = d^2$.

The same caveat as before applies: the equivalence of categories is not completely canonical..

The metaplectic tensor product

Let β be a composition $r = r_1 + r_2$.
Let $\underline{G}_\beta \simeq \mathrm{GL}_{r_1}(F) \times \mathrm{GL}_{r_2}(F)$ be the corresponding standard Levi subgroup of $\mathrm{GL}_r(F)$.
Let G_β be the inverse image of $\underline{G}_\beta$ in G_r.
Let $\underline{H}_\beta = \{\mathrm{diag}(g_1, g_2) \in \underline{G}_\beta \mid \det g_1, \det g_2 \in (F^*)^n\}$.
Let H_β be the inverse image of $\underline{H}_\beta$ in G_r.
Then, H_β is a special subgroup of G_β.
At the same time, H_β is a special subgroup of $G'_\beta = G_{r_1} \underset{\mu_n}{\times} G_{r_2}$ since the two blocks commute in H_β.
Thus for compatible choices of central characters we get an equivalence of categories

$$\mathcal{M}_{\chi'}(G'_\beta) \to \mathcal{M}_\omega(H_\beta) \to \mathcal{M}_\chi(G_\beta)$$

and therefore a biadditive, biexact functor

$$\mathcal{M}_{\omega_1}(G_{r_1}) \times \mathcal{M}_{\omega_2}(G_{r_2}) \to \mathcal{M}_\omega(G_\beta) \xrightarrow{I} \mathcal{M}_\omega(G_r)$$

where I denotes parabolic induction.

Remark Associativity still holds (but it is a bit more painful to prove then in the linear, banal case).

Consider the following situation

- $G = G_{2r}$, $P = M \ltimes \underline{U}$ the standard parabolic of type (r, r).
- ρ an irreducible representation of G_r.
- $\pi = \rho \otimes_\omega \rho$, an irreducible representation of M.
- $I_P(\pi, s) = \mathrm{Ind}_P^G(\pi \cdot (|\det|^{s/2} \otimes |\det|^{-s/2}))$, $s \in \mathbb{C}$.
- $w \in N_G(M) \setminus M$. We have an intertwining operator

  $$M(w, s) : I(\pi, s) \to I(\pi^w, -s)$$

  which is given for $\Re s \gg 0$ by

  $$M(w, s)\varphi(g) = \int_{\underline{U}(F)} \varphi(w\underline{u}g)\ d\underline{u}$$

- $T_w : \pi^w \to \pi$ is the intertwining operator (defined up to a sign) such that $T_w^2 = \pi(w^{-2})$.

Theorem (Analogue of Olshanski (1974))

Suppose that ρ is square-integrable. Then,

$$I_P(T_w, 0) \circ \lim_{s \to 0} (1 - q^{-rns}) M(w, s) = \pm c_{r,n} \cdot \mathrm{id}_{I_P(\pi,0)}$$

provided that the Haar measure on $\underline{U}(F) \simeq \mathrm{Mat_r(F)}$ defining $M(w,s)$ is chosen to be $|\det|^r$ times the formal degree of ρ.
Here $c_{r,n}$ is an explicit constant depending only on r and n.

The group of characters of F^* of order dividing n acts on the set of irreducible representations of G_r. We write $[\pi]$ for the orbit of π.

Corollary

Let ρ_i be cuspidal representations of G_{r_i}, $i = 1, 2$. Then,

$$\rho_1 \times \rho_2 \text{ is reducible } \iff [\rho_1] = [\rho_2 \cdot |\det|^{\pm s}] \Rightarrow r_1 = r_2.$$

Here s is a positive real number which depends only on ρ_1.

The classification procedure now follows almost word-for-word the linear case (à la Tadić, Mínguez, Sécherre). We just have to change an "irreducible representation" by its orbit.

- Define a segment as a set $\{\rho_1, \dots, \rho_k\}$ of distinct cuspidal representations such that $\rho_i \times \rho_{i+1}$ is reducible.
- Attach an irreducible representation to a segment Δ as $Z(\Delta) = \mathrm{soc}(\rho_1 \times \cdots \times \rho_k)$ for a suitable ordering.
- $Z(\Delta_1) \times Z(\Delta_2)$ is reducible $\iff \Delta_1 \prec \Delta_2$ or $\Delta_2 \prec \Delta_1$.
- A multisegment is a finite multiset of segments. Writing $\mathrm{m} = \Delta_1 + \cdots + \Delta_k$ such that $\Delta_i \not\prec \Delta_j$ for all $i < j$ define

$$\zeta(\mathrm{m}) = Z(\Delta_1) \times \cdots \times Z(\Delta_k).$$

  This depends only on m. The map $\mathrm{m} \mapsto Z(\mathrm{m}) = \mathrm{soc}(\zeta(\mathrm{m}))$ defines a bijection between multisegments and the orbits of irreducible representations of G_r, $r \geq 0$.
- Depending on the orientation chosen for defining $Z(\Delta)$ this gives rise to either Zelevinsky classification or (a slightly refined version of) Langlands classification.