Revisiting a Conjecture of Salamanca-Riba and Vogan

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Background

- Let G be a connected real reductive Lie group. A longstanding problem in representation theory of G is to classify $\widehat{G}$, the set of all unitarizable irreducible $(\mathfrak{g}, K)$-modules.
- In their 1998 Annals paper, Salamanca-Riba and Vogan proposed that one can reduce the study of $\widehat{G}$ to representations π with *unitarily small* lowest K-types.
- In order for their reduction argument to work, one needs to prove a (non-)unitarity conjecture involving the infinitesimal character of π.
- In this talk, we propose a slightly stronger non-unitary conjecture, which immediately implies the SV-conjecture.
- A proof of the generalized conjecture when $G = GL(n, \mathbb{C})$ will be given, where $\widehat{G}$ is known by Vogan. This approach is applicable to other groups such as $U(p, q)$, where $\widehat{G}$ is not yet known.
- **If this talk makes any sense at all, it is due to Professor Adams and Vogan for their great efforts to run the weekly** `atlas` **seminar and making it available online!**

Vogan's definition of λ_a

- Let G be connected real reductive Lie group with maximal compact K. Let $H = TA$ be the fundamental Cartan subgroup (T is maximal torus of K).
- Write $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ be the Lie algebras, and $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ be their complexifications.
- Fix root systems $\Delta(\mathfrak{g},\mathfrak{t}) = \Delta(\mathfrak{p},\mathfrak{t}) \cup \Delta(\mathfrak{k},\mathfrak{t})$, and a positive root system $\Delta^+(\mathfrak{k},\mathfrak{t})$.
- For all dominant weights $\mu \in \mathfrak{t}^*$, choose $\Delta^+(\mathfrak{g},\mathfrak{t}) \supseteq \Delta^+(\mathfrak{k},\mathfrak{t})$ making $\mu + 2\rho_{\mathfrak{k}}$ dominant. Define $\lambda_a(\mu) := P(\mu + 2\rho_{\mathfrak{k}} - \rho_{\mathfrak{g}})$, where P is the projection onto the dominant $W(\mathfrak{g},\mathfrak{t})$-chamber (see Vogan's green book for more details).

Theorem (Vogan)

Let $\Pi_a^{\lambda_a}(G) := \left\{ \begin{matrix} \pi \text{ adm. irred.} \\ (\mathfrak{g},K)\text{-module} \end{matrix} \middle| \begin{matrix} \text{a lowest } K\text{-type } V_\mu \text{ of } \pi \\ \text{satisfies } \lambda_a(\mu) = \lambda_a \end{matrix} \right\}$. *Then there is a bijection*

$$\Phi : \Pi_a^{\lambda_a - \rho(\mathfrak{u}(\lambda_a))}(G(\lambda_a)) \longrightarrow \Pi_a^{\lambda_a}(G),$$

where $\mathfrak{p}(\lambda_a) = \mathfrak{g}(\lambda_a) + \mathfrak{u}(\lambda_a)$ *be the theta-stable parabolic subalgebra defined by* $\lambda_a \in \mathfrak{t}^*$, *and* Φ *is given by cohomological induction and picking the appropriate composition factor.*

Salamanca-Riba and Vogan's definition of λ_u

- Unfortunately, the bijection Φ in the previous page does not preserve unitarity.
- In [SV], Salamanca-Riba and Vogan tried to remedy the problem by 'enlarging the theta-stable Levi' or equivalently, projecting more μ to 0.
- For all $\mu \in \mathfrak{t}^*$, define $\lambda_u(\mu) := P(\mu + 2\rho_{\mathfrak{k}} - 2\rho_{\mathfrak{g}})$.

Theorem (SV)

Let $\Pi_h^{\lambda_u}(G) := \left\{ \begin{matrix} \pi \text{ adm. irred. Hermitian} \\ (\mathfrak{g}, K)\text{-module} \end{matrix} \middle| \begin{matrix} \text{a lowest } K\text{-type } V_\mu \text{ of } \pi \\ \text{satisfies } \lambda_u(\mu) = \lambda_u \end{matrix} \right\}$. Then there is a bijection $\Pi_h^{\lambda_u}(G(\lambda_u)) \longrightarrow \Pi_h^{\lambda_u}(G)$.

Conjecture (SV)

The above bijection preserves unitarity.

- Assuming the conjecture holds, then one can reduce the study of $\widehat{G}$ to the representations π whose lowest K-types V_μ satisfies $G(\lambda_u(\mu)) = G$.
- Such K-types are called **unitarily small**.

The conjecture

It was proved in [SV] that the following conjecture implies the reduction argument:

Conjecture (SV, Conjecture 5.7)

Let π be an admissible, Hermitian, irreducible representation with a unitarily small lowest K-type μ and real infinitesimal character $\Lambda \in \mathfrak{h}^*$. If Λ does not lie in

$$\lambda_u(\mu) + \text{(convex hull of } W(\mathfrak{g}, \mathfrak{h}) \cdot \rho_{\mathfrak{g}}\text{)},$$

then the Hermitian form of π has opposite signatures on the level of unitarily small K-types.

Conjecture (Dong, Vogan)

Under the same hypothesis, suppose $\langle \Lambda, \alpha_i^\vee \rangle > 1$ for some simple coroot $\alpha_i^\vee$ of the root system $\Delta(\mathfrak{g}, \mathfrak{h})$, then the same conclusion of the above conjecture holds.

- In the special case when G is split and $\mu = 0$ is the trivial representation, the above statements are precisely what Vogan conjectured on Tuesday!
- We will prove this refined conjecture for $G = GL(n, \mathbb{C})$.

Unitarily small K-types for complex groups

- When G is a complex group treated as a real group, one has the complexifications $\mathfrak{g} \cong \mathfrak{g}_0 \times \mathfrak{g}_0$, $\mathfrak{h} \cong \mathfrak{h}_0 \times \mathfrak{h}_0$, and the identifications:

$$\mathfrak{k} = \{(X, -X^t) \in \mathfrak{g} : X \in \mathfrak{g}_0\} \cong \mathfrak{g}_0; \quad \mathfrak{t} = \{(H, -H) : x \in \mathfrak{h}_0\} \cong \mathfrak{h}_0.$$

- Using the above dictionary, we 'translate' information of the positive roots $\Delta^+(\mathfrak{k}, \mathfrak{t})$ into our more familiar coordinates $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$:

	$\Delta^+(\mathfrak{k}, \mathfrak{t})$		$\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$
Positive compact root $\Delta^+(\mathfrak{k}, \mathfrak{t})$:	$(\mu/2, -\mu/2)$	$\longleftrightarrow$	μ
Sum of positive compact roots $2\rho_{\mathfrak{k}}$:	$(\rho, -\rho)$	$\longleftrightarrow$	2ρ
Half sum of all positive roots $\rho_{\mathfrak{g}}$:	$(\rho, -\rho)$	$\longleftrightarrow$	2ρ

(here $\mu \in \Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$, and $\rho := \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)} \alpha$)

- Therefore, $\lambda_a(\mu) = \mu$, and $\lambda_u(\mu) = P(\mu - 2\rho)$ in $\mathfrak{h}_0^*$-coordinates.
- In other words, the K-type μ is unitarily small iff it lies inside the convex hull with vertices equal to the $W(\mathfrak{g}_0, \mathfrak{h}_0)$-orbit of 2ρ.

Hermitian modules for $GL(n, \mathbb{C})$

By the classification of irreducible representations of complex groups, all irreducible, Hermitian representations of $GL(n, \mathbb{C})$ with real infinitesimal characters are characterized by the Zhelobenko parameters $\pi = J(\lambda_L; \lambda_R)$, with

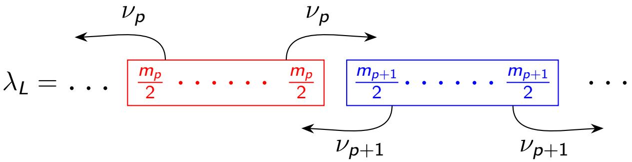
$$\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \left(\cdots \left| \begin{matrix} \left(\frac{m_p}{2}, \dots, \frac{m_p}{2}\right) + \underline{\nu}_p \\ -\left(\frac{m_p}{2}, \dots, \frac{m_p}{2}\right) + \underline{\nu}_p \end{matrix} \right| \begin{matrix} \left(\frac{m_{p+1}}{2}, \dots, \frac{m_{p+1}}{2}\right) + \underline{\nu}_{p+1} \\ -\left(\frac{m_{p+1}}{2}, \dots, \frac{m_{p+1}}{2}\right) + \underline{\nu}_{p+1} \end{matrix} \right| \cdots \right),$$

where:

- $\cdots > m_p > m_{p+1} > \cdots$ are integers;
- $\underline{\nu}_p = (\nu_{p,1}, \nu_{p,2}, \dots, -\nu_{p,2}, -\nu_{p,1}) \in \mathbb{R}^{(\#\ of\ \frac{m_p}{2}'s)}$ is symmetric.
- The lowest K-type of π is $\mu = (\cdots; m_p, \dots, m_p; m_{p+1}, \dots, m_{p+1}; \cdots)$.
- By earlier discussions, μ is unitarily small $\Leftrightarrow P(\mu - 2\rho) = 0 \Leftrightarrow$ $m_p - m_{p+1} \leq 2$ for all p.

SV-Conjecture for $GL(n, \mathbb{C})$

Consequently, if $\pi = J(\lambda_L; \lambda_R)$ is irreducible, Hermitian with real infinitesimal character and unitarily small lowest K-type, λ_L must look like:



or

$$\ldots \left(\frac{m_p}{2} + \nu_{p,1} \cdots\cdots \frac{m_p}{2} - \nu_{p,1}\right) \quad \left(\frac{m_{p+1}}{2} + \nu_{p+1,1} \cdots \frac{m_{p+1}}{2} - \nu_{p+1,1}\right) \ldots$$

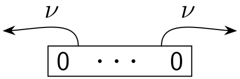
with $\frac{m_p}{2} - \frac{m_{p+1}}{2} \leq 1$, and our refined conjecture says the following:

Conjecture

Let $\pi = J(\lambda_L; \lambda_R)$ be as given above, so that the lowest K-type of π is unitarily small. Reorder the coordinates of $\lambda_L \sim (\ell_1 \geq \ell_2 \geq \cdots \geq \ell_n)$ in descending order. Suppose there exists i such that $\ell_i - \ell_{i+1} > 1$, then the Hermitian form on π is indefinite on some unitarily small K-types of π.

Proof of SV-Conjecture for $GL(n, \mathbb{C})$ - Spherical Case

We begin by proving the conjecture in the spherical case $\pi_{sp} = J(\lambda; \lambda)$, i.e. there

is only one block $\lambda = \nu =$ 

$= (\ell_1 \geq \cdots \geq \ell_n)$.

We apply the following algorithm:

(1) Starting from the largest coordinate L of λ, find the longest string of descending integers $\mathcal{S} = (L, L-1, \ldots, L-k)$ in λ.

(2) Remove a copy of the elements in $\mathcal{S}$ from λ, and repeat Step (1) until there are no coordinates left.

Theorem

Let $\pi_{sp} = J(\lambda; \lambda)$, and $\mathcal{S}_i$ are the strings obtained from λ in the above algorithm. Then the induced module

$$I(\lambda) := Ind_{\prod_i GL(p_i)}^{GL(n)}(\bigotimes_i |\det|^{s_i})$$

(here $s_i := 2\cdot$ (mean of the entries of $\mathcal{S}_i$)) has the same trivial and adjoint K-type multiplicities and signatures as π_{sp}. **Consequently, if $I(\lambda)$ has opposite signatures on these two K-types, then so does π_{sp}!**

Proof of SV-Conjecture for $GL(n, \mathbb{C})$ - Spherical Case

Here is a sketch proof of the refined SV-conjecture for $\pi_{sp} = J(\lambda; \lambda)$:

- Suppose $\lambda \sim (\ell_1, \ldots, \ell_n)$ has a gap $\ell_j - \ell_{j+1} > 1$ for some j, then $\ell_{n-j} - \ell_{n-j+1} > 1$ by symmetry of λ.
- So the strings of λ must 'break' at both ℓ_j, ℓ_{j+1} and ℓ_{n-j}, ℓ_{n-j+1}:

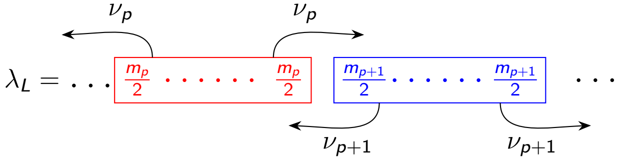
$$\begin{array}{ccccccc} \mathcal{S}_1 & & & & & & \mathcal{S}'_1 \\ (\ell_1 \ \cdots) \ \cdots & & & & & & \cdots \ (\cdots \ \ell_n) \\ & (\cdots \ \ \ell_j) & & \ell_{j+1}, \cdots, \ell_{n-j} & & (\ell_{n-j+1} \ \cdots) & \\ & \mathcal{S}_q & & & & \mathcal{S}'_q & \end{array}$$

- Deform $\mathcal{S}_1, \ldots, \mathcal{S}_q$, to $t \to \infty$ and $\mathcal{S}'_q, \ldots, \mathcal{S}'_1$ to $t \to -\infty$ simultaneously.
- Then we get a family of induced modules $I(\lambda(t))$ corresponding to the deformed strings for all $t \geq 0$, with $I(\lambda(0)) = I(\lambda)$.
- Since $\ell_j - \ell_{j+1} = \ell_{n-j} - \ell_{n-j+1} > 1$, the multiplicity and signature of $I(\lambda(t))$ remain the same on the level of adjoint K-type for all $t \geq 0$.
- When t is big, $I(\lambda(t))$ has 'big' infinitesimal character, so one can apply Dirac inequality to $I(\lambda(t))$ to conclude that $I(\lambda(0)) = I(\lambda)$, and hence π_{sp} have indefinite forms on the trivial and adjoint K-type.

Side note: Arguments of this kind appeared in Bang-Jensen's work in the early 90s!

Proof of SV-Conjecture for $GL(n, \mathbb{C})$ - General Case

Now go to the general case, i.e. $\pi = J(\lambda_L; \lambda_R)$ with

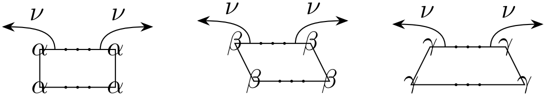


- By Vogan's λ_a-bijection $\Phi : \Pi_a^{\lambda_a'}(L) \xrightarrow{\cong} \Pi_a^{\lambda_a}(G)$, $L = \prod_p GL(\#\text{ of } \frac{m_p}{2})$ and
$$\Phi\left(\wedge^{top}\overline{\mathfrak{u}} \otimes (\boxtimes_p \pi_p)\right) = \pi, \quad \text{where } \pi_p = \det^{m_p} \otimes J(\nu_p; \nu_p).$$
- Consider π_p. If there is a '> 1 gap' in the coordinates ν_p, then π_p has opposite signatures on the lowest K-type $(m_p, \ldots, m_p)$ and the adjoint K-type $(m_p+\mathbf{1}, m_p, \ldots, m_p, m_p-\mathbf{1})$.
- These K-types are L-**bottom layer**, so Φ preserves their signatures.
- Therefore, π has indefinite forms on the lowest K-type and $(\cdots, m_{p-1}, m_p+\mathbf{1}, m_p, \ldots, m_p, m_p-\mathbf{1}, m_{p+1}, \cdots)$.

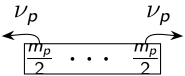
Consequently, the coordinates of ν_p has gap ≤ 1 for all p. So all > 1 gaps in λ_L (if any) must occur between two different blocks. However, by unitarily small condition, the values of $\frac{m_p}{2}$, $\frac{m_{p+1}}{2}$ blocks differ by ≤ 1!

Other groups?

- We expect similar techniques can be applied to other groups. For instance, all complex reductive groups have Levi subgroups consisting of Type A factors.
- The same goes for $U(p,q)$: Any real Levi subgroup of $U(p,q)$ is a product of $GL(k,\mathbb{C})$'s with $U(p',q')$.
- By Barbasch, Salamanca-Riba (which I learnt from Vogan's talks), the Langlands parameters for representations of $U(p,q)$ can be represented by 'blocks' such as:



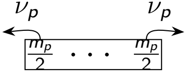
with $\alpha, \beta, \gamma, \cdots \in \frac{1}{2}\mathbb{N}$.

- These blocks just behaves like the  blocks in $GL(n,\mathbb{C})$.

- There are several theoretical obstacles to overcome, such as bottom layer arguments may not work nicely, composition factors of induced modules and so on...

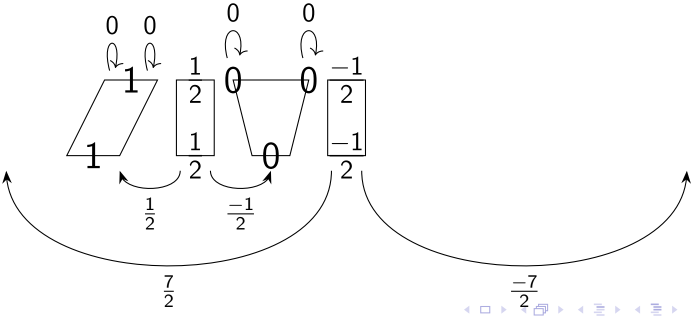
Fundamental blocks for $U(p, q)$

- But these difficulties can be overcome!
- Recall that in $GL(n, \mathbb{C})$, the fundamental case to study is the



(pseudo)-spherical block

- Once the above case is understood, the general case for $GL(n, \mathbb{C})$ follows from bottom layer K-type arguments.
- In $U(p, q)$, the fundamental cases to study are the **semi-spherical blocks**, for example:



Thank you!