Theta correspondence and unitary representations

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Contents

1 Dual pairs and Howe correspondence

Basic notion: (Howe)

- W: a finite-dimensional real symplectic vector space.
- (G, G'): a reductive dual pair in $\mathrm{Sp}(W)$, i.e., a pair of subgroups such that
  - G and G' are mutual centralizers of each other;
  - G and G' act reductively on W.

Irreducible reductive dual pairs (seven families):

- Type II: correspond to a division algebra D

$$(\mathrm{GL}_m(\mathbb{R}), \mathrm{GL}_n(\mathbb{R})) \subseteq \mathrm{Sp}_{2mn}(\mathbb{R})$$

$$(\mathrm{GL}_m(\mathbb{C}), \mathrm{GL}_n(\mathbb{C})) \subseteq \mathrm{Sp}_{4mn}(\mathbb{R})$$

$$(\mathrm{GL}_m(\mathbb{H}), \mathrm{GL}_n(\mathbb{H})) \subseteq \mathrm{Sp}_{8mn}(\mathbb{R})$$

- Type I: correspond to a division algebra D with involution $\natural$

$$(\mathrm{O}_{p,q}, \mathrm{Sp}_{2n}(\mathbb{R})) \subseteq \mathrm{Sp}_{2(p+q)n}(\mathbb{R})$$

$$(\mathrm{O}_p(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C})) \subseteq \mathrm{Sp}_{4pn}(\mathbb{R})$$

$$(\mathrm{U}_{p,q}, \mathrm{U}_{r,s}) \subseteq \mathrm{Sp}_{2(p+q)(r+s)}(\mathbb{R})$$

$$(\mathrm{Sp}_{p,q}, \mathrm{O}^*_{2n}) \subseteq \mathrm{Sp}_{4(p+q)n}(\mathbb{R})$$

(G, G'): a reductive dual pair in $\mathrm{Sp}(W)$.

- Fix an oscillator (or Weil) representation $\widehat{\omega}$ (by fixing a nontrivial unitary character on $\mathbb{R}$). This is a unitary representation of $\widetilde{\mathrm{Sp}}(W)$ (the real metaplectic group), constructed by Segal, Shale and Weil.
- Let ω be the associated smooth representation, called a smooth oscillator representation.
- For any reductive subgroup E of $\mathrm{Sp}(W)$, denote $\widetilde{E}$ its inverse image in $\widetilde{\mathrm{Sp}}(W)$, and by $\mathrm{Irr}(\widetilde{E}, \omega)$ the subset of $\mathrm{Irr}(\widetilde{E})$ which are realizable as quotients by $\omega(\widetilde{E})$-invariant closed subspaces of ω.

- **Howe duality theorem**: The set $\mathrm{Irr}(\widetilde{G} \cdot \widetilde{G'}, \omega)$ is the graph of a bijection between $\mathrm{Irr}(\widetilde{G}, \omega)$ and $\mathrm{Irr}(\widetilde{G'}, \omega)$. Moreover any element $\pi \otimes \pi'$ of $\mathrm{Irr}(\widetilde{G} \cdot \widetilde{G'}, \omega)$ occurs as a quotient of ω in a unique way.

- The Howe duality conjecture also holds true for p-adic local fields:
  - works of Waldspurger, Minguez, Gan-Takeda, Gan-Sun

A variant formulation of Howe duality:

- Let $\pi \in \mathrm{Rep}(\widetilde{G})$ (finitely generated admissible quasisimple).
- The full theta lift of π:

$$\Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi) := (\omega \widehat{\otimes} \pi^{\vee})_{\widetilde{G}} \in \mathrm{Rep}(\widetilde{G'})$$

(the subscript $\widetilde{G}$ indicating the Hausdorff $\widetilde{G}$-coinvariant space).

- The theta lift $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ of π:

the largest semisimple quotient of $\Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$.

- If π is irreducible, then $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ is irreducible or zero.

Fundamental tasks:

- Given $\pi \in \mathrm{Irr}(\widetilde{G})$, determine $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$; or at least determine whether $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi) \neq 0$.
- Construct interesting (e.g. unitary) representations from this formalism.

2 Compact dual pairs

Let $(G, G') \subseteq \mathrm{Sp}$ be a reductive dual pair, with G compact.

- G' is of Hermitian symmetric type.
- There is a character χ of $\widetilde{G}$ such that $\chi^{-1} \otimes \widehat{\omega}|_{\widetilde{G}}$ factors through the linear group G. Representations which occur in $\omega|_{\widetilde{G}}$ are of the form $\chi \otimes \sigma$. where $\sigma \in \mathrm{Irr}(G)$.

Kashiwara-Vergne and Howe: (1970's)

- The restriction of the unitary representation $\widehat{\omega}$ to $\widetilde{G} \cdot \widetilde{G'}$ is a discrete and multiplicity-free sum.
- We have the decomposition

$$\widehat{\omega} \simeq \sum_{\sigma \in \mathrm{Irr}(G)} (\chi \otimes \sigma) \boxtimes L(\sigma),$$

  where $L(\sigma)$ is an irreducible unitary lowest weight representation of $\widetilde{G'}$ or zero.
- $L(\sigma) \neq 0$ if and only if σ occurs in the space of harmonics, which is a multiplicity-free representation of the compact group $G \times \widetilde{K'}$.
- The map $\sigma \mapsto L(\sigma)$ is injective in its domain and is described explicitly by the pairing of σ with the lowest $\widetilde{K'}$-type of $L(\sigma)$ in the space of harmonics.

Example: $(G, G') = (\mathrm{O}(p), \mathrm{Sp}_{2n}(\mathbb{R}))$.

- If $p \leq n$, then

$$L(\sigma) \neq 0, \quad \forall \sigma \in \mathrm{Irr}(G).$$

- If $p > n$, then

$$L(\sigma) \neq 0 \iff \sigma \text{ occurs in } \mathrm{L}^2(\mathrm{O}(p)/\,\mathrm{O}(p-n)),$$

  where $\mathrm{O}(p-n)$ embeds in $\mathrm{O}(p)$ in the standard way.

- If $p > 2n$, then the $L(\sigma)$'s ($\neq 0$) are members of the holomorphic discrete series.

More on unitary lowest weight representations

- **Enright-Parthasarathy**: For $G' = \mathrm{Sp}_{2n}(\mathbb{R})$ or $\mathrm{U}_{r,s}$, all irreducible unitary lowest weight representations of $\widetilde{G'}$ arise in this manner (by varying all possible G's which are compact).
- For $G' = \mathrm{O}^*_{2n}$, there are some minor exceptions.
- **Enright-Howe-Wallach** classifies irreducible unitary lowest weight representations of any covering of G' (internally).
- **Nishiyama-Ochiai-Taniguchi** determines the associated cycles of unitary lowest weight representations (via the model from the Howe correspondence)

3 Stable range theta lifting

- Consider a type I irreducible dual pair

$$(G, G') = (G(V), G(V')),$$

where V, V' are the standard modules over D (which resp. carry ϵ-Hermition and ϵ'-Hermitian forms, with $\epsilon\epsilon' = -1$).

- The dual pair (G, G') is said to be in stable range with G the smaller member, if
  - there exists a totally isotropic subspace V_1' of V' such that $\dim_{\mathrm{D}}(V) \leq \dim_{\mathrm{D}}(V_1')$.
  - **Notation**: $2G \leq G'$.

Works of Howe, Li: (1980's)

- Suppose that (G, G') is in the stable range with G the smaller member, then Howe correspondence gives rise to an injection

$$\theta : \quad (\widetilde{G})^{\wedge}_{gen} \hookrightarrow (\widetilde{G'})^{\wedge}_{gen}.$$

(The symbol $\wedge$ indicates the unitary dual; the subscript *gen* indicates the genuine part.)

- There are two messages:
  - $(\widetilde{G})^{\wedge}_{gen} \subset \mathrm{Irr}(\widetilde{G}, \omega)$, namely the whole set $(\widetilde{G})^{\wedge}_{gen}$ is in the domain of Howe correspondence;
  - Stable range theta lifting preserves the unitarity.

A more concrete way to describe the stable range correspondence:

- Fix a totally isotropic subspace V_1' of V' such that $\dim_{\mathrm{D}}(V) = \dim_{\mathrm{D}}(V_1')$, and let P' be the parabolic subgroup of G' preserving $(V_1')^*$, with the unipotent radical N'. We have

$$P' \supset H' := \mathrm{GL}((V_1')^*) \cdot N' \supset G \cdot N'.$$

Then $\theta(\pi)|_{\widetilde{H'}}$ is <u>irreducible</u> and

$$\theta(\pi)|_{\widetilde{H'}} \simeq \mathrm{Ind}_{\widetilde{G}\cdot N'}^{\widetilde{\mathrm{GL}}((V_1')^*)\cdot N'} (\pi^\vee \otimes \rho'),$$

where ρ' is a certain oscillator-Heisenberg representation of $\widetilde{G} \cdot N'$.

Notion of rank: Howe

- This is defined via the ZN-spectrum, where N is the unipotent radical of a certain maximal parabolic subgroup P of G.
- For $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$, we have the decomposition of its unitary dual according to the N-rank: (P: the Siegel parabolic subgroup)

$$(\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}))^\wedge = \bigcup_{r \le n} (\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}))^\wedge_r$$
$$= (\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}))^\wedge_n \bigcup (\bigcup_{r<n} (\bigcup_{\mathrm{rank}\,\beta = r} (\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}))^\wedge_\beta)).$$

  - $(\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}))^\wedge_r$: N-rank r
  - $(\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}))^\wedge_\beta$: N-spectral type β (β: a symmetric $n \times n$ matrix)
- The larger the rank of an irreducible unitary representation, the faster its matrix coefficients tend to decay.
  - This is a quantitative version of the Howe-Moore Theorem.

"Classification" of low rank representations: (Li)

- For $G' = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$, a symmetric matrix β of rank $r < n$ determines an orthogonal group $G = \mathrm{O}(p, q)$, with $p + q = r$. Representations in $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})^{\wedge}_{\beta}$ consist precisely of theta lifts from irreducible unitary representations of $\mathrm{O}(p, q)$.
- Similar results hold for low rank irreducible unitary representations of other classical groups.

4 Concrete models via integration

The following is a general idea to establish unitarity preservation, and it first appeared in Li's Yale thesis.

- Given $\pi \in \mathrm{Rep}(\widetilde{G})_{gen}$, consider the integral

$$\begin{array}{rcl} \omega \times \pi^\vee \times \bar{\omega} \times \pi & \to & \mathbb{C}, \\ (\phi, v', \phi', v) & \mapsto & \int_G \langle \tilde{g} \cdot \phi, \phi' \rangle \cdot \langle \tilde{g} \cdot v', v \rangle \, dg. \end{array}$$

- If it is absolutely convergent, it yields a continuous bilinear map

$$(\omega \widehat{\otimes} \pi^\vee) \times (\bar{\omega} \widehat{\otimes} \pi) \to \mathbb{C},$$

  and (by a slight variant) a $\widetilde{G'}$-invariant Hermitian form on $\omega \widehat{\otimes} \pi^\vee$, if π is unitary.

- Define
$$\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi) := \frac{\omega \widehat{\otimes} \pi^{\vee}}{\text{the left kernel of the bilinear map}}.$$
This is a quotient of $\Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$, and hence in $\mathrm{Rep}(\widetilde{G'})_{gen}$.

- The challenge in establishing unitary preservation:
  - Prove that $\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ is non-zero.
  - Prove that the induced Hermitian form on $\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ is positive definite.

- Success cases (for unitarity preservation):
  - **Li**: for any unitary π when $2G \leq G'$, and for (most) π in the discrete series when $\dim_{\mathrm{D}}(V) \leq \dim_{\mathrm{D}}(V')$.
  - **He**: for (most) π is in the so-called strongly semistable range.
  - Need some additional constraints on π to prove nonvanishing.

Barbasch-Ma-Sun-Z: a general result on unitarity preservation

- Notion of convergent range, based on a suitable bound of matrix coefficients (the benchmark function is defined in terms of the standard module V).
- Assume a mild condition on sizes (on $\mathrm{rank}_{\mathrm{D}}(V)$ and $\mathrm{rank}_{\mathrm{D}}(V')$), and a mild condition on π ("overconvergent"):

$$\pi \text{ is unitary} \Longrightarrow \bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi) \text{ is unitary.}$$

Remark:

- Given that $\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ is unitary, it is a semisimple quotient of $\Theta_{\widetilde{G}}^{\widetilde{G}'}(\pi)$. Thus if π is irreducible and $\bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi) \neq \{0\}$, then Howe Duality Theorem implies that $\theta_{\widetilde{G}}^{\widetilde{G}'}(\pi) = \bar{\theta}_{\widetilde{G}}^{\widetilde{G}'}(\pi)$ and is irreducible.

Harris-Li-Sun: (source of unitary structure)

Let G be a real reductive group with a maximal compact subgroup K.

- π_1, π_2: two unitary representations of G such that π_2 is <u>weakly contained</u> in the regular representation.
- $u_1, u_2, \cdots, u_r$: vectors in π_1 such that the integral $\int_G \langle g u_i, u_j \rangle \, \Xi_G(g) \, \mathrm{d}g$ is <u>absolutely convergent</u>.
- $v_1, v_2, \cdots, v_r$: K-finite vectors in π_2.

Then the integral $\int_G \langle g u, u \rangle \; \mathrm{d}g$ absolutely converges to a <u>nonnegative</u> real number, where $u := \sum_{i=1}^r u_i \otimes v_i \in \pi_1 \otimes \pi_2$.

Remark: Applied to the Howe duality setting, it implies convergence will ensure unitarity presevation after passing the point of G-temperedness.

5 Theta lifting and invariants of representations

- To have good control of the lifting process, a basic technique is to understand how fundamental invariants such as infinitesimal characters and K-types (joint harmonics) behalf under theta lifting.
- Other fundamental invariants such as (Vogan's) associated cycles and generalized Whittaker models should also be utilized.
  - The key lies in the moment maps.
  - $K_{\mathbb{C}}$-equivariant version:

$$\begin{array}{ccc} & \mathcal{X} \subset \mathrm{Hom}(V_{\mathbb{C}}, V'_{\mathbb{C}}) & \\ {\scriptstyle \phi} \swarrow & & \searrow {\scriptstyle \phi'} \\ \mathfrak{p} & & \mathfrak{p}' \end{array}$$

    where $\phi(T) = T^*T$ and $\phi'(T) = TT^*$.

- The associated cycles in the theta lifting setting have an upper bound via geometric theta lift (for nilpotent $K_{\mathbb{C}}$-orbits).

  Barbasch-Ma-Sun-Z: If $\mathcal{O} = \nabla(\mathcal{O}')$ is regular descent, then

  $$\mathrm{AC}_{\mathcal{O}'}(\Theta(\pi^\vee)) \preceq \check{\vartheta}_{\mathcal{O}}^{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}}(\pi)).$$

  (This generalizes earlier work of Nishiyama-Ochiai-Taniguchi, Nishiyama-Zhu and Loke-Ma.)

- The generalized Whittaker models in the theta lifting setting have an equality via geometric theta lift (for nilpotent G-orbits).

  Gomez-Z: If $\mathcal{O} = \nabla(\mathcal{O}')$ is regular descent, then

  $$\mathrm{Wh}_{\mathcal{O}',\tau'}(\Theta(\pi^\vee)) \simeq \mathrm{Wh}_{\mathcal{O},\Theta(\tau')^\vee}(\pi).$$

  (This is an effective tool for showing nonvanishing.)

6 Application: special unipotent representations

- Work of **Barbasch-Ma-Sun-Z**, stated informally: by starting from unitary characters and applying iterated theta lifting (in a controlled fashion), one can obtain all special unipotent representations of a real classical group G (attached to a nilpotent orbit $\check{\mathcal{O}}$ satisfying some parity condition).
- This also holds for the real metaplectic group, where we replace the term "special" by a notion called "metaplectic special".
  - We have an associated notion of metaplectic Barbasch-Vogan duality, similar to the Barbasch-Vogan duality for reductive linear groups.

Label $\star$	Classical Lie Group G	Langlands dual group $\check{G}$
$A^{\mathbb{R}}$	$\mathrm{GL}_n(\mathbb{R})$	$\mathrm{GL}_n(\mathbb{C})$
$A^{\mathbb{H}}$	$\mathrm{GL}_{\frac{n}{2}}(\mathbb{H})$ (n even)	$\mathrm{GL}_n(\mathbb{C})$
A	$\mathrm{U}(p,q)$	$\mathrm{GL}_{p+q}(\mathbb{C})$
$\widetilde{A}$	$\widetilde{\mathrm{U}}(p,q)$	$\mathrm{GL}_{p+q}(\mathbb{C})$
B	$\mathrm{O}(p,q)$ ($p+q$ odd)	$\mathrm{Sp}_{p+q-1}(\mathbb{C})$
D	$\mathrm{O}(p,q)$ ($p+q$ even)	$\mathrm{O}_{p+q}(\mathbb{C})$
C	$\mathrm{Sp}_{2n}(\mathbb{R})$	$\mathrm{O}_{2n+1}(\mathbb{C})$
$\widetilde{C}$	$\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$	$\mathrm{Sp}_{2n}(\mathbb{C})$
D^*	$\mathrm{O}^*(2n)$	$\mathrm{O}_{2n}(\mathbb{C})$
C^*	$\mathrm{Sp}(\frac{p}{2},\frac{q}{2})$ (p,q even)	$\mathrm{O}_{p+q+1}(\mathbb{C})$

- Given a $\check{G}$-orbit $\check{\mathcal{O}}$ in $\mathrm{Nil}(\check{\mathfrak{g}})$, one attaches an infinitesimal character $\chi_{\check{\mathcal{O}}}$ (via an $\mathfrak{sl}_2$-triple containing $\check{\mathcal{O}}$).
- By a theorem of Dixmier, there exists a unique maximal G-stable ideal of $\mathcal{U}(\mathfrak{g})$ that contains the kernel of $\chi_{\check{\mathcal{O}}}$. Write $I_{\check{\mathcal{O}}}$ for this ideal.
- The associated variety of $I_{\check{\mathcal{O}}}$ is the closure of a nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathfrak{g}$.
  - $\mathcal{O}$ is called the Barbasch-Vogan dual of $\check{\mathcal{O}}$ and is special in the sense of Lusztig.
- Everything works for the metaplectic group (replaced with metaplectic Barbasch-Vogan duality and metaplectic special).

Definition: (Barbasch-Vogan) An irreducible Casselman-Wallach representation π of G is said to be special unipotent attached to $\check{\mathcal{O}}$ if $I_{\check{\mathcal{O}}}$ annihilates π.

Remark: The notion was motivated by Arthur's conjecture on unipotent automorphic forms.

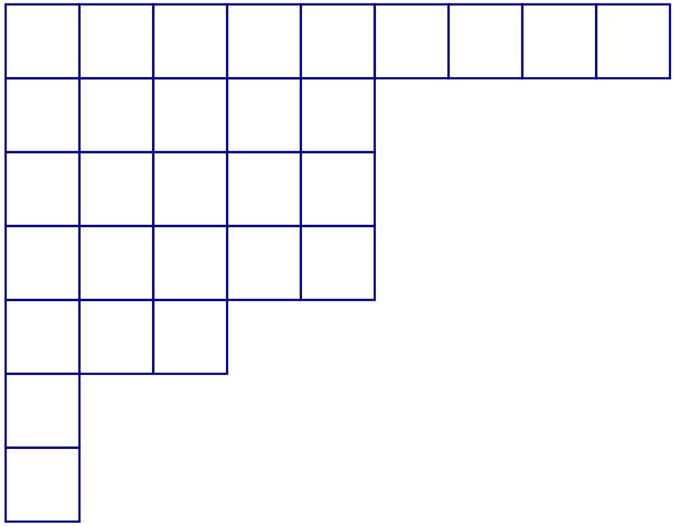
Notation: $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$, the set of equivalent classes of irreducible Casselman-Wallach representations of G that are special unipotent attached to $\check{\mathcal{O}}$.

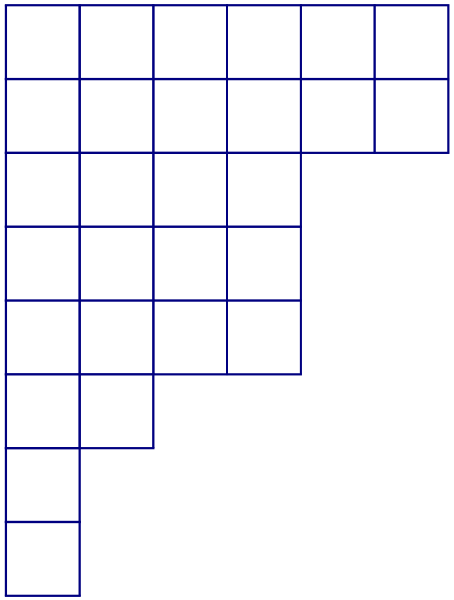
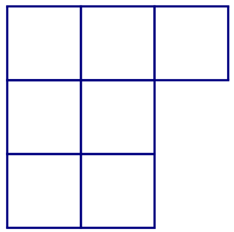
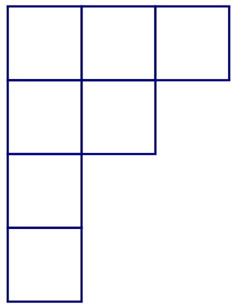
- In [BMSZ1] and [BMSZ2], we parameterize and explicitly construct all special unipotent representations of the real classical groups $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{H})$, $\mathrm{U}(p,q)$, $\mathrm{O}(p,q)$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{O}^*(2n)$, $\mathrm{Sp}(p,q)$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, as well as all metaplectic special unipotent representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$.
  - **BMSZ1**: Special unipotent representations of real classical groups: counting and reduction to good parity, arXiv:2205.05266.
  - **BMSZ2**: Special unipotent representations of real classical groups: construction and unitarity, arXiv:1712.05552.

For groups of type B, C or D, the steps involved are as follows:

- *Count the set $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ via combinatorial objects
  - Tools: coherent continuation representations, theory of primitive ideals, double cells and Harish-Chandra cells, branching laws of Weyl group representations, ... (Kazhdan-Lusztig, Lusztig, Joseph, Vogan, Barbasch-Vogan, Casian, ...)
- Reduce the problem of construction to the case when $\check{\mathcal{O}}$ has good parity (via irreducible parabolic induction)
  - Tools: Kazhdan-Lusztig-Vogan, Renard-Trapa
- Construct representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ by iterated theta lifting when $\check{\mathcal{O}}$ has good parity
  - Tool: combinatorial descent (chasing combinatorial parameters)
- *Distinguish representations via associated cycles
  - Tools: moment maps, geometric theta lifting, doubling method, degenerate principal series, ...
- This establishes the exhaustion.

Example: $G = \mathrm{Sp}(28, \mathbb{R})$, $\check{G} = \mathrm{O}(29, \mathbb{C})$.

$\check{\mathcal{O}} =$ 

$\mathcal{O} =$  $\xrightarrow{\text{Springer}} (\imath, \jmath) =$  $\times$ 

$$\mathrm{PP}_\star(\check{\mathcal{O}}) = \{(1,2), (5,6)\}$$

Painted bipartition (with symbols $\bullet, s, r, c, d$) and descent:

$$\begin{array}{ccc} \bullet & \bullet & r \\ \bullet & \bullet & \\ d & d & \end{array} \times \begin{array}{ccc} \bullet & \bullet & s \\ \bullet & \bullet & \\ s & & \\ s & & \end{array} \times C \xrightarrow{\nabla} \begin{array}{ccc} \bullet & \bullet & r \\ \bullet & s & \\ d & d & \end{array} \times \begin{array}{cc} \bullet & \bullet \\ \bullet & \end{array} \times D$$

$$\xrightarrow{\nabla} \begin{array}{cc} \bullet & r \\ \bullet & \\ d & \end{array} \times \begin{array}{cc} \bullet & s \\ \bullet & \end{array} \times C \xrightarrow{\nabla} \begin{array}{cc} \bullet & r \\ s & \\ d & \end{array} \times \begin{array}{c} \bullet \end{array} \times D$$

$$\xrightarrow{\nabla} \boxed{r} \times \boxed{s} \times C \xrightarrow{\nabla} \boxed{r} \times \emptyset \times D \xrightarrow{\nabla} \emptyset \times \emptyset \times C.$$

Corresponding Lie groups

$$\begin{aligned} & \mathrm{Sp}(28, \mathbb{R}) \to \mathrm{O}(10, 10) \\ \to \quad & \mathrm{Sp}(14, \mathbb{R}) \to \mathrm{O}(5, 5) \\ \to \quad & \mathrm{Sp}(4, \mathbb{R}) \to \mathrm{O}(2, 0) \to \mathrm{Sp}(0, \mathbb{R}). \end{aligned}$$

A themetic diagram: combinatorics, analysis and geometry

$$
\begin{array}{ccccc}
 & \text{Classical} & & \text{geometry} & \\
\nearrow & & \downarrow & & \nwarrow \\
\text{Comb. parameter} & \underset{descent}{\longrightarrow} & \odot & \overset{lift}{\longrightarrow} & \text{Representation} \\
\searrow & & \downarrow & & \swarrow \\
 & \text{Associated} & & \text{cycle} &
\end{array}
$$

Barbasch-Ma-Sun-Z: (confirming the Arthur-Barbasch-Vogan conjecture for real classical groups)

- All special unipotent representations of the real classical groups are unitarizable;
- all metaplectic special unipotent representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$ are also unitarizable.

Remarks:

- The unitarizability of special unipotent representations for quasisplit classical groups is independently due to Adams, Arancibia Robert and Mezo, as a consequence of their result

$$\text{Arthur packet} = \text{ABV packet}.$$

- The result is also true for the real Spin groups.

Other findings: (besides construction and unitarity)

- We determine the associated cycle of any special unipotent representation. (This is actually very difficult.)
- If $\check{\mathcal{O}}$ is quasi-distinguished, then the associated cycle map induces a bijection

$$\mathrm{AC}_{\mathcal{O}} : \ \mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathrm{AOD}(\mathcal{O}).$$

  - $\check{\mathcal{O}}$ is called quasi-distinguished if there is no odd i if $\star \in \{C, \widetilde{C}, C^*\}$, and no even i if $\star \in \{B, D, D^*\}$ such that

$$\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) > 0.$$

  - $$\mathrm{AOD}(\mathcal{O}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathrm{AOD}(\mathscr{O}),$$

    where $\mathrm{AOD}(\mathscr{O})$ is the set of isomorphism classes of admissible orbit data over $\mathscr{O}$.

Thank you!