How to compute the unitary dual

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Outline

Slides eventually at
`http://www-math.mit.edu/~dav/paper.html`

What's this about really?

$G(\mathbb{R})$ any real reductive algebraic group.

$\widehat{G(\mathbb{R})}_u$ = (equiv classes of) irr unitary reps of $G(\mathbb{R})$.

I'll assume that studying this set (the unitary dual problem) is the most world's best problem.

How can you approach it?

Goal for today: focus on a small piece of the unitary dual problem for which the answer involves some interesting and accessible mathematics; and which displays many ideas from the general case.

Two important subgroups for $GL(n,\mathbb{R})$

$$\begin{aligned} K(\mathbb{R}) &= O(n) = \text{orthogonal group,} \\ A &= \text{positive diagonal matrices,} \\ A^+ &= \text{positive diag mats with decreasing entries.} \end{aligned}$$

Any invertible $n \times n$ real g has a polar decomposition

$$g = k_1 a k_2, \qquad (a \in A^+, \quad k_i \in O(n)).$$

Matrix a is unique. Diag entries are the singular values of g. Largest singular value is

$$a_1 = \max_{v \in \mathbb{R}^n \backslash 0} \frac{||gv||}{||v||},$$

the largest amount that g can stretch a vector.

Similarly, a_n is the least that g can shrink a vector.

Since $K(\mathbb{R})$ is compact, polar decomp says that A—better, A^+—enumerates all ways to go to infinity in $G(\mathbb{R})$.

So what can you do with KAK?

$$K = O(n) = \text{orthogonal group},$$
$$A = \text{positive diagonal matrices},$$
$$A^+ = \text{positive diag mats with decreasing entries}.$$

Study harmonic functions on the unit disc by boundary values: limiting behavior in radial directions.

Same applies to functions on $GL(n,\mathbb{R}) = KAK$: helps to study limiting behavior in the A variable, particularly along A^+.

(approximate) Theorem (Harish-Chandra). If ϕ nice function on $GL(n,\mathbb{R})$, (say matrix coeff of irr rep) then there is an asymptotic expansion at infinity on A^+

$$\phi(k_1 a k_2) \sim c(k_1,k_2)a^\nu + \text{ lower terms}, \quad (a \in A^* \to \infty)$$

with $\nu \in \mathbb{C}^n$. Here $a^\nu = a_1^{\nu_1} \cdot \cdots \cdot a_n^{\nu_n}$, and "lower terms" are

$$c_m(k_1,k_2)a^{\nu-m}, \qquad m \in \mathbb{Z}^n \text{ sum of } e_i - e_j \text{ with } i < j.$$

Condition on m makes a^{-m} decay exponentially on A^+.

$$\text{irr repn } \pi \overset{\text{mat coeff}}{\longrightarrow} \text{ function } \phi \overset{\text{asymp}}{\longrightarrow} \nu \in \mathbb{C}^n = \mathfrak{a}^*.$$

This is a hint of what the Langlands classification looks like.

Two important subgroups for $G(\mathbb{R})$

Suppose $G(\mathbb{R})$ real reductive algebraic. Define

$$\begin{aligned} K(\mathbb{R}) &= \text{maximal compact subgroup},\\ G(\mathbb{R}) &= K(\mathbb{R})AN(\mathbb{R}) \quad \text{Iwasawa decomposition}\\ A &\simeq \mathfrak{a} = \mathrm{Lie}(A) \quad \text{vector group}\\ A^+ &= \text{subgroup acting on } \mathfrak{n}(\mathbb{R}) \text{ by eigvals} \geq 1. \end{aligned}$$

Any $g \in G(\mathbb{R})$ has a Cartan decomposition

$$g = k_1 a k_2, \qquad (a \in A^+, \;\; k_i \in K(\mathbb{R})).$$

Element a is unique. Measures distance of g from $K(\mathbb{R})$.

Since $K(\mathbb{R})$ is compact, polar decomp says that A^+ enumerates ways to go to infinity in $G(\mathbb{R})$.

So what can you do with *KAK*?

Study nice functions on $G(\mathbb{R}) = KAK$ via their limiting behavior in the the A variable, particularly along the cone A^+.

(approximate) Theorem (Harish-Chandra). If ϕ nice function on $G(\mathbb{R})$, (say matrix coeff of irr rep) then there is an asymptotic expansion at infinity on A^+

$$\phi(k_1 a k_2) \sim c(k_1, k_2) a^\nu + \text{ lower terms}, \quad (a \in A^* \to \infty)$$

with $\nu \in \mathfrak{a}^*$. Here $a^\nu = \exp(\nu(\log(a)))$; "lower terms" are

$$c_m(k_1, k_2) a^{\nu - m}, \qquad m \in \mathfrak{a}^* \text{ sum of weights of } \mathfrak{a} \text{ on } \mathfrak{n}(\mathbb{R}).$$

Condition on m makes a^{-m} decay exponentially on A^+.

$$\text{irr repn } \pi \overset{\text{mat coeff}}{\longrightarrow} \text{ function } \phi \overset{\text{asymp}}{\longrightarrow} \nu \in \mathbb{C}^n = \mathfrak{a}^*.$$

This display is the idea of the Langlands classification: irreducible representations of $G(\mathbb{R})$ are approximately indexed by complex-valued linear functionals on the real vector space $\mathfrak{a}$.

Langlands classification for spherical reps

$G(\mathbb{R}) = K(\mathbb{R})AN(\mathbb{R})$ Iwasawa decomposition.

Real vector space $\mathfrak{a}$ comes with (maybe not reduced) restricted root datum $(X^*, R, X_*, R^\vee)$, so small Weyl group W_A.

Repn (π, V_π) of $G(\mathbb{R})$ called spherical if $V_\pi^{K(\mathbb{R})} \neq 0$.

Theorem (Harish-Chandra)

1. Irreducible (not necessarily unitary) spherical reps of $G(\mathbb{R})$ are in bijection with $\mathfrak{a}^*/W_A$.
2. Suppose π is such a representation, $v \in V_\pi^{K(\mathbb{R})}$, $\lambda \in (V_\pi^d)^{K(\mathbb{R})}$, and $\lambda(v) = 1$. Then the function
$$\phi_\pi(g) = \lambda(\pi(g)v) \in C^\infty(G)$$
is $K(\mathbb{R})$-bi-invariant, indep of choices of v and λ.
3. The function ϕ_π has an asymptotic expansion along A^+ with a leading term
$$a \mapsto a^{\nu-\rho}, \quad \nu \in \mathfrak{a}^*, \quad \mathrm{Re}(\nu(H_\alpha)) \geq 0 \quad (\text{all } H_\alpha \in R^{\vee,+}).$$
4. The correspondence in (1) is $\pi \mapsto W_A \cdot \nu$.

What does that tell you?

Function $\phi_\pi(g) = \lambda(\pi(g)v)$ is a matrix coeff of π.

Representation π is unitarizable iff ϕ_π is positive definite, so that's the big question.

Reduction from Knapp's book *Overview*: write

$$\nu = \nu_{\text{Re}} + i\nu_{\text{im}}, \qquad \text{with } \nu_{\text{Re}} \text{ and } \nu_{\text{im}} \text{ real-valued linear functionals}$$

$$P_{\nu_{\text{im}}}(\mathbb{R}) = L_{\nu_{\text{im}}}(\mathbb{R}) U_{\nu_{\text{im}}}(\mathbb{R}) \quad \text{parabolic def by } \nu_{\text{im}}.$$

$$\pi_L = \text{spherical rep of } L_{\nu_{\text{im}}(\mathbb{R})} \text{ defined by } \nu_{\text{Re}}.$$

Then π is unitary for $G(\mathbb{R})$ iff π_L is unitary for $L_{\nu_{\text{im}}}(\mathbb{R})$;

and in this case π_L is unitarily induced from $P_{\nu_{\text{im}}}(\mathbb{R})$.

Reduced big question: for which real $\nu \in \mathfrak{a}^*$ is π unitary?

Theorem (Helgason-Johnson): ϕ_π is bdd iff $\nu \in \text{cvx hull}(W_A \cdot \rho)$.

So need to run the unitarity algorithm on all $\nu \in$ cvx hull.

Good news: that's a compact polyhedron.

Bad news: it's enormous.

Worst news: it's uncountably infinite.

Polygon Pollyanna

$$\mathfrak{a}_0^* \supset \text{cvx hull } \langle W \cdot \rho \rangle \supset \langle W \cdot \rho \rangle \cap \mathfrak{a}_0^{*,+} =_{\text{def}} HJ$$

Worst news was that we need to check unitarity for uncountably many points in HJ.

"Pollyanna" is one who looks at huge polytope and says, "There must be a root datum in here somewhere."

Recall restricted root datum $(X^*, R, X_*, R^\vee)$.

Root datum $\rightsquigarrow$ aff coroot hyperplanes and aff reflections

$$H_{\alpha^\vee,m} = \{\nu \in \mathfrak{a}_0^* \mid \langle \nu, \alpha^\vee \rangle = m\}, \quad s_{\alpha^\vee,m}(\nu) = \nu - (\langle \nu, \alpha^\vee \rangle - m)\alpha.$$

Affine Weyl group $W_{A,\text{aff}} = \langle s_{\alpha^\vee,m} \rangle$, a Coxeter group.

Like any loc fin hyperplane arrangement, affine coroot hyperplanes partition $\mathfrak{a}_0^*$ into facets, each the interior of a (probably lower dimensional) convex polytope.

Any compact set (like HJ) meets only finitely many facets.

Unitarity status is constant on facets.

How to compute (spherical) unitary dual

1. Find a compact set (like $HJ = \langle W \cdot \rho \rangle \cap \mathfrak{a}_0^{*,+}$) containing all the unitary points.
2. Compute the (finite) partition of your set into facets.
3. On each facet, test the unitarity of one point.

Barbasch and Ciubotaru have an improvement of (1):

The fundamental parallelepiped is

$$FPP =_{\text{def}} \left\{ \nu \in \mathfrak{a}_0^* \mid 0 \le \langle \nu, \alpha^\vee \rangle \le 1, \text{ all simple restricted coroots } \alpha^\vee \right\}$$

At least for $G(\mathbb{R})$ split, they prove

$$\text{if } \nu \in \mathfrak{a}_0^{*,+} \text{ and } \pi(\nu) \text{ unitary, then } \nu \in FPP.$$

The set *FPP* is much smaller than *HJ*, so computationally this is a big improvement.

Connection with the Dirac inequality

The original abstract promised that there would be a connection with Dirac operators.

Parthasarathy's Dirac inequality says

$$\text{if } \nu \in \mathfrak{a}_0^{*,+} \text{ and } \pi(\nu) \text{ unitary, then } \langle \nu, \nu \rangle \leq \langle \rho, \rho \rangle.$$

This statement is slightly weaker than the Helgason-Johnson bound appearing two slides earlier.

It would be interesting to use ideas related to the unitary representation $V_\pi \otimes \text{Spin}$ of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$—that is, to the world of the Dirac operator—to prove the B-C statement about *FPP* on the previous slide.

Really what I would like is a proof of a statement like that of Barbasch-Ciubotaru applicable to not-necessarily spherical representations.