On some special Diophantine quintuples in $\mathbb{Z}[\sqrt{D}]$

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Let $\mathcal{R}$ be a commutative ring with the unity.
A Diophantine m-tuple in $\mathcal{R}$ is a set of m elements in $\mathcal{R}\backslash\{0\}$ with the property that the product of any two of its distinct elements increased by the unity is a perfect square in $\mathcal{R}$.

Examples:

- $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ (Diophantus)
- $\{1, 3, 8, 120\}$ (Fermat)
- $\{n-1, n+1, 4n, 16n^3-4n\}$ (Euler) or more general

  $$\{a, b, a+b+2r, 4r(r+a)(r+b)\},$$

  where $ab+1=r^2$
- $\{\frac{5}{36}, \frac{5}{4}, \frac{32}{9}, \frac{189}{4}, \frac{665}{1521}, \frac{3213}{676}\}$ (Gibbs)
- Infinitely many rational Diophantine sixtuples (Dujella, Kazalicki, Mikić, Szikszai)
- $\{1, 3, 8, 120, 1678\}$ in $\mathbb{Z}[\sqrt{201361}]$

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The most frequently observed problem:

How large these sets can be?

A brief chronology of research in $\mathbb{Z}$ / Dioph. quintuple conjecture

- Baker,Davenport (1969):
  $\{1, 3, 8\}$ uniquely extends to $\{1, 3, 8, 120\}$
- Dujella (1997):
  $\{k - 1, k + 1, 4k\}$ uniquely extends to a quadruple
- Dujella, Petho (1998):
  $\{1, 3\}$ cannot be extended to a quintuple
- Dujella (2004):
  There are only finitely many Diophantine quintuples.
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What do we know about the size of these sets in other rings?

- $\mathbb{Q}$: no example of a seventuple (infinitely many sextuples)
- $\mathbb{Z}[\sqrt{D}]$, $D < 0$: no example of a quintuple
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Any Diophantine triple $\{a_1, a_2, a_3\}$ can be extended to a Diophantine quadruple by adding one of the following two elements

$$d_{\pm} = a_1 + a_2 + a_3 + 2a_1a_2a_3 \pm 2rst,$$

where $a_1a_2 + 1 = r^2$, $a_1a_3 + 1 = s^2$, $a_2a_3 + 1 = t^2$, but just in case

$$d_-, d_+ \notin \{a_1, a_2, a_3, 0\}.$$

A Diophantine quadruple $\{a_1, a_2, a_3, a_4\}$ such that $a_4 = d_-$ or $a_4 = d_+$, where $d_{\pm}$ are given by (6), is called regular.

$\{a_1, a_2, a_3, a_4\}$ is regular Diophantine quadruple if and only if

$$(a_1 + a_2 - a_3 - a_4)^2 = 4(a_1a_2 + 1)(a_3a_4 + 1)$$

Conjecture: All Diophantine quadruples in $\mathbb{Z}$ are regular.

A Diophantine quintuple in $\mathcal{R}$ containing two regular Diophantine quadruples is called biregular[a].

[a] Biregular quintuples in $\mathbb{Q}$ were applied to construction of high-rank elliptic curves and rational Diophantine sextuples.

Obviously, $\{a_1, a_2, a_3, d_-, d_+\}$ is a biregular Diophantine quintuple if $d_-d_+ + 1 = \square$ in $\mathcal{R}$ and $d_-, d_+ \notin \{a_1, a_2, a_3, 0\}$.

Example 1 (Simply obtained family of quintuples in $\mathbb{Z}\sqrt{D}$)

We take

$$a_1 = n-1,\ a_2 = n+1,\ a_3 = 16n^3 - 4n, n \in \mathbb{N},$$

and get

$$a_4 = d_- = 4n,\ a_5 = d_+ = 64n^5 - 48n^3 + 8n.$$

$\{a_1, \ldots, a_5\}$ *is a Diophantine quintuple in* $\mathbb{Z}[\sqrt{D}]$,
$D = 256n^6 - 192n^4 + 32n^2 + 1$.[1]

Therefore, we are especially interested in families of D's which are asymptotically larger than this simply obtained family, i.e. in parametric families of D's where involved polynomials have degree smaller than 6.

[1] D is a perfect square only for $n = 0$

Gibbs listed 160 examples of Diophantine quintuples in real quadratic number rings $\mathbb{Z}[\sqrt{D}]$ for all square free D with $1 < D < 50$ except for $D \in \{23, 35, 42, 43, 47\}$:
`researchgate.net/publication/323176085_Diophantine_Quintuples_over_Quadratic_Rings`
(We found the example of Diophantine quintuple in $\mathbb{Z}[\sqrt{43}]$[2]).
Among these, we observed the examples of biregular Diophantine quintuples of the "special" form

$$\{e, a + b\sqrt{D}, a - b\sqrt{D}, c + d\sqrt{D}, c - d\sqrt{D}\}$$

where $e, a, b, c, d \in \mathbb{Z}$, $c, d \neq 0$, i.e., Diophantine quintuples containing two pairs of conjugate elements. Also, we assumed that the elements $c \pm d\sqrt{D}$ are regular extensions of the set $\{e, a \pm b\sqrt{D}\}$.

[2] $\{(0, 848), (\pm 195, 30), (\pm 7512908; 1145708)\}$

Tablica: Gibbs' examples of quituples containing two pairs of conjugate elements

D	e	(a, b)	$(a, -b)$	(c, d)	$(c, -d)$
2	3	(7, 4)	(7, -4)	(119, -84)	(119, 84)
2	3	(39, 20)	(39, -20)	(4407 ,3116)	(4407 ,-3116)
2	6	(31, 15)	(31, -15)	(6200, -4384)	(6200, 4384)
2	6	(403 279)	(403 -279)	(81536 -57652)	(81536 57652)
2	10	(13 9)	(13 -9)	(176 124)	(176 -124)
2	21	(97 68)	(97 -68)	(6977 4932)	(6977 -4932)
5	4	(7 3)	(7 -3)	(50 -22)	(50 22)
5	28	(148 30)	(148 -30)	(974948 -436010)	(974948 436010)
13	6	(8 2)	(8 -2)	(166 46)	(166 -46)
13	6	(86 20)	(86 -20)	(26530 7358)	(26530 -7358)
13	6	(268 22)	(268 -22)	(786926 -218254)	(786926 218254)
13	10	(148 34)	(148 -34)	(137826 38226)	(137826 -38226)
17	12	(21 5)	(21 -5)	(438 -106)	(438 106)
29	4	(41 7)	(41 -7)	(2166 -402)	(2166 402)
29	20	(17 3)	(17 -3)	(1174 218)	(1174 -218)
29	44	(331 23)	(331 -23)	(8292066 -1539798)	(8292066 1539798)
29	112	(17 1)	(17 -1)	(58386 -10842)	(58386 10842)
34	5	(81 12)	(81 -12)	(16817 -2884)	(16817 2884)
37	4	(43 5)	(43 -5)	(7482 -1230)	(7482 1230)

The set $\{e, a + b\sqrt{D}, a - b\sqrt{D}, c + d\sqrt{D}, c - d\sqrt{D}\}$ is a biregular Diophantine quintuple if the following conditions hold:

$$e(a + b\sqrt{D}) + 1 = (u + v\sqrt{D})^2, \tag{1}$$

$$(a + b\sqrt{D})(a - b\sqrt{D}) + 1 = x^2 D \text{ or } \cancel{x^2}, \tag{2}$$

$$c \pm d\sqrt{D} = e + 2a + 2e(a^2 - Db^2) \pm 2(u^2 - Dv^2)x\sqrt{D}, \tag{3}$$

$$(c + d\sqrt{D})(c - d\sqrt{D}) + 1 = y^2 \text{ or } = y^2 D, \tag{4}$$

for some $u, v, x, y \in \mathbb{Z}$.
(1) & (2): "triple condition", (3) & (4): "biregularity condition"

There exists a biregular Diophantine quintuple of the form

$$\{e, a+b\sqrt{D}, a-b\sqrt{D}, c+d\sqrt{D}, c-d\sqrt{D}\}$$

in the ring $\mathbb{Z}[\sqrt{D}]$ for

- $D = n^2(n+1)^2+1,\ n \in \mathbb{N}$,
- $D = \dfrac{a^2+1}{10}$, where $a > 3$ is an integer solution of the equation
$$5\chi^2 - 2a^2 = 27,$$
- $D = n^2(8n \pm 1)^2 + 1,\ n \in \mathbb{N}$.

1.Family of triples: From eqs. (1), (2) we have

$$a = \frac{u^2+Dv^2-1}{e},\ b = \frac{2uv}{e},\ (u^2+Dv^2-1)^2-D(2uv)^2+e^2 = Dx^2e^2.$$

The last eq. implies that , $D \mid 1+e^2-2u^2+u^4$. So,

$$1+e^2-2u^2+u^4 = kD,\ k \in \mathbb{Z},$$

which gives us

$$\underbrace{\frac{1}{k}(k^2-2kv^2+v^4+e^2v^4-2v^2(k+v^2)u^2+v^4u^4)}_{p(u)} = x^2e^2.$$

For $k = \dfrac{e^2v^2}{4}$, $p(u) = \square$.

$$\{e, a \pm b\sqrt{D} = \frac{e^2\left(u^2+3\right)+4\left(u^2-1\right)^2}{e^3} \pm \frac{2uv}{e}\sqrt{D}\} \qquad (5)$$

is a Diophantine triple in $\mathbb{Q}(\sqrt{D})$, for

$$D = \frac{4(1+e^2-2u^2+u^4)}{e^2v^2}.$$

2. Extension to a quintuple: Assume that $c + d\sqrt{D}, c - d\sqrt{D}$ are obtained by regular extension of (5). The condition

$$(c+d\sqrt{D})(c-d\sqrt{D})+1 = y^2$$

gives us

$$\underbrace{-16e^2\left(u^4-6u^2+1\right)+e^4\left(e^2-4u^2-15\right)+64u^2\left(u^2-1\right)^2}_{q(u)} = \boxed{e^4y^2}.$$

For $e = 4$ (the discriminant $q(u)$ equals 0) we get

$$q(u) = \frac{1}{4}u^2(-3+u^2)^2 = \square.$$

3. Integer condition: The set

$$\{4, a \pm b\sqrt{D} = \frac{13+2u^2+u^4}{16} \pm \frac{uv}{2}\sqrt{D}, \underbrace{c \pm d\sqrt{D}}_{reg.\ extensions}\}$$

is a quintuple in $\mathbb{Q}(\sqrt{D})$, $D = \dfrac{17-2u^2+u^4}{4v^2}$. For $u = 2n+1$ and $v = 2$ we have

$$D = 1 + n^2(1+n)^2,\ a = 1 + n + 2n^2 + 2n^3 + n^4,\ b = 1 + 2n,$$

$$c = 6 - 14n + 4n^2 + 20n^3 - 22n^4 - 16n^5 + 32n^6 + 32n^7 + 8n^8,$$

$$d = -6 + 14n - 2n^2 - 24n^3 + 8n^4 + 24n^5 + 8n^6$$

D	e	(a, b)	(c, d)
5	4	(7, 3)	(50, 22)
37	4	(43, 5)	(7482, 1230)
145	4	(157, 7)	(140670, 11682)
401	4	(421, 9)	(1158926, 57874)
901	4	(931, 11)	(6063786, 202014)

"Brute force strategy" / Less beautiful examples

We assume that

$$D = D(n) = d_4 n^4 + d_3 n^3 + d_2 n^2 + d_1 n + d_0,$$

$$u = u(n) = u_1 n + u_0,$$

where $d_0, d_1, d_2, d_3, d_4, u_1, u_0 \in \mathbb{Z}$. Similar to the previous case, we get that the triple $\{e = 4, a \pm b\sqrt{D}\}$,

$$D = \frac{1}{4v^2}(17 - 2u_0^2 + u_0^4 + (-4u_0u_1 + 4u_0^3u_1)n + (-2u_1^2 + 6u_0^2u_1^2)n^2 + 4u_0u_1^3n^3 + u_1^4n^4),$$

$$a = \frac{1}{16}(13 + 2u_0^2 + u_0^4 + (4u_0u_1 + 4u_0^3u_1)n + (2u_1^2 + 6u_0^2u_1^2)n^2 + 4u_0u_1^3n^3 + u_1^4n^4),$$

$$b = \frac{1}{2}(u_0 + u_1 n)v.$$

extends "biregularly" to a quintuple in $\mathbb{Q}(\sqrt{D})$

"Integer condition" is s satisfied for e.g. $v = 10$, $u_0 = 23 + 50k$, $u_1 = 50\ell$, $n := k + \ell n$

$$D = 697 + 6072n + 19825n^2 + 28750n^3 + 15625n^4,$$

$$a = 17557 + 152375n + 496250n^2 + 718750n^3 + 390625n^4,$$

$$b = 115 + 250n,$$

$$\begin{aligned} c &= 2392278510 + 41841233150n + 319909592500n^2 + 1396567187500n^3 \\ &\quad + 3807366406250n^4 + 6637656250000n^5 + 7226562500000n^6 \\ &\quad + 4492187500000n^7 + 1220703125000n^8 \end{aligned}$$

$$\begin{aligned} d &= 90614010 + 1190152250n + 6504293750n^2 + 18931875000n^3 \\ &\quad + 30953125000n^4 + 26953125000n^5 + 9765625000n^6. \end{aligned}$$

1. Solutions of $5\chi^2 - 2a^2 = 27$ with $a > 0$ are given by:

$$\begin{aligned}
\chi_n^{\pm}\sqrt{5} + a_n^{\pm}\sqrt{2} &= (3\sqrt{5} \pm 3\sqrt{2})(19 + 6\sqrt{10})^n, \\
{\chi'}_n^{\pm}\sqrt{5} + {a'}_n^{\pm}\sqrt{2} &= (5\sqrt{5} \pm 7\sqrt{2})(19 + 6\sqrt{10})^n, \ n \in \mathbb{N}_0.
\end{aligned}$$

Sequences $(a_n^{\pm})$, $({a'}_n^{\pm})$, $(\chi_n^{\pm})$, $({\chi'}_n^{\pm})$ satisfy the same binary recurrence

$$X_{n+2} = 38X_{n+1} - X_n, \ n \geq 0, \tag{6}$$

with initial conditions:

$$\begin{aligned}
&(a_0^+, \chi_0^+) = (3, 3), && (a_1^+, \chi_1^+) = (147, 93), \\
&(a_0^-, \chi_0^-) = (-3, 3), && (a_1^-, \chi_1^-) = (33, 21), \\
&({a'}_0^+, {\chi'}_0^+) = (7, 5), && ({a'}_1^+, {\chi'}_1^+) = (283, 179), \\
&({a'}_0^-, {\chi'}_0^-) = (-7, 5), && ({a'}_1^-, {\chi'}_1^-) = (17, 11).
\end{aligned} \tag{7}$$

2. $D = \dfrac{a^2+1}{10}$ is well-defined, $a \in \{a_n^{\pm}, {a'}_n^{\pm} : n \in \mathbb{N}_0\}$, $a > 3$
(D is an integer which is not a perfect square)

3. We construct a Diophantine triple $\{e, a+b\sqrt{D}, a-b\sqrt{D}\}$
   - a
   - $b = 3$
   - $e = \dfrac{4u}{3}$, where

$$u = \begin{cases} \frac{1}{3}\left(2a - \frac{\sqrt{2a^2+27}}{\sqrt{5}}\right), & \text{if } a \in (a_n^-), ({a'}_n^+), \\ \frac{1}{3}\left(2a + \frac{\sqrt{2a^2+27}}{\sqrt{5}}\right), & \text{if } a \in (a_n^+), ({a'}_n^-). \end{cases}$$

We check:

$$e(a \pm b\sqrt{D}) + 1 = (\underbrace{u \pm 2\sqrt{D}}_{=r,s})^2, \ (a+b\sqrt{D})(a-b\sqrt{D}) + 1 = (\underbrace{\sqrt{D}}_{=t})^2.$$

4. Regular extension of $\{e, a-b\sqrt{D}, a+b\sqrt{D}\}$

$$e+(a+b\sqrt{D})+(a-b\sqrt{D})+2e(a+b\sqrt{D})(a-b\sqrt{D})\pm 2rst$$

$$=\underbrace{2a+e(2D-1)}_{=c}\pm\underbrace{2(u^2-4D)}_{=d}\sqrt{D}$$

5. We check: $(c+d\sqrt{D})(c-d\sqrt{D})+1=\square$.

$$(c+d\sqrt{D})(c-d\sqrt{D})+1=\left(\frac{5\chi\pm 8a}{9}\right)^2.$$

D	e	(a, b)	(c, d)
5	4	(7, 3)	(50, 22)
29	20	(17, 3)	(1174, 218)
109	20	(33, 3)	(4406, 422)
2161	172	(147, 3)	(743506, 15994)
8009	172	(283, 3)	(2755490, 30790)
42641	764	(653, 3)	(65155990, 315530)

1. Constructing parametric family of triples

Assume that $\{e, a + b\sqrt{D}, a - b\sqrt{D}\}$ is a Diophantine triple s. t.

$$e(a + b\sqrt{D}) + 1 = (u + v\sqrt{D})^2, \; (a + b\sqrt{D})(a - b\sqrt{D}) + 1 = x^2 D,$$

for some $u, v, x \in \mathbb{Z}$. For $x = 2$ ("small") we have

$$ea + 1 = u^2 + v^2 D, \; eb = 2uv, \; D = \frac{a^2 + 1}{b^2 + 4}.$$

Also assume that there is a *linear* connection between variables:

$$a = u + k, \; b = v + \ell, \; k, \ell \in \mathbb{Z}.$$

We get a quadratic equation in u:

$$(4+\ell^3-4v+\ell^2 v)u^2-2kv(4+\ell^2+\ell v)u-(\ell+v)(4+\ell^2+2\ell v-k^2v^2) = 0$$

Since $u \in \mathbb{Z}$, the discriminant of the previous equation should be a perfect square. This is fulfilled for $\ell = 2k - 2$ and solutions are

$$\cancel{u^- = \frac{1}{2}(v-2)},$$
$$u^+ = \frac{2k^3(v+2) + k^2\left(v^2 - 12\right) - 4k(v-4) + 4(v-2)}{4k^3 + 2k^2(v-6) - 4k(v-4) - 8}.$$

For "small" $k = 2$ we obtain

$$u^+ = \frac{1}{2}(2 + 3v + v^2)$$

which is an integer for all $v \in \mathbb{Z}$ and corresponding

$$D = \frac{1}{4}(5 + 2v + v^2)$$

is an integer for $v = 2m - 1$, $m \in \mathbb{Z}$.

Taking all that ($k = \ell = 2$, $v = 2m - 1$, $u = 2m^2 + m$) into account we get that

$$\{e, a \pm b\sqrt{D}\} = \{2m(2m-1), 2m^2+m+2 \pm (2m+1)\sqrt{m^2+1}\} \quad (8)$$

is a Diophantine triple in $\mathbb{Z}[\sqrt{m^2+1}]$, for $m \in \mathbb{Z}$, $m \neq 0$.

2. Regular extensions of a triple:

$$c \pm d\sqrt{D} = e + 2a + 2e(a^2 - Db^2) \pm 4(u^2 - v^2 D)\sqrt{D}$$

extend the triple (8) to a quintuple. This is fulfilled, if

$$c^2 - d^2 D + 1 = 32m + 1 = y^2, \; y \in \mathbb{Z},$$

i.e.

$$y = 16n \pm 1, \; n \in \mathbb{Z}.$$

(Assuming that $32m + 1 = y^2 D = y^2(m^2 + 1)$, $y \in \mathbb{Z}$, we get only finitely many solutions of which only one corresponds to a Diophantine quintuple, $m = 32$.)

$$D = n^2(8n \pm 1)^2 + 1$$

$$\begin{aligned}
e &= 2n(8n \pm 1)(2n(8n \pm 1) - 1),\\
a &= n(8n \pm 1)(2n(8n \pm 1) + 1) + 2 = 128n^4 \pm 32n^3 + 10n^2 \pm n + 2,\\
b &= 2n(8n \pm 1) + 1 = 16n^2 \pm 2n + 1,\\
c &= 4(8n^4(8n \pm 1)^4 - 4n^3(8n \pm 1)^3 + 8n^2(8n \pm 1)^2 - 3n(8n \pm 1) + 1\\
&= 4(32768n^8 \pm 16384n^7 + 1024n^6 \mp 512n^5 + 424n^4 \pm 124n^3 - 16n^2 \mp 3n -\\
d &= 4(8n^3(8n \pm 1)^3 - 4n^2(8n \pm 1)^2 + 4n(8n \pm 1) - 1),\\
&= 4(4096n^6 \pm 1536n^5 - 64n^4 \mp 56n^3 + 28n^2 \pm 4n - 1)
\end{aligned}$$

D	e	(a, b)	(c, d)
50	182	(107, 15)	(72832, 10300)
82	306	(173, 19)	(200776, 22172)
901	3540	(1832, 61)	(25516444, 850076)
1157	4556	(2348, 69)	(42170476, 1239772)

```
For all square free D, 1 < D < 2000, 1 ≤ u, v ≤ 10000:
  For all e ∈ Divisors(gcd(u^2 + Dv^2 − 1, 2uv)):
    z = ((u + v√D)^2 − 1)/e
    If (N(z) + 1)/D = □, then
       {e, z, z̄} is a triple
       If (z + z̄ − e)^2 − 4Dx^2 + 1 = □ or D·□, then
         {e, z, z̄} extends to a biregular quintuple
```

D	e	(a, b)	(c, d)
53	4	(33307,675)	(8681731610,1192527550)
58	90	(17,1)	(41704,5476)
61	1482	(782,100)	(4520182,578750)
73	4	(27,3)	(634,74)
73	8	(27,3)	(1214,142)
73	8	(162452,17803)	(52056888864,6092797992)
82	306	(173,19)	(200776,22172)
85	14	(132,6)	(402470,43654)
85	4	(3277,113)	(77233470,8377146)
97	3792	(1239,115)	(1913419134,194278278)
109	20	(33,3)	(4406,422)
113	1680	(1228,113)	(218696456,20573232)
130	6	(203,11)	(306160,26852)
145	4	(157,7)	(140670,11682)
229	1992	(15007,719)	(425594736326,28124091802)
401	4	(421,9)	(1158926,57874)
401	232	(782,25)	(167458932,8362500)
409	20	(143,7)	(16626,822)
493	15924	(11037,497)	(1271792334,57278646)
586	590	(3671,71)	(12416221632,512909388)
697	4	(17557,115)	(2392278510,90614010)
769	1400	(5321,187)	(3981276042,143568486)
901	4	(931,11)	(6063786,202014)
901	3540	(1832,61)	(25516444,850076)
1093	1056	(563,17)	(2308486,69826)
1765	4	(1807,13)	(23739330,565062)
1961	2	(1030,10)	(3461262,78162)

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